Problem 1 (4 points)

A variant of the Cayley transformation between $H = \{x + iy \in \mathbb{C} : y > 0\}$ and $B_1(0) = \{x + iy \in \mathbb{C} : |x + iy| < 1\}$ is $\phi(z) = i\frac{z-i}{z+i}$. Show:

(a) In real variables, ϕ can be written as

$$\phi(x, y) = \left(\frac{2x}{|x|^2 + (y+1)^2}, \frac{|x|^2 + |y|^2 - R^2}{|x|^2 + (y+1)^2} \right).$$

(b) ϕ is an isometry between the half-plane model and the Poincaré model of the hyperbolic space.

Problem 2 (4 points)

A set $\Gamma = \{\sum_{i=1}^n k_i e_i : k_i \in \mathbb{Z}\}$ for a basis $(e_i)_{i=1,\dots,n} \subset \mathbb{R}^n$ is called a lattice. It is known from differential geometry that $\mathbb{R}^n/\Gamma$ is diffeomorphic to the torus $\mathbb{T}^n$. Let $\langle \cdot, \cdot \rangle_{eucl}$ be the Euclidean metric on $\mathbb{R}^n$. Show:

(a) There exists a unique Riemannian metric on g_Γ on $\mathbb{R}^n/\Gamma$ such that the projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ is a local isometry, i.e., $\pi^* g_\Gamma = \langle \cdot, \cdot \rangle_{eucl}$.

(b) $(\mathbb{R}^n/\Gamma, g_\Gamma)$ is isometric to $(\mathbb{R}^n/\Gamma', g_{\Gamma'})$ if and only if there exists an isometry F of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{eucl})$ with $F(\Gamma) = \Gamma'$.

Problem 3 (4 points)

A Riemannian metric g on a Lie group G is called left-invariant (right-invariant) if for each $h \in G$, the left translation l_h (or right translation r_h) is an isometry. A metric that is both left- and right-invariant is called bi-invariant.

(a) Show: If b is a scalar product on $T_e G$, then there exists a unique left-invariant Riemannian metric g on G with $g_e = b$.

(b) Let $G = SO(n)$ and $b(A, B) := \text{trace}(A^T B)$ for $A, B \in T_{E_n} SO(n)$. Show that the left-invariant metric induced by b is bi-invariant.

Abgabe am Montag, 15. Mai bis 12 Uhr beim Assistenten.