Aufgabe 1 (4 Punkte)

Let (M, g) be a Riemannian manifold, $f \in C^\infty(M)$, and $X \in \Gamma(TM)$ a smooth, nowhere vanishing vector field. Show that $X = \nabla f$ if and only if $X(f) = |X|_g^2$ and X is orthogonal to all level sets of f at all regular points of f.

Aufgabe 2 (4 Punkte)

Let (M, g) and (N, h) be connected Riemannian manifolds, and let d_g and d_h be the corresponding Riemannian distance functions. Show that:

(a) If $\phi : M \to N$ is a local Riemannian isometry, then $d_h(\phi(x), \phi(y)) \leq d_g(x, y)$.

(b) If $\phi : M \to N$ is a Riemannian isometry, then $d_h(\phi(x), \phi(y)) = d_g(x, y)$.

Aufgabe 3 (4 Punkte)

(a) Let $x \in \mathbb{H}^n$ and $v \in \mathbb{R}^{n+1}_1$ with $\langle x, v\rangle_1 = 0$ and $\langle v, v\rangle_1$. Let $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}_1$ be given by $\gamma(t) = \cosh t \cdot x + \sinh t \cdot v$. Show that: $\gamma(t) \in \mathbb{H}^n$ for all $t \in \mathbb{R}$, $\gamma(0) = x$ and $\gamma'(0) = (x, v) \in T_x\mathbb{H}^n$, and γ is parameterized by arc length.

(b) Show that if $c : [a, b] \to \mathbb{H}^n$ is a C^1 curve with $c(a) = e_0$ and $c(b) =: (p_0, \dots, p_n)$, then

$$L^h(c) \geq \operatorname{arcosh} p_0 - 1$$

where $h = i^*\langle\cdot,\cdot\rangle_1$ with $i : \mathbb{H}^n \to \mathbb{R}^{n+1}_1$, $i(x) = x$. When does equality hold?

Abgabe am Montag, 08. Mai bis 12 Uhr beim Assistenten.