
Exercise 1 (4 points)
Let (M,g) be a Riemannian manifold, $\gamma : [a,b] \to M$ be a smooth curve, and X be a vector field along γ. Let $\phi = (x^1,\dots,x^m) : U \to V$ be a chart. Then we have $X(t) = \sum_{i=1}^m X^i(t)\frac{\partial}{\partial x^i} \circ \gamma(t)$ and $\phi \circ \gamma(t) =: (\gamma^1(t),\dots,\gamma^m(t))$. Show that using Exercise 3, Sheet 4, the definition

$$\nabla_{\gamma'(t)} X|_t = \sum_{k=1}^m \left((X^k)'(t) + \sum_{i,j=1}^m {}^{\phi}\Gamma_{ij}^k \circ \gamma(t)(\gamma')^i X^j(t) \right) \frac{\partial}{\partial x^k} \circ \gamma(t)$$

is independent of the chart ϕ.

Exercise 2 (4 points)
Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection, and $\gamma : (a,b) \to M$ be a smooth curve. Show that:

(a) If $\varphi : (c,d) \to (a,b)$ and $V \in \Gamma(\gamma^* TM)$ is parallel along γ, then $\tilde{V}(t) := V \circ \varphi(t)$ is parallel along $\tilde{\gamma} := \gamma \circ \varphi$.

(b) For $X \in \Gamma(TM)$, we have

$$\nabla_{\gamma'(s)} X = \frac{d}{dt}\Big|_{t=s} P^{\gamma}_{t,s} X \circ \gamma(t)$$

(*The parallel transport determines the connection*).

Exercise 3 (4 points)
Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular C^∞ map (a regular smooth surface), $X_i = \frac{\partial X}{\partial x^i}$, $i = 1,2$, and $N := \frac{X_1 \times X_2}{|X_1 \times X_2|_{eucl}}$, where $\times$ denotes the cross product between vectors in $\mathbb{R}^3$. For $i,j,k = 1,2$, let Γ_{ij}^k be defined as

$$X_{ij} := \frac{\partial^2 X}{\partial x^i \partial x^j} = \sum_{k=1}^2 \Gamma_{ij}^k X_k + h_{ij} N.$$

Show that the coefficients Γ_{ij}^k are the Christoffel symbols of the induced Riemannian metric $X^*\langle \cdot,\cdot \rangle_{eucl}$ on U with respect to the chart $\phi(x^1,x^2) = (x^1,x^2)$.

Additional Exercise (4 extra points)

(a) Compute the Christoffel symbols of the standard connection $\bar{\nabla}$ on $T\mathbb{R}^3$ with respect to the coordinates defined in Sheet 1, Exercise 1 (polar coordinates).

(b) Let (M, g) be a Riemannian manifold. Prove the statement in Lemma 3.6: Let $\gamma \in C^\infty(I, M)$, $V, W \in \Gamma(\gamma^* TM)$. Then we have

$$\frac{d}{dt}\Big|_{t=0} g_{\gamma(t)}(V(t), W(t)) = g_{\gamma(t)}(\nabla_t V|_t, \nabla_t W|_t).$$

Abgabe am Dienstag, 6. Juni bis 12 Uhr beim Assistenten.