
Exercise 1 (4 points)

Let (M,g) and $(\tilde{M},\tilde{g})$ be Riemannian manifolds, and let $F : M \to \tilde{M}$ be an isometry. Denote $\nabla, \tilde{\nabla}$ as the Levi-Civita connections, and $R, \tilde{R}$ as the curvature tensors of g and $\tilde{g}$, respectively. Show that:

(a) If $\tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M})$ are F-related to $X, Y \in \Gamma(TM)$ (i.e., for all $p \in M$, $DF_p(X_p) = \tilde{X}_{F(p)}$ and $DF_p(Y_p) = \tilde{Y}_{F(p)}$), then $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ is F-related to $\nabla_X Y$.

(b) If $v, w, z \in T_pM$, $\tilde{v} = DF_p(v)$, $\tilde{w} = DF_p(w)$, and $\tilde{z} = DF_p(z)$, then $DF_p(R(v,w)z) = \tilde{R}(\tilde{v},\tilde{w})\tilde{z}$.

Exercise 2 (4 points)

Let (M,g) be a Riemannian manifold, $\pi : TM \to M$ be the tangent bundle, and $\phi : U \to V$ be a chart. ϕ induces a local trivialization of TM over U, i.e., $\pi^{-1}(U) = U \times \mathbb{R}^m$, where $E_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, m$, is a basis field of TM on U. Let ω_i^j be the connection 1-forms and Ω_i^j be the curvature 2-forms defined by

$$\nabla_X E_i =: \sum_{i=1}^{m} \omega_i^j(X) E_j \text{ und } R(X,Y)E_i =: \sum_{i=1}^{m} \Omega_i^j(X,Y)E_j \quad \forall X, Y \in \Gamma(TU).$$

Show that

$$\Omega_i^j = d\omega_i^j - \sum_{k=1}^{m} \omega_i^k \wedge \omega_k^j.$$

Exercise 3 (4 points)

A vector field X on a Riemannian manifold (M,g) is called a Killing vector field if the (local) flow Φ_t^X of X is isometric for every $t \in \mathbb{R}$. Show that:

(a) A vector field $X(x) = (x, V(x))$ on $\mathbb{R}^n$ with the Euclidean metric $\langle \cdot, \cdot \rangle_{eucl}$ is a Killing vector field if and only if there exists a skew-symmetric matrix $A \in \mathcal{M}(n, \mathbb{R})$ and a vector $v \in \mathbb{R}^n$ such that $V(x) = Ax + v$.

(b) A vector field X on the Riemannian manifold (M,g) is a Killing vector field if and only if

$$g(\nabla_v X|_p, w) + g(v, \nabla_w X|_p) = 0 \; \forall p \in M, \; \forall v, w \in T_pM$$

i.e., the bilinear form $(v,w) \mapsto g(\nabla_v X|_p, w)$ is skew-symmetric.

Abgabe am Dienstag, 13. Juni bis 12 Uhr beim Assistenten.