Problem 1 (4 Punkte)
Let (X, d) be a metric space. We consider a continuous paths $\gamma : [a, b] \to X$. Recall that a partition Z is a tuple $(t_0, \ldots, t_N)$ with $t_0 = a \leq t_1 \leq \cdots \leq t_N = b \in [a, b]$. We define $|Z| = \max_{i=1,\ldots,N} |t_i - t_{i-1}|$ and $L^Z(\gamma) = \sum_{i=1}^N d(\gamma(t_{i-1}), \gamma(t_i))$. The induced length of γ is $L_d(\gamma) = \sup_{Z \text{ partition}} L^Z(\gamma)$. Show that

$$L(\gamma) = \lim_{i\to\infty} L^{Z_i}(\gamma)$$

where $\{Z_i\}_{i\in\mathbb{N}}$ are partitions such that $|Z_i| \downarrow 0$.

Problem 2 (4 Punkte)
Let $(V, |\cdot|)$ be a finite dimensional normed vector space and $\gamma : [a, b] \to V$ a differentiable path. Let (V, d) be the metric space associated to $(V, |\cdot|)$, i.e. $d(\cdot, \cdot) = |\cdot - \cdot|$. Let $L_d(\gamma)$ be the induced lenght. Show that

$$L_d(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Problem 3 (4 Punkte)
Consider $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with the induced Euclidean metric d_2. Show that

(a) $(\mathbb{S}^1, d_2)$ is not a length space.

(b) The induced intrinsic distance $\hat{d}_2$ is given by $\hat{d}_2(x, y) = \arccos\langle x, y\rangle_2$ where $\langle x, y\rangle_2$ is the Euclidean inner product.

Problem 4 (Bonus) (1 Punkte)
Show: If a length space (X, d) is homeomorphic to a segment, then it is isometric to a segment.

Abgabe am Mittwoch, 15. November bis 12 Uhr beim Assistenten.