
Problem 1 (4 Punkte)
Let X be a metric space and let $\mathfrak{M}(X)$ the collection of closed subsets in X. Consider a sequence $A_i \in \mathfrak{M}(X)$ converging to a set $A \in \mathfrak{M}(X)$ w.r.t. the Hausdorff distance in X (in short: $A_i \overset{H}{\to} A$ in X, or $A_i \overset{H}{\to} A$ in $\mathfrak{M}(X)$). Prove that

(a) A is the set of limits of all converging sequences $\{a_n\}$ in X such that $a_n \in A_n$ for all n.

(b) $A = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} A_m}$.

(c) Assume X is compact. If $A_{i+1} \subset A_i$ for all $i \in \mathbb{N}$, then $\{A_i\}$ converges in $\mathfrak{M}(X)$ to the intersections $\bigcap_{i\in\mathbb{N}} A_i$. If $A_{i+1} \supset A_i$ for all $i \in \mathbb{N}$, then $\{A_i\}_{i\in\mathbb{N}}$ converges in $\mathfrak{M}(X)$ to the closure of the union $\bigcup_{i\in\mathbb{N}} X_i$.

(d) Let $A_i \overset{H}{\to} A$ in $\mathfrak{M}(\mathbb{R}^n)$ and all sets A_i are convex. Prove that A is convex.

Problem 2 (4 Punkte)

(a) Prove that $d_{GH}(X, Y) < \infty$ if X and Y are bounded metric spaces.

(b) Let X and Y be metric spaces and $\operatorname{diam} X < \infty$. Prove that $d_{GH}(X, Y) \geq \frac{1}{2}|\operatorname{diam} X - \operatorname{diam} Y|$.

(c) Let P be a metric space consisting of one point. Prove that $d_{GH}(X, P) = \operatorname{diam}(X)/2$ for any metric space X.

Problem 3 (4 Punkte)
Let X, Y be two metric spaces. Recall that the dilatation of a Lipschitz map $f : X \to Y$ is defined by

$$\operatorname{dil} f = \sup_{x,x'\in X} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

A homeomorphism $f : X \to Y$ is called bi-Lipschitz if both f and f^{-1} are Lipschitz maps.
The Lipschitz distance d_L between two metric spaces X and Y is defined by

$$d_L(X, Y) = \inf_{f:X\to Y} \log(\max\{\operatorname{dil} f, \operatorname{dil} f^{-1}\})$$

where the infimum is taken over all bi-Lipschitz homeomorphisms $f : X \to Y$. If there is no bi-Lipschitz homeomorphism from X to Y, then one sets $d_L(X, Y) = \infty$.

(a) Show that d_L is nonnegative, symmetric and satisfies the triangle inequality. Moreover, for compact metric spaces X and Y, $d_L(X, Y) = 0$ if and only if X is isometric to Y.

(b) Show that convergence of compact metric spaces w.r.t. d_L implies uniform convergence.

(c) Prove that convergence w.r.t. d_L is equivalent to uniform convergence within the class of finite metric spaces.

Abgabe am Donnerstag, 18. Januar bis 12 Uhr beim Assistenten.