
Aufgabe 1 (4 Punkte)
Let $g^{eucl} = \langle \cdot, \cdot \rangle$ be the Euclidean metric on $\mathbb{R}^3$, i.e.

$$g_x^{eucl}(v, w) = \langle v, w \rangle = \sum_{i=1}^{3} v^i w^i$$

for $x \in \mathbb{R}^3$ and $v, w \in T_x\mathbb{R}^3 \simeq \mathbb{R}^3$. Let $F : V = (0, \infty) \times (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$ be given by

$$F(r, \varphi, \theta) = (r \sin\varphi \sin\theta, r \cos\varphi \sin\theta, r \cos\theta) \text{ für } (r, \varphi, \theta) \in V.$$

The map $F^{-1} = \phi : F(V) \to V$ is a chart of $\mathbb{R}^3$.

(a) Calculate the coefficients of the Euclidean metric g^{eucl} in this chart. Calculate the lengths of the curve $c : (0, 2\pi) \to \mathbb{R}^3$, $t \mapsto F(r_0, t, \theta_0)$ for fixed r_0 and fixed θ_0 in this chart.

(b) Calculate the volume of the *shell* $x \in \mathbb{R}^3 : r_0 < |x|_{eucl} < r_1$ where $|x|_{eucl} = \langle x, x \rangle^{\frac{1}{2}}$.

Aufgabe 2 (4 Punkte)

(a) Let (M, g) be a Riemannian manifold, $\phi, \psi : U \subset M \to \mathbb{R}^n$ be charts, and let $A \subset U$ be measurable. Show that the Riemannian volume $\mathrm{vol}^g(A)$ of A does not depend on the choice of chart.

(b) Show that the Riemannian volume $\mathrm{vol}^g(A)$ of a measurable set $A \subset M$ is independent of the choice of measurable partition used in the definition of $\mathrm{vol}^g(A)$.

Aufgabe 3 (4 Punkte)

(a) Verify the statements in Example 2.7 from the lecture.

(b) Apply the Example 2.7 to the parametrization $X : B_1(0) = x \in \mathbb{R}^2 : (x^1)^2 + (x^2)^2 < 1 \to \mathbb{R}^3$, $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$, of the upper hemisphere.

Abgabe am Montag, 02. Mai bis 12 Uhr im Postfach des Assistenten.