
Exercise 1 (4 points) Let V be a vector space. The contraction $c_{ij} : \bigotimes_{k=1}^{s} V^* \otimes \bigotimes_{l=1}^{r} V \to \bigotimes_{k=1}^{s-1} V^* \otimes \bigotimes_{l=1}^{r-1} V$ is the linear mapping uniquely defined by the assignment

$$c_{ij}(v_1 \otimes \cdots \otimes v_r \otimes v^1 \otimes \cdots \otimes v^s) := v^j(v_i)(v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_r \otimes v^1 \otimes \cdots \otimes \hat{v}^j \otimes \cdots \otimes v^s).$$

on the monomials. Let M be a manifold. Show that:

(a) If ∇ is a linear connection on TM, then for $r, s \geq 0$ there exists a unique linear connection $\nabla^{r,s}$ on $T^r_s M$ satisfying the following properties:

(i) For all $S \in \Gamma(T^r_s M)$, $X \in \Gamma(TM)$, and any contraction c,

$$\nabla_X^{r-1,s-1}(c(S)) = c(\nabla_X^{r,s} S).$$

(ii) For all $S \in \Gamma(T^r_s M)$, $T \in \Gamma(T^p_q M)$, and $X \in \Gamma(TM)$,

$$\nabla_X^{r+p,s+q}(S \otimes T) = (\nabla_X^{r,s} S) \otimes T + S \otimes (\nabla_X^{p,q} T).$$

(b) Let (M, g) be a Riemannian manifold, and ∇ be a torsion-free connection. Show that ∇ is the Levi-Civita connection if and only if $\nabla_w^{0,2} g = 0$.

Exercise 2 (4 points)

Let g be a bi-invariant Riemannian metric on the Lie group G (cf. Sheet 3, Exercise 3) and ∇ be the corresponding Levi-Civita connection. Show that:

(a) Every left-invariant vector field is a Killing vector field. (Sheet 6, Exercise 3)

(b) If X is a left-invariant vector field, then $\nabla_X X = 0$ and every flow line of X is a geodesic.

(c) If X and Y are left-invariant vector fields, then $\nabla_X Y = \frac{1}{2}[X, Y]$.
Hints: Use the torsion-freeness of the Levi-Civita connection.

Exercise 3 (4 points)

Let (M, g) be a Riemannian manifold and $\phi : U \to V$ be a chart. Denote by ${}^{\phi}g_{ij}$ the coefficients of the local representation of g. Let $p \in U$ with $\phi(p) = 0$ and assume ${}^{\phi}g_{ij}(p) = \delta_{ij}$. Show that:

(a) $d\left({}^{\phi}g_{ij}\right)|p = 0$ if and only if ${}^{\phi}\Gamma ij^k(p) = 0$ for all $i, j, k = 1, \ldots, m$.

(b) There exists a diffeomorphism $\varphi : V \subset \mathbb{R}^m \to \tilde{V}$ such that for $\psi = \varphi \circ \phi$ it holds ${}^{\psi}\Gamma^k_{ij} = 0$ for all $i, j, k = 1, \ldots, m$.
Hints: Consider a polynomial of the form $\varphi(x) = \sum_{k=1}^{m} (x^k + C^k_{ij} x_i x_j) e_k$.

Abgabe am Dienstag, 20. Juni bis 12 Uhr beim Assistenten.