
Exercise

We consider $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric $g^H_{(x,y)} = \frac{1}{y^2}\langle \cdot, \cdot \rangle_{eukl}$ (*Half-space model of Hyperbolic space*). The mapping $\phi(x, y) = (x, y)$ is a chart.

(a) Compute the Christoffel symbols with respect to ϕ: ${}^{\phi}\Gamma^1_{11} = {}^{\phi}\Gamma^2_{12} = {}^{\phi}\Gamma^1_{22} = 0$, ${}^{\phi}\Gamma^2_{11} = \frac{1}{y}$, and ${}^{\phi}\Gamma^1_{12} = {}^{\phi}\Gamma^2_{22} = -\frac{1}{y}$.

(b) Let $v_0 = (0, 1) \in T_{(0,1)}H$, and let $P^{\gamma}_{0,t} v_0 = v(t)$ denote the parallel transport along the curve $\gamma(t) = (t, 1)$. Show that $g^H_{\gamma(t)}((0, 1), v(t)) = \cos t$.

Hints: $v(t) = (a(t), b(t))$ satisfies an ordinary differential equation. To solve it, assume $a(t) = \cos\theta(t)$, $b(t) = \sin\theta(t)$. Determine θ.