
Problem 1 (4 Punkte)
Consider the region $E = \{(x,y) : 0 \le y \le e^{-x}, x \ge 0\}$ equipped with the induced Euclidean metric. This is the region enclosed between the graph of e^{-x}, $x \ge 0$, the x-axis, and the segment $\{(0,y) : y \in [0,1]\}$. We glue E together by identifying $(x,0)$ with $(x+1, e^{-(x+1)})$ for all $x \ge 0$. Show that the diameter of the resulting glued space is finite.

Hint: Consider paths that connect (n, e^{-n}) with $(n,0)$. Use the following estimate $\sum_{n=1}^{\infty} e^{-n} \le \int_0^{\infty} e^{-x} dx = 1$.

Problem 2 (4 Punkte)
One says a group G acts on a set X if there exists a map $\phi : G \times X \to X$, $(g,x) \to \phi(g,x) =: g(x)$ such that

(i) $gh(x) = g(h(x))$, and

(ii) $e(x) = x$

for every $g, h \in G$, $x \in X$. Here e is the unit of G.

Now let (X,d) be a length space and $G \subset \mathrm{Iso}(X)$ where $\mathrm{Iso}(X)$ is the isometry group of (X,d). We introduce an equivalence relation R_G via xR_Gy if and only if $\exists g \in G$ such that $x = g(y)$. We define $\bar{d}(\bar{x}, \bar{y}) := \inf\{d(x,y) : x \in \bar{x}, y \in \bar{y}\}$ for $\bar{x}, \bar{y} \in X/G$. Recall that the equivalence class $\bar{x}$ is given by the orbit $\{g(x) \in X : g \in G\}$.

Show that $\bar{d}$ coincides with d_{R_G}.

Problem 3 (4 Punkte)
Let (X,d) be a length space and let $C(X)$ be the metric cone over X. Let $\bar{\gamma} : [a,b] \to C(X)$ be a curve in the cone given by $\bar{\gamma}(t) = (r(t), \gamma(t))$ where γ is a curve in X. Prove that

$$L(\bar{\gamma}) \ge \sqrt{r(a)^2 + r(b)^2 - 2r(a)r(b)\cos L(\gamma)}$$

if $L(\gamma) \le \pi$, and

$$L(\bar{\gamma}) \ge r(a) + r(b)$$

if $L(\gamma) \ge \pi$.

Bonus Problem (2 Punkte)
Use ϵ-midpoints to show that the direct product of two length spaces (X, d_X) and (Y, d_Y) is a again a length space.

Abgabe am Dezember, 06. November bis 12 Uhr beim Assistenten.