
Problem 1 (4 Punkte)

(a) Let (X,d) be an ∞-pseudo-metric space. Show that the relation $x \sim y \Leftrightarrow d(x,y) < \infty$ is an equivalence relation, and the restriction of d to each equivalence class is a (finite) pseudo-metric.

(b) Let (X,d) be a pseudo-metric space. Show that the relation $xRy \Leftrightarrow d(x,y) = 0$ is an equivalence relation, and $\hat{d}([x],[y]) = d(x,y)$ is metric on $\hat{X}/R$.

Prolem 2 (4 Punkte)

(a) The diameter of a set S in a metric space X is defined by $\operatorname{diam}(S) = \sup_{x,y} |xy|$. Prove that the metric space X is complete if and only if it possesses the following property. If $\{X_n\}$ is sequence of closed subsets of X such that $X_{n+1} \subset X_n$ for all n, and $\operatorname{diam}(X_n) \to 0$ as $n \to 0$, then the sets X_n have a common point.

(b) Prove: a metric space is compact if and only if every sequence has converging subsequence.

Problem 3 (4 Punkte) Let X be a metric space. Prove that

(a)
1. If X is compact, then $\operatorname{diam}_X < \infty$.
2. There exist two points $x, y \in X$ such that $|xy| = \operatorname{diam}_X$.

(b) Define $\operatorname{rad}_X = \inf_{x \in X} \sup_{y \in X} |xy|$. Show that $\operatorname{rad}_X = \inf\{r > 0 : X \subset B_r(x) \text{ for } x \in X\}$.

(c) For two subsets $A, B \subset X$ define $d(A,B) = \inf\{|xy| : x \in A, y \in B\}$. Show that there exist $x \in A$ and $y \in B$ such that $|xy| = d(A,B)$ if A, B are compact. Does d satisfy the triangle inequality?

Problem 4 (Bonus) (2 Punkte)
Let X be a complete metric space, $0 < \lambda < 1$ and $f : X \to X$ a map such that $|f(x)f(y)| \leq \lambda |xy|$ for all $x, y \in X$. Show that there exists a point $x_0 \in X$ such that $f(x_0) = x_0$.

Abgabe am Mittwoch, 1. November bis 12 Uhr beim Assistenten.