
Exercise 1 (4 points)
Let $(M, g = \langle \cdot, \cdot \rangle)$ be a Riemannian manifold.

(a) Let $f \in C^\infty(M)$. The tensor $\nabla^2 f \in \Gamma(T_2^0 M)$ given by

$$\nabla^2 f(X, Y) := \langle \nabla_X \nabla f, Y \rangle,$$

is called the Hessian form of f. Compute $\nabla^2 f$ in local coordinates and show that ∇^2 is symmetric and coincides with the second derivative in the case $(M, g) = (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{eucl})$.

(b) Let $f \in C^\infty(M)$ and $\langle \nabla f, \nabla f \rangle = const$. Show: The integral curves of ∇f are geodesics.

Exercise 2 (4 points)
Let (M_i, g_i), $i = 0, 1$, be two Riemannian manifolds, and let $M_0 \times M_1$ be the product manifold together with the product metric $g = \pi_0^* g_0 + \pi_1^* g_1$, where $\pi_i : M_0 \times M_1 \to M_i$ denotes the projection onto the ith factor. Let $\phi_i : U_i \to V_i$ be a chart on M_i, $i = 0, 1$.

(a) Compute the Christoffel symbols of the Levi-Civita connection of $(M_0 \times M_1, g)$ with respect to the chart $\phi = (\phi_0, \phi_1)$ using the Christoffel symbols of g_i with respect to the chart ϕ_i.

(b) Show: $c = (c_0, c_1) : (a, b) \to (M_0 \times M_1, g)$ is a geodesic if and only if $c_0 : (a, b) \to (M_0, g_0)$ and $c_1 : (a, b) \to (M_1, g_1)$ are geodesics.

Exercise 3 (4 points) Regarding the *Noether's theorem.*

(a) Let (M, g) be a Riemannian manifold and $X \in \Gamma(TM)$. Show: The function $F : TM \to \mathbb{R}$ defined by $F(v) = \langle X \circ \pi(v), v \rangle$ is an integral of the geodesic flow (i.e., $\langle X(c(t)), c'(t) \rangle = const$ for every geodesic c) if and only if X is a Killing vector field.

(b) *Theorem of Clairaut.* Let $M \subset \mathbb{R}^3$ be a surface of revolution. If c is a curve on M, let $\theta(t)$ be the angle at $c(t)$ between $c'(t)$ and the circle of revolution passing through $c(t)$. $r(t)$ is the distance from $c(t)$ to the axis of revolution. Show using part (a) that $r(t) \cos \theta(t)$ is constant if c is a geodesic.

Abgabe am Dienstag, 27. Juni bis 12 Uhr beim Assistenten.