**Exercise 1** (4 points)
Let $(M,g)$ be a Riemannian manifold, and let $f \in C^\infty(M)$. $\nabla^2 f \in \Gamma(T^0_2 M)$ denotes the Hessian form of $f$. $f$ is called convex if, for every $p \in M$, the symmetric bilinear form $\nabla^2 f|p$ is positive semidefinite. A subset $U \subset M$ is called totally convex if every geodesic $c : [a,b] \to M$ with $c(a), c(b) \in U$ lies entirely within $U$. Show that:

(a) If $f \in C^\infty(M)$ and $c : (a,b) \to M$ is a geodesic, then $\nabla^2 f|c(t)(c'(t), c'(t)) = (f \circ c)''(t)$.

(b) $f$ is convex if and only if $f \circ c : (a,b) \to \mathbb{R}$ is convex for every geodesic $c : (a,b) \to M$.

(c) If $f$ is convex, then for every $t \in \mathbb{R}$, the set $f^{-1}((-\infty, t))$ is totally convex.

**Exercise 2** (4 points)
Let $\mathbb{S}^n \subset \mathbb{R}^n$ be the unit sphere with the induced metric. Show that:

(a) For every $p \in \mathbb{S}^n$, $\exp_p$ is defined on the entire $T\mathbb{S}^n_p$ and $\exp_p |B\pi(0_p) : B_\pi(0_p) \to \mathbb{S}^n \backslash {-p}$ is a diffeomorphism.

(b) For $q \in \mathbb{S}^n \backslash {-p}$, compute the Hessian form of

$$Q(x) := \langle (\exp_p |_{B_\pi(0_p)})^{-1}(x), (\exp_p |_{B_\pi(0_p)})^{-1}(x) \rangle.$$

For which $q$ is the Hessian form $\nabla^2 Q|q$ positive definite, and for which $q$ is it positive semidefinite? *Hint: First show that $Q(x) = (\arccos\langle x, p\rangle)^2$. Then compute the Hessian form using Exercise 1 (a). Consider the radial and orthogonal components in $T_x\mathbb{S}^n$ with respect to $c_v'(1)$, where $v = (\exp_p |B_\pi(0_p))^{-1}(x)$.*

**Exercise 3** (4 points) Let $(M,g)$ be a connected Riemannian manifold.

(a) Show that $(M,g)$ is complete if and only if the following holds: For any subset $A \subset M$, $A$ is compact if and only if $A$ is closed and bounded with respect to $d_g$.

(b) Let $g$ and $\tilde{g}$ be Riemannian metrics on $M$. Show: If $\tilde{g}(v,v) \geq g(v,v)$ for all $v \in TM$ and $(M,g)$ is complete, then $(M,\tilde{g})$ is also complete.

(c) Let $M \subset \mathbb{R}^n$ be an embedded submanifold with the inclusion map $\iota : M \to \mathbb{R}^n$. Let $g = \iota^*\langle \cdot, \cdot \rangle$ be the Riemannian metric induced by the Euclidean metric. Show: If $M$ is closed as a subset of $\mathbb{R}^n$, then $(M,g)$ is complete.

*Abgabe am Dienstag, 04. Juli bis 12 Uhr beim Assistenten.*