
Exercise 1 (4 points) *Existence of convex neighborhoods.*
Let (M, g) be a Riemannian manifold, $p \in M$, and let $U \subset T_pM$ be open such that $\exp_p|_U : U \to V = \exp_p(U)$ is a diffeomorphism. Let $Q : V \to \mathbb{R}$ be defined by

$$Q(x) = \langle (\exp_p|_U)^{-1}(x), (\exp_p|U)^{-1}(x)\rangle.$$

Show that:

(a) For $v \in T_pM$, we have $\nabla^2 Q(v, v) = 2\langle v, v\rangle$.

(b) There exists a neighborhood $V' \subset V$ of p such that $Q|V' : V' \to \mathbb{R}$ is convex.

(c) There exists a neighborhood W of p such that for all $x, y \in W$, there is a unique shortest path between x and y in M, and it lies in W.

Hint: Exercise 1, Sheet 9.

Exercise 2 (4 points)

(a) Let $(V, \langle\cdot,\cdot\rangle)$ be an m-dimensional Euclidean vector space, and let $b : V \times V \to \mathbb{R}$ be a symmetric bilinear form. Let $S^{m-1} = v \in V : |v| = 1$ be the unit sphere in V. Show that:

$$\frac{1}{\mathrm{vol}_{m-1}(S^{m-1})} \int_{S^{m-1}} b(v, v) d\,\mathrm{vol}^{S^{m-1}} = \frac{1}{m}\mathrm{Tr}(b)$$

where $d\,\mathrm{vol}^{S^{m-1}}$ is the volume element on S^{m-1} induced by $(V, \langle\cdot,\cdot\rangle)$.

(b) Let (M, g) be an m-dimensional Riemannian manifold, $p \in M$, and $v \in T_pM$ with $|v| = 1$. Show that

$$\mathrm{ric}(v, v) = \frac{m-1}{\mathrm{vol}_{m-2}(S^{m-2})} \int_S K(\mathrm{Span}(u, v)) d\,\mathrm{vol}^S(u)$$

where $S = u \in T_pM : \langle u, v\rangle = 0, |u| = 1$ is the unit sphere in the orthogonal complement of $v \in T_pM$.

Exercise 3 (4 points)
Let G be a Lie group equipped with a bi-invariant metric $\langle\cdot,\cdot\rangle$. Show that:

(a) For $u, v, w, z \in T_eG$, we have:

$$R(u, v)w = -\frac{1}{4}[[u, v], w] \quad \text{and} \quad \langle R(u, v)w, z\rangle = \frac{1}{4}\langle [u, v], [z, w]\rangle.$$

If u and v are orthonormal, then $K(\mathrm{Span} u, v) = \frac{1}{4}|[u, v]|^2$.

(b) $SO(3)$ equipped with the bi-invariant metric from Exercise 3, Sheet 3, has constant sectional curvature. Calculate this constant.

Abgabe am Dienstag, 11. Juli bis 12 Uhr beim Assistenten.