
Exercise 1 (4 points)

Let $\rho : \mathbb{R}^+ \to \mathbb{R}$ be defined by $\rho(r) = r$, and $\mathbb{R}^+ \times_\rho \mathbb{S}^{n-1}$ denotes the corresponding *warped product* where $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \langle x, x\rangle_{eucl} = 1\}$ with the restricted Euclidean metric. We define $\Phi : \mathbb{R}^+ \times_\rho \mathbb{S}^{n-1} \to \mathbb{R}^n \backslash \{0\}$ by $\Phi(r, w) = rw$. Show that Φ is an isometry between the warped product metric and the Euclidean metric $\langle \cdot, \cdot \rangle_{eucl}$ on $\mathbb{R}^n \backslash \{0\}$.

Exercise 2 (4 points)

Let M be a submanifold of $\mathbb{R}^n$, and $i^*\langle \cdot, \cdot\rangle_{eucl} = g$ be the Riemannian metric induced by the embedding $i : M \to \mathbb{R}^n$. We denote by $TM^\perp = \bigcup_{p\in M} T_pM^\perp$ the normal bundle with the corresponding projection map $\pi^\perp : TM^\perp \to M$, where $T_pM^\perp = v \in T_p\mathbb{R}^n \simeq \mathbb{R}^n : \langle v, w\rangle_{eucl} = 0 \; \forall w \in TM$. Show that a connection can be defined on the normal bundle as follows:

$$(X, Y) \in \Gamma(TM) \times \Gamma(TM^\perp) \mapsto \nabla^\perp_X Y := \left(\bar{\nabla}_X Y\right)^\perp \in \Gamma(TM^\perp),$$

where $\bar{\nabla}$ denotes the standard connection on $T\mathbb{R}^n$.

Exercise 3 (4 points)

Let M be a smooth manifold, and ∇ be a connection on TM. Let $(E_i)_{i=1,\dots,m}$ and $(\tilde{E}_i)_{i=1,\dots,m}$ be smooth local basis vector fields for TM on an open set $U \subset M$. There exists a matrix-valued function $A : U \to Gl(m, \mathbb{R})$ with smooth entries $A^j_i \in C^\infty(U)$, such that $\tilde{E}_i = A^j_i E_i$ on U. Let Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$ be the corresponding Christoffel symbols. Show that

$$\tilde{\Gamma}^k_{ij} = (A^{-1})^k_l A^r_i A^s_j \Gamma^l_{rs} + (A^{-1})^k_l A^r_i E_r(A^l_j).$$

Abgabe am Montag, 23. Mai bis 12 Uhr beim Assistenten.