
Problem 1 (4 Bonuspunkte)
Let X be a compact Alexandrov space with curvature ≥ 1 and $\mathrm{diam}_X = \pi$. Prove that X is isometric to a spherical suspension over a compact Alexandrov space of curvature ≥ 1. Recall that a spherical suspension over some metric space Y with $\mathrm{diam}_Y \leq \pi$ is the quotient space $[0,\pi] \times Y/\sim$ built from $(0,x) \sim (0,y)$ and $(\pi,x) \sim (\pi,y)$ equipped with the metric d_S given by

$$\cos d_S((s,x),(t,y)) = \cos s \cos t + \sin s \sin t \cos d_Y(x,y).$$

Hint: Let $p,q \in X$ be points with $|pq| = \pi$. Define $Y = \{x \in X : |px| = |qx| = \pi/2\}$. Then prove the following facts: (1) $\forall x,y \in Y$ shortest paths $[px]$ and $[qx]$ are unique and $\angle pxy = \angle qxy = \angle pyx = \angle qyx = \pi/2$. In particular the triangles are isometric to comparison triangles in the 1-plane. (2) Y is a convex subset of X. (3) Y is an Alexandrov space of curvature ≥ 1. (4) Every point in X belongs to some shortest path connecting p and q.

Problem 2 (4 Bonuspunkte)
The radius rad_X of a compact metric space X is the minimal number $r > 0$ such that $X = \overline{B}_r(p)$ for some $p \in X$. Prove that

(a) $\frac{1}{2}\mathrm{diam}_X \leq \mathrm{rad}_X \leq \mathrm{diam}_X$ for every metric space X.

(b) If X is an n-dimensional Alexandrov space of curvature ≥ 1 and $\mathrm{rad}_X = \pi$, then X is isometric to $\mathbb{S}^n$.

HInt: Use the result of Problem 1.

Problem 3 (4 Bonuspunkte)
Let $n \in \mathbb{N}$. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that for any Alexandrov space X with curvature ≥ 1 and $\dim_H X \leq n$ the following holds.

1. If $\mathrm{diam}_X > \pi - \delta$, then X is ϵ-close in the Gromov-Hausdorff sense to a spherical suspension over some Alexandrove space Y with curvature ≥ 1 and $\dim_H Y \leq n-1$.
2. If $\mathrm{rad}_X > \pi - \delta$, then X is ϵ-close in the Gromov-Hausdorff sense to $\mathbb{S}^k$ for $k \in \{2,\ldots,n\}$.

Problem 3 (4 Bonuspunkte)
Let Y be an $(n-1)$-dimensional Alexandrov space of curvature ≥ 1 containing n pairs of points $\{(x_i,y_i\}$ such that $|x_iy_i| = \pi$ for all i and the determinant of the $n \times n$ matrix $(\cos|x_ix_j|)_{i,j}$ is nonzero. Prove that Y is isometric to $\mathbb{S}^{n-1}$.