Problem 1 (4 Punkte)
Let X, Y and Z be metric spaces, and let $\mathfrak{R}_1$ be a correspondence between X and Y, and $\mathfrak{R}_2$ be a correspondence between Y and Z. A composition of $\mathfrak{R}_1$ and $\mathfrak{R}_2$, denoted as $\mathfrak{R}_1 \circ \mathfrak{R}_2$, is the set of all $(x, z) \in X \times Z$ for which there is $y \in Y$ such that $(x, y) \in \mathfrak{R}_1$ and $(y, z) \in \mathfrak{R}_2$.

(a) Prove that $\mathfrak{R}_1 \circ \mathfrak{R}_2$ is a correspondence between X and Z.

(b) Prove that $\operatorname{dist} \mathfrak{R}_1 \circ \mathfrak{R}_2 \leq \operatorname{dist} \mathfrak{R}_1 + \mathfrak{R}_2$.

(c) Use (b) to give an alternative proof of the triangle inequality for the Gromov-Hausdorff distance.

Problem 2 (4 Punkte)
Prove the following generalization of Theorem 5.19 from the lecture: if X and Y are metric spaces with $d_{GH}(X, Y) = 0$, X is compact and Y is complete, then X and Y are isometric.

Problem 3 (4 Punkte)
Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of metric spaces, and let X be a finite metric space of cardinality N, $X = \{x_i : 1 \leq i \leq N\}$.

(a) Assume $X_n \overset{GH}{\to} X$. Prove that, for all n sufficiently large, the cardinality of X_n is at least N.

(b) Prove that $X_n \overset{GH}{\to} X$ if and only if the following holds. For all sufficiently large n, X_n can be split into a disjoint union of N nonempty sets $X_{n,1}, X_{n,2}, \dots, X_{n,N}$ so that for all i, j we have

$$\operatorname{diam} X_{n,i} \to 0, \quad d(X_{n,i}, X_{n,j}) \to |x_i x_j| \quad \text{as } n \to \infty.$$

Bonus Problem (4 Punkte)
Let N be a fixed natural number. Prove that the Lipschitz, uniform and Gromov-Hausdorff convergence determine the same topology on the class of finite metric spaces of cardinality N.

Abgabe am Donnerstag, 25. Januar bis 12 Uhr beim Assistenten.