
Task 1 (4 points)
Let $(X, |\cdot,\cdot|)$ be a metric space, and $x_1, x_2 \in X$, and $r_1, r_2 \in (0, \infty)$. Prove:

(a) $|x_1 x_2| \geq r_1 + r_2 \quad \Rightarrow \quad B_{r_1}(x_1) \cap B_{r_2}(x_2) = \emptyset$;

(b) $|x_1 x_2| \leq r_1 - r_2 \quad \Rightarrow \quad B_{r_2}(x_2) \subset B_{r_1}(x_1)$.

Task 2 (4 points)

(a) Let (X, d) be a metric space and $S \subset X$. We define the distance to S as the function
$$d_S(x) := d(x, S) := \inf_{s \in S} d(x, s).$$
Prove: $d_S : X \to [0, \infty)$ is Lipschitz with a Lipschitz constant less than or equal to 1.

(b) Prove:

1. Lipschitz maps are continuous.
2. If $f : X \to Y$ and $g : Y \to Z$ are Lipschitz maps, then $g \circ f : X \to Z$ is a Lipschitz map, and $\operatorname{dil}(g \circ f) \leq \operatorname{dil} f \cdot \operatorname{dil} g$.
3. The set of Lipschitz maps from a metric space to a normed space is a vector space, and one has $\operatorname{dil}(f + g) \leq \operatorname{dil} f + \operatorname{dil} g$ and $\operatorname{dil}(\lambda f) = |\lambda| \operatorname{dil} f$ for $\lambda \in \mathbb{R}$.

Task 3 (4 points)
Let $(X, |\cdot,\cdot|)$ be a metric space. Prove:

$$|p, q| + |x, y| \leq |p, x| + |q, x| + |p, y| + |q, y| \ \forall x, y, p, q \in X.$$

Submission on Wednesday, October 25th, by 12 p.m. to the assistant.