Exercise 1 (4 points)

Let (M, g) and (N, h) be complete, connected Riemannian manifolds, and let N additionally be simply connected. Show: If $f : M \to N$ is a smooth mapping such that $Df_p : (T_pM, g_p) \to (T_{f(p)}N, h_{f(p)})$ is an isometry for all $p \in M$, then f is an isometry.

Exercise 2 (4 points)

Let (M, g) be a compact connected Riemannian manifold, and let d^g be the distance function with respect to g.

(a) Let (M, d) be a compact metric space and $V \subset M \times M$ be open with $\Delta \subset V$, where $\Delta = (x, x) \in M \times M$. Then there exists an $\epsilon > 0$ such that $(p, q) \in M \times M : d^g(p, q) < \epsilon$.

(b) There exists an $\epsilon > 0$ such that for all $p, q \in M$ with $d^g(p, q) < \epsilon$, there exists a unique geodesic $c_{p,q} : [0, 1] \to M$ with $c_{p,q}(0) = p$ and $c_{p,q}(1) = q$ and $L^g(c) < \epsilon$. Moreover, $c'p, q(0) \in TM$ depends smoothly on $(p, q) \in M \times M$.

(c) For all compact $K \subset M$, there exists an $\epsilon > 0$ such that for all $p \in K$, the map $\exp_p |B_\epsilon(0_p)$ is a diffeomorphism.

(d) If N is another manifold and $f, h \in C^\infty(N, M)$ with $d^g(f(x), h(x)) < \epsilon$ for all $x \in N$, where $\epsilon > 0$ is from (b), then f and h are homotopic.

Exercise 3 (4 points)

Compute the curvature of $\mathbb{H}^n_r$ using Jacobi fields. *Hint: Use the fact that $\mathbb{H}^n_r$ is a a frame homogeneous space.*

Freiwillig Abgabe beim Dozenten.