Geometry of Metric Spaces

Christian Ketterer

February 11, 2024

Contents

16.10.2023

1 Metric Spaces

1.1 Definition. Let X be a set. A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is a pseudo-metric (also semi-metric) if

1. $d(x,x) = 0 \ \forall x \in X$,
2. $d(x,y) = d(y,x) \ \forall x,y \in X$, (*Symmetry*)
3. $d(x,y) \leq d(x,z) + d(z,y) \ \forall x,y,z \in X$. ($\Delta$-*inequality*)

The pair (X,d) is a *pseudo-metric space.*
If the first property is replaced with

4. $d(x,y) = 0 \ \Leftrightarrow \ x = y \ \ \forall x,y \in X$,

we call (X,d) *a metric space.*

Notation. Unless different metrics on the same set X are considerd we will omit an explicite reference to the metric d and we will just say "a metric space X", and we frequently write $d(x,y) = |xy|$ for $x,y \in X$.

1.2 Proposition. *Let d be a pseudo-metric on X. Consider the equivalence relation $x \sim y$ $\Leftrightarrow d(x,y) = 0$. If $x \sim x_1$ and $y \sim y_1$, then $d(x,y) = d(x_1,y_1)$. Hence, the projection $\hat{d}$ of d onto $X/\sim = \hat{X}$ is well-defined and $(\hat{X},\hat{d})$ is a metric space.*

1.3 Definition. Let X,Y be two metric spaces. A map $f : X \to Y$ is called distance preserving if $|f(x)f(y)| = |xy| \ \forall x,y \in X$. If f is bijective then f called an isometry. Two metric spaces are isometric if there exists an isometry from one to the other.

1.4 Examples. (a) Let X be a set and define

$$d(x,y) = |xy| = \begin{cases} 0 & x = y; \\ 1 & x \neq y. \end{cases}$$

(X,d) is a metric space.

(b) Let X and Y be metric spaces. A metric on $X \times Y$ is defined via

$$|(x,y),(x',y')| = \sqrt{|xx'|^2 + |yy'|^2}.$$

In particular, if $X = Y = \mathbb{R}$ equipped with $|xx'| = |x - x'|$, then $|(x,y),(x',y')| = \|(x,y) - (x',y')\|_{eucl}$ on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

(c) Let X be a metric space and $\lambda > 0$. The dilated (or rescaled) metric space λX is the same set X equipped with the metric $d_{\lambda X}(x,y) = \lambda d(x,y)$.

Let $Y \subset X$ be a subset. The restricted metric on Y is defined as $d_Y := d_X|_{Y \times Y}$.

(d) Let V be a vector space. A function $|\cdot| : V \to \mathbb{R}$ is called a norm if $\forall v, w \in V$ and $\forall \lambda \in \mathbb{R}$ it holds

1. $|v| = 0 \ \Leftrightarrow v = 0$,
2. $|\lambda v| = |\lambda||v|$,
3. $|v + w| \leq |v| + |w|$.

Then (V, d) with $d(v, w) := |v - w|$ is a metric space.

Remark. One may also consider (pseudo)-metrics with values in $[0, \infty]$. We call them ∞-metrics. The following shows how to reduce questions about ∞-metrics to genuine metrics. Define an equivalence relation $x \sim y$ via $|xy| < \infty$. The equivalence class X_x of a point $x \in X$ will be called metric component of x. By the triangle inequality $d|_{X_x \times X_x}$ is then a finite (pseudo)-metric space. On the other, if $\{X_\alpha\}$ is a collection of (pseudo)-metric spaces, then the disjoint union $X = \bigcup_\alpha X_\alpha$ equipped with

$$|xy| := \begin{cases} d_{X_\alpha}(x, y) & \text{if } x, y \in X_\alpha \text{ for some } \alpha, \\ \infty & \text{otherwise} \end{cases}$$

is an ∞-(pseudo)-metric space.

18.10.2023

1.5 Definition. Let X be a metric space. For $r \in (0, \infty]$ let $B_r(x) = \{y \in X : |xy| < r\}$ be the ball of radius r with center x and let $\overline{B}_r(x) = \{y \in X : |xy| \le r\}$ be the corresponding closed ball. The topology associated to a metric space is define as follows: $U \subset X$ is open if $\forall x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. $A \subset X$ is closed if $X \backslash A$ is open. This topology is a Hausdorff topology.

The standard definitions of convergence, continuous function, etc. admit straightforward generalizations.

1.6 Definition. Let X be a metric space. A sequence (x_n) is a Cauchy sequence if $|x_n x_m| \to 0$ if $n, m \to \infty$. X is called complete if every Cauchy sequence has a limit.

Remark. Completeness is not a topological property. There exist homeomorphic metric spaces X and Y such that X is complete but Y is not.

1.7 Definition. Let X and Y be metric spaces. A map $f : X \to Y$ is called Lipschitz if $\exists C \ge 0$ such that $|f(x_1)f(x_2)| \le C|x_1x_2|\ \forall x_1, x_2 \in X$. C is called a Lipschitz constant of f and the minimal C is called dilatation of f and denoted with $\operatorname{dil} f$.

A Lipschitz function with Lipschitz constant less than 1 is called non-expanding.

1.8 Proposition. *Let X, Y be metric spaces such that Y is complete, let $X' \subset X$ be dense in X and let $f : X' \to Y$ be a Lipschitz map. Then there exists a unique continuous map $\widetilde{f} : X \to Y$ such that $\widetilde{f}|_{X'} = f$, $\widetilde{f}$ is Lipschitz and $\operatorname{dil} f = \operatorname{dil} \widetilde{f}$.*

Proof. Consider $x \in X$ and $(x_i) \subset X' \to x \in X$. We define $\widetilde{f}(x) := \lim_{i\to\infty} f(x_i)$. Observe that $(f(x_i))_{i\in\mathbb{N}}$ is a Cauchy sequence and therefore indeed convergent in Y because $|f(x_i)f(x_j)| \le \operatorname{dil} f |x_ix_j|$ for all $i, j \in \mathbb{N}$ and $|x_ix_j| \to 0$ for $i, j \to \infty$ since the sequence (x_i) converges.

Hence we have a map $\widetilde{f} : X \to Y$. We have the inequality

$$|\widetilde{f}(x)\widetilde{f}(x')| = \lim_{i\to} |f(x_i)f(x_i')| \le \lim_{i\to\infty} \operatorname{dil} f |x_ix_i'| = \operatorname{dil} f |x_ix_i'|.$$

Hence $\widetilde{f}$ is Lipschitz with Lipschitz constant $\operatorname{dil} f$ and $\operatorname{dil} \widetilde{f} \le \operatorname{dil} f$. Moreover $\operatorname{dil} \widetilde{f} = \operatorname{dil} f$. Otherwise this would contradict the definition of $\operatorname{dil} f$.

The uniqueness of $\widetilde{f}$ is easily verified. □

1.9 Theorem. *Let X be a metric space. $\exists$ a complete metric space $\widetilde{X}$, such that X is dense subset of X. $\widetilde{X}$ is unique in the following sense: If $\widetilde{X}'$ is another metric space with these properties, then there exists a unique isometry $f : \widetilde{X} \to \widetilde{X}'$ such that $f|_X = \mathrm{id}$.*

Proof. We consider the set $\mathfrak{X}$ of all Cauchy sequences (x_n). A function on $\mathfrak{X}^2$ is given by

$$d((x_n), (y_n)) = \lim_{n\to\infty} |x_ny_n|.$$

The limit always exists in $[0, \infty)$. It is elementary to check that d is a pseudo-metric on $\mathfrak{X}$. We define $\widetilde{X} = \mathfrak{X}/\infty$ where $(x_n) \sim (y_n)$ if and only if $d((x_n), (y_n)) = 0$.

A natural map from X to $\widetilde{X}$ is given by $x \mapsto (x_n)$ with $x_n = x\ \forall n \in \mathbb{N}$. This map is distance preserving and we can identify X with its image in $\widetilde{X}$. X is also dense in $\widetilde{X}$ since every element $[(x_n)] \in \widetilde{X}$ is the limit of squence $([(x_n^i)])_{i\in\mathbb{N}} \subset \widetilde{X}$ where (x_n^i) is the sequence w.r.t. n given by $x_n^i = x_i$ for all $n \in \mathbb{N}$.

The uniqueness follows from Proposition 1.8: The inclusion map $i : X \subset \widetilde{X} \to \widetilde{X}'$ has unique extension to $\widetilde{X}$. □

1.1 Compact spaces

Recall that a topological space X is called compact if any open covering of X has a finite sub-collection that still covers X.

1.10 Fact. *Let X be a Hausdorff topological space.*

1. *If $S \subset X$ is compact, then S is closed in X.*
2. *if X is compact and $S \subset X$ is closed, then S is compact.*
3. *A subset $S \subset \mathbb{R}^n$ is compact if and only if S is closed and bounded.*
4. *If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.*
5. *If X is compact and $f : X \to Y$ is continuous and bijective, then f is a homeomorphism.*

1.11 Definition.

- Let X be a metric space and $\epsilon > 0$. A set $S \subset X$ is called an ϵ-net if $\forall x \in X\ \exists y \in S$ such that $|xy| \le \epsilon$.
- The metric space X is called totally bounded if $\forall \epsilon > 0$ there is a finite ϵ-net in X.
- A set S in a metric space X is called ϵ-separated for $\epsilon > 0$, if $|xy| \ge \epsilon$ for any two points $x, y \in S$.
- A set $S \subset X$ is called maximal ϵ-separated in X, if $S \cup \{x\}$ is not ϵ-separated for every $x \in X \backslash S$.

1.12 Lemma. *A maximal ϵ-separated set $S \subset X$ is an ϵ-net.*

Proof. Assume S is not an ϵ-net. Then $\exists x \in X$ such that $B_\epsilon(x) \cap S = \emptyset$. This contradicts with S being maximal ϵ-separated. □

Let X be a metric space. We define

1. The diameter $\mathrm{diam}_X := \sup_{x,y \in X} |xy|$ of X,
2. The radius $\mathrm{rad}_X := \inf_{x \in X} \sup_{y \in X} |xy|$ of X.

Remark. It holds $\mathrm{rad}_X = \inf\{r > 0 : B_r(x) \supset X, x \in X\}$ (Exercise).

23.10.2023

1.13 Theorem. *Let X be a metric space. Then the following statements are equivalent:*

1. *X is compact.*

2. *Any sequence in X has a converging subsequence.*

3. *Any infinite subset of X has an accumulation point.*

4. *X is complete and totally bounded.*

In particular, if X is compact, then $\operatorname{diam}_X < \infty$ *(Exercise).*

Proof (Sketch). 1. $\Rightarrow$ 2. (Exercise), 2. $\Rightarrow$ 3. (easy)

3. $\Rightarrow$ 4. We show that for every $\epsilon > 0$ there exists a finite ϵ-net.

Assume this is not true. Then we construct a sequence of points $(x_i) \subset X$ such that $|x_i x_{i+1}| \geq \epsilon \ \forall i \in \mathbb{N}$. But this will contradict 3.

The sequence (x_i) is constructed inductively. Assume we have already found points $(x_i)_{i=1,\dots,n}$ with $|x_i x_j| \geq \epsilon$. Since $\{x_1, \dots, x_n\}$ cannot be a ϵ-net, there exists a point $x_{n+1} \in X$ such that $d(x, x_i) > \epsilon \ \forall i = 1, \dots, n$.

Hence, $\forall \epsilon > 0$ we find a finite ϵ-net in X and therefore X is totally bounded. Moreover, given a Cauchy sequence 3. implies that there exists a converging subsequence. But since the original sequence was Cauchy it already converges.

4. $\Rightarrow$ 1. Assume X is not compact. Hence there exists an open cover $\{U_\alpha\}$ of X without a finite subcover.

We define a sequence (x_n) as follows. For $\epsilon = \frac{1}{2}$ $\exists$ a finite ϵ-net S_1. Hence there exists a least one point $x_1 \in S$ such that no finite subset of $\{U_\alpha\}$ covers $B_{\frac{1}{2}}(x_1)$.

In the next step, for $\epsilon = \frac{1}{2^2} = \frac{1}{4}$ there exists a finite ϵ-net S_2. We can find a least one point $x_2 \in S_2$ such that $B_{\frac{1}{4}}(x_2) \cap B_{\frac{1}{2}}(x_1) \neq \emptyset$ and there is no finite subset of $\{U_\alpha\}$ covering $B_{\frac{1}{4}}(x_2)$.

Iteratively we find a sequence (x_n) such that $B_{\frac{1}{2^{n+1}}}(x_{n+1}) \cap B_{\frac{1}{2^n}}(x_n) \neq \emptyset$ for all $n \in \mathbb{N}$.

Hence $|x_n, x_n + 1| \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} \leq \frac{1}{2^{n-1}}$ and therefore

$$|x_n, x_m| \leq \sum_{k=n}^{m} \frac{1}{2^{k-1}} = \frac{1}{2^{n-1}} \sum_{k=0}^{m} \frac{1}{2^k} \leq \frac{1}{2^{n-1}} \ \forall m > n.$$

It follows (x_n) is a Cauchysequence. Thus $(x_n) \to a \in X$.

$\exists \alpha$ such that $a \in U_\alpha$. We choose $\epsilon > 0$ such that $B_\epsilon(a) \subset U_\alpha$. Now we choose $n \in \mathbb{N}$ big enough such that $4\frac{1}{2^{n-1}} \leq \epsilon$ and $|x_n, a| \leq \frac{1}{2^{n-1}}$. It follows for $y \in B_{\frac{1}{2^n}}(x_n)$ that

$$|y, a| \leq |y, x_n| + |x_n, a| \leq 2\frac{1}{2^{n-1}} < \epsilon.$$

Hence $B_{\frac{1}{2^n}}(x_n) \subset B_\epsilon(a) \subset U_\alpha$ in contradiction to the choice of x_n. $\square$

Remark. The theorem remains true for ∞-metric spaces. One can show that a compact ∞-metric space is the finite union of compact metric spaces.

1.14 Theorem (Lebesgue's Lemma)**.** *Let X be a compact metric space, and let $\{U_\alpha\}_{\alpha\in a}$ be an open covering of X. Then $\exists\delta > 0$ such that any ball of radius $\delta > 0$ in X is contained in U_α for some $\alpha \in A$.*

Proof. We assume $X\backslash U_\alpha \neq \emptyset \ \forall\alpha \in A$. Otherwise we are done. We define

$$f(x) = \sup\{r \in \mathbb{R} : \exists\alpha \in A \text{ s.t. } B_r(x) \subset U_\alpha\} \in [0, \infty].$$

Claim: $f(x) \in (0, \infty)\ \forall x \in X$. If $f(x) = \infty$, then $\forall\ n \in \mathbb{N}\ \exists$ an index α_n such that $B_n(x) \subset U_\alpha$. On the other hand $\exists y_n \notin U_{\alpha_n}$. Hence $|y_n x| \geq n\ \forall n \in \mathbb{N}$. This contradicts $\operatorname{diam}_X < \infty$. For all $x \in X\ \exists r > 0$ sucht that $B_r(x) \subset U_\alpha$ for some α. Hence $f(x) > 0$.

Claim: The function f is continuous. In fact we show that f is nonexpanding. We notice that for all $x \in X$ it holds $B_{f(x)}(x) \subset U_\alpha$ for some α. Let $x, y \in X$.

1st case: If $y \notin B_{f(x)}(x)$ and $x \notin B_{f(y)}(y)$, then $|f(x)-f(y)| \leq \max\{f(x), f(y)\} \leq |xy|$.

2nd case: If $x \in B_{f(y)}(y)$ and $B_{f(x)}(x) \subset B_{f(y)}(y)$, then $f(x) = \sup\{r > 0 : B_r(x) \subset B_{f(y)}(y)\}$ and hence $f(x) = f(y) - |xy|$. Since $f(x) < f(y)$, we have

$$|f(y) - f(x)| = f(y) - f(x) = |xy|.$$

3rd case: If $x \in B_{f(y)}(y)$, $B_{f(x)}(x)\backslash B_{f(y)}(y) \neq \emptyset$ and $|xy| \geq f(x)$. Then $f(x) \geq \sup\{r > 0 : B_r(x) \subset B_{f(y)}(y)\} = f(y) - |xy| > 0$ and hence

$$|xy| \geq f(y) - f(x) \geq |xy| - f(x) \geq 0 \text{ since } |xy| \geq f(x).$$

Hence $|xy| \geq |f(x) - f(y)|$.

4th case: If $x \in B_{f(y)}(y)$, $B_{f(x)}(x)\backslash B_{f(y)}(y)$ and $|xy| < f(x)$, then $y \in B_{f(x)}(x)$. From the 2nd and 3rd step we get $|xy| \geq f(y) - f(x)$. Since $y \in B_{f(x)}(x)$ we can also apply the 2nd and 3rd step with x and y in reversed roles to obtain $|xy| \geq f(x) - f(y)$, and hence $|xy| \geq |f(x) - f(y)|$.

Since f is continuous on a compact space, there exists $\delta > 0$ such that $f(x) \geq \delta$ $\forall x \in X$. This is the statement. □

1.15 Corollary. *Let X and Y be metric spacces and let X be compact. If $f : X \to Y$ is continuous, then $\forall\epsilon > 0\ \exists\delta > 0$ such that $|f(x_1)f(x_2)| < \epsilon$ for all $x_1, x_2 \in X$ with $|x_1x_2| < \delta$.*

Proof. Continuity implies that for all $x \in X$ there exists an open neighborhood U_x of x such that $f(U_x) \subset B_\epsilon(f(x))$. By Lebesgue's covering lemma there exists $\delta > 0$ such that $B_\delta(x) \subset U_x\ \forall x \in X$. Hence, if $x, y \in X$ such that $|x, y| < \delta$ then $|f(x)f(y)| < \epsilon$. □

1.16 Theorem. *A compact metric space X cannot be isometric to a proper subset of itself, i.e. if $f : X \to X$ is distance preserving, then $f(X) = X$.*

Proof. We argue by contradiction. Let $p \in X\backslash f(X)$. Since $f(X)$ is compact and hence closed, there exists $\epsilon > 0$ such that $B_\epsilon(p) \cap f(X) = \emptyset$. Let n be the maximal cardinality of an ϵ-separated set in X and let $S \subset X$ be an ϵ-separated set of cardinality n. Since f is distance preserving $f(S)$ is also ϵ-separated. But $d(p, f(S)) \geq d(p, f(X)) \geq \epsilon$ and therefore $f(S) \cup \{p\}$ is an ϵ-separated set of cardinality $n+1$. This is a contradiction. □

1.17 Theorem. *Let X be a compact metric space. Then*

1. *Any nonexpanding surjective map $f : X \to X$ is an isometry.*
2. *If a map $f : X \to X$ is such that $|f(x)f(y)| \geq |xy|\ \forall x, y \in X$, then f is an isometry.*

Proof. **1.** We show that f is distance preserving and apply the previous theorem. Assume this is not the case. Then $\exists p, q \in X$ such that $|f(p)f(q)| < |pq|$. Then we can pick $\epsilon > 0$ such that $|f(p)f(q)| < |pq| - 5\epsilon$.

Let n be a natural number such that $\exists$ an ϵ-net in X of cardinality n, and consider the set $\mathfrak{N} \subset X^n$ of all n-tuples of points in X that form an ϵ-net in X. This is a closed set in X^n and therefore also compact. Define $D : X^n \to \mathbb{R}$ by

$$D(x_1, \dots, x_n) = \sum_{i,j=1}^{n} |x_i x_j|.$$

This function is continuous and therefore attains a minimum on $\mathfrak{N}$. Let $(x_1, \dots, x_n)$ be such a minimum. Since f is nonexpanding and surjective also $(f(x_1), \dots, f(x_n))$ is in $\mathfrak{N}$. Moreover $D(f(x_1), \dots, f(x_n)) \leq D(x_1, \dots, x_n)$ and since $(x_1, \dots, x_n)$ is a minimum of D on $\mathfrak{N}$, it holds $D(f(x_1), \dots, f(x_n)) = D(x_1, \dots, x_n)$, and in fact $|f(x_i)f(x_j)| = |x_i x_j|$ since f is nonexpanding.

On the other hand $\exists i, j \in \{1, \dots, n\}$ such that $|x_i p| \leq \epsilon$ and $|x_j q| \leq \epsilon$. Hence

$$|pq| \leq |px_i| + |x_i x_j| + |x_j q| \leq |x_i x_j| + 2\epsilon$$

and

$$\begin{aligned} |f(x_i)f(x_j)| &\leq |f(p)f(q)| + |f(p)f(x_i)| + |f(q)f(x_j)| \\ &\leq |pq| - 5\epsilon + |px_i| + |qx_j| \leq |pq| - 3\epsilon \leq |x_i x_j| - \epsilon. \end{aligned}$$

Hence $|f(x_i)f(x_j)| < |x_i x_j| - \epsilon$. This is a contradiction with what we just proved.

2. Define $Y = f(X)$. Then Y is dense in X. Assume this would not be true. Then there exists $p \in X$ and $\epsilon > 0$ such that $B_\epsilon(p) \cap Y = \emptyset$. Let S be a maximal ϵ-separating set in X. Since $|f(x)f(y)| \geq |xy|\ \forall x, y \in X$, it follows that also $f(S)$ is maximal ϵ-separating, and hence $f(S) \subset Y$ is an ϵ-net in X. Therefore $d(p, F(S)) \leq \epsilon$, or in other words $\exists y \in f(S)$ such that $d(p, y) \leq \epsilon$. This is a contradiction.

Consider now the map $g = f^{-1} : Y \to X$. g is nonexpanding and defined on a dense subset in X. Hence g there exists a unique nonexpanding map $\widetilde{g} : X \to X$. By **1.** we have that $\widetilde{g}$ is an isometry, and in particular f is distance preserving. By **1.** again f itself is an isometry. $\square$

25.10.2023

1.2 Hausdorff measure and Hausdorff dimension

1.18 Definition (Hausdorff measure)**.** Let X be a metric space and let n be a nonnegative real number. Let $\{S_i\}_{i\in I}$ be a finite or countable family of sets in X. The n-weight of this family is defined as

$$w_n(\{S_i\}_{i\in I}) = \sum_{i\in I} (\operatorname{diam} S_i)^n.$$

If $n = 0$, substitute any term 0^0 in the formula by 1.

Given $\epsilon > 0$ and a subset $A \subset X$ define

$$\mathcal{H}_{n,\epsilon}(A) = \inf\{w_n(\{S_i\})_{i\in I} : A \subset \bigcup_{i\in I} \text{ and } \operatorname{diam}(S_i) < \epsilon \ \forall i\}.$$

The infimum is taken over all finite or countable coverings of A by sets of diameter If there is no such covering we set $\mathcal{H}_{n,\epsilon}(A) = \infty$.

The n-dimensional Hausdorff measure of A is defined by the formula

$$\mathcal{H}^n(A) = C(n) \lim_{\epsilon\to 0} \mathcal{H}_{n,\epsilon}(A)$$

where $C(n)$ is a positive normalisation constant.

Moreover we define $\mathcal{H}^n(\emptyset) = 0$.

Remark.

- The value $\mathcal{H}_{n,\epsilon}$ is nonincreasing in ϵ ($\mathcal{H}_{d,\epsilon} \uparrow$ for $\epsilon \downarrow 0$). Hence $\mathcal{H}^n(A)$ is well-defined for any subset $A \subset X$ and any metric space X. It may be either a nonnegative real number or $+\infty$.
- The constant $C(n)$ is introduced for one reason only: if n is an integer, one chooses $C(n)$ such that the n-dimensional Hausdorff measure of sets in $\mathbb{R}^n$ has the property $\mathcal{H}^n([0,1]^n) = 1$.

1.19 Proposition. *Let X be a metric space and $A, B \subset X$ as well as $\{A_i\}_{i\in I} \subset 2^X$.*

1. *$A \subset B$, then $\mathcal{H}^n(A) \le \mathcal{H}^n(B)$,*
2. *$\mathcal{H}^n(\bigcup_{i\in I} A_i) \subset \sum_{i\in I} \mathcal{H}^n(A_i)$ for any finite or countable collection of sets $A_i \subset X$,*
3. *$d(A, B) > 0$, then $\mathcal{H}^n(A \cup B) = \mathcal{H}^n(A) + \mathcal{H}^n(B)$.*

Proof. 1. A cover $\{S_i\}_i$ of B is always also a cover of A. Hence $\mathcal{H}_{n,\epsilon}(A) \le \mathcal{H}_{n,\epsilon}(A)$ $\forall \epsilon > 0$ and consequently $\mathcal{H}^n(A) \le \mathcal{H}^n(B)$.

2. Let $\{S^i_j\}_j$ be a cover of A_i, $i \in I = \mathbb{N}$, such that $\mathcal{H}_{n,\epsilon}(A_i) \ge w_n(\{S^i_j\}) - \frac{\epsilon}{2^i}$. Hence $\{S^i_j\}_{i,j}$ is a cover of $\bigcup_{i\in\mathbb{N}} A_i = A$. Hence

$$\mathcal{H}_{n,\epsilon}(A) \le \sum_i w_n(\{S^i_j\}) = \sum_i w_n(\{S^i_j\}) \le \sum_i \mathcal{H}_{n,\epsilon}(A_i)) + \epsilon \sum_i \frac{1}{2^i}.$$

If $\epsilon \downarrow 0$, then $\mathcal{H}^n(A) \le \sum_{i\in\mathbb{N}} \mathcal{H}^n(A_i)$.

3. Given any cover $\{S_j\}$ of A and $\{\widetilde{S}_i\}$ of B, we can intersect every S_j with $B_\epsilon(A) = \{y \in X : \exists x \in A \text{ s.t. } |xy| < \epsilon\}$ and every $\widetilde{S}_i$ with $B_\epsilon(B)$ where $\epsilon \in (0, d(A, B))$. This procedure only decreases the diameter. Hence w.l.og. we can assume sets in a covering of A are disjoint from the sets in a covering of B. $\square$

1.20 Remark. Caratheodory's theorem yields that by these three properties $\mathcal{H}^n$ restricted to the Borel σ-algebra of X is measure for any $n \geq 0$.

We recall some definitions from measure theory.

Let X be a set. A family $\mathfrak{A} \subset 2^X$ is called a σ-algebra if

1. $\emptyset, X \in \mathfrak{A}$,
2. $A, B \in \mathfrak{A} \Rightarrow A \backslash B \in \mathfrak{A}$,
3. $\{A_i\}_{i \in N} \subset \mathfrak{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathfrak{A}$.

A measure on $\mathfrak{A}$ is a function $\mu : \mathfrak{A} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. $\{A_i\}_{i \in \mathbb{N}} \subset \mathfrak{A} \Rightarrow \mu(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i)$ where $\{A_i\}$ is a finite or countable family of disjoint sets (σ-additivity)

If $\mathfrak{T} \subset 2^X$ is an arbitrary collection of subsets of X, then there exists a smallest σ-algebra $\sigma(\mathfrak{T})$ that contains $\mathfrak{T}$. We say $\sigma(\mathfrak{T})$ is generated by $\mathfrak{T}$. The σ-algebra generated by open sets in a topological space X is called the Borel σ-algebra of X and the elements of the Borel σ-algebra are called Borel sets.

1.21 Theorem. *Let $\mathcal{H}_n$ be the n-dimensional Hausdorff measure on $(\mathbb{R}^n, d_{eucl})$. Let ω_n be the volume of the Euclidean, n-dimensional ball of radius 1. If we choose the normalisation constant $C(n) = \frac{1}{2}\omega_n$, then $\mathcal{H}^n = \mathcal{L}^n$ where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure.*

1.22 Lemma. *Let X, Y be metric spaces and $f : X \to Y$ a Lipschitz map with diletation $\leq C$. Then $\mathcal{H}^n(f(X)) \leq C^n \mathcal{H}^n(X)$.*

Proof. If $\{S_i\}$ is a covering of X with $\operatorname{diam} S_i \leq \epsilon \ \forall i$, then $\{f(S_i)\}_i$ is a covering of $f(X)$ with $\operatorname{diam}_{f(S_i)} \leq C \operatorname{diam}_{S_i}$. Indeed, we first have that $\operatorname{diam}_{f(S_i)} \leq C\epsilon$. If $f(x), f(y) \in f(S_i)$ such that $|f(x)f(y)| \geq \operatorname{diam}_{f(S_i)} - \delta$ for $\delta > 0$ arbitrarily small, then $\operatorname{diam}_{f(S_i)} - \delta \leq |f(x)f(y)| \leq |xy| \leq \operatorname{diam}_{S_i}$.

Hence it follows that $w_n(\{f(S_i)\}) \leq C^n w_n(\{S_i\})$. Consequently we have $\mathcal{H}_{n,C\epsilon}(X) \leq C^n \mathcal{H}_{n,\epsilon}(X)$. Letting $\epsilon \downarrow 0$ we also get $\mathcal{H}^n(f(X)) \leq C^n \mathcal{H}^n(X)$. □

1.23 Theorem. *For every metric space X there exists a $n_0 \in [0, \infty]$ such that $\mathcal{H}^n(X) = 0$ for all $n > n_0$ and $\mathcal{H}^n(X) = \infty$ for all $n < n_0$.*

Proof. Define $n_0 = \inf\{n \geq 0 : \mathcal{H}^n(X) \neq \infty\}$. By definition $\mathcal{H}^n(X) = \infty \ \forall n < n_0$. If $n > n_0$, there exists $n' \in (n_0, n)$ such that $\mathcal{H}^{n'}(X) = M < \infty$. Hence, for $\epsilon > 0$ there exists a covering $\{S_i\}$ of X such that $\operatorname{diam} S_i < \epsilon \ \forall i$ and $\sum_i (\operatorname{diam} S_i)^{n'} < 2M$. It follows

$$\sum (\operatorname{diam} S_i)^n = \sum (\operatorname{diam} S_i)^{n-n'} (\operatorname{diam} S_i)^{n'} \leq \epsilon^{n-n'} \sum (\operatorname{diam} S_i)^{n'} \leq 2M\epsilon^{n-n'}.$$

Hence $\mathcal{H}^n_\epsilon(X) \leq 2\epsilon^{n-n'} M$. Since $n > n'$ and since $\epsilon > 0$ was arbitrary, it follows $\mathcal{H}^n(X) = 0$. □

1.24 Definition. The number $n_0 \in [0, \infty]$ in the previous theorem is called Hausdorff dimension of X and denoted with $\dim_{\mathcal{H}} X$.

1.25 Proposition. *Let X be a metric space. Then*

1. *$Y \subset X$, then $\dim_{\mathcal{H}} Y \leq \dim_{\mathcal{H}} X$.*

2. *If X is covered by a finite or countable collection $\{X_i\}_i$ of subsets in X, then $\dim_{\mathcal{H}} X = \sup_i \dim_{\mathcal{H}} X_i$.*

3. *If $f : X \to Y$ is a Lipschitz map, then $\dim_{\mathcal{H}} f(X) \leq \dim_{\mathcal{H}} X$.*

Proof. Exercise □

29.10.2023

2 Length Spaces

2.1 Length structures

Let X be a topological space.

A *path* γ in X is a continuous map $\gamma : I \to X$ where I is interval in $\mathbb{R}$ that may be open, closed, finite or infinite. A single point is counted as an interval.

2.1 Remark. Two paths $\gamma_i : I_i \to X$, $i = 0, 1$, are equivalent or reparametrizations of each other if there exists an interval $I \subset \mathbb{R}$, a path $\gamma : I \to X$ and continuous nondecreasing functions $\varphi_i : I_i \to I$ such that $\gamma_i = \gamma \circ \varphi_i$, $i = 0, 1$.

An (unparametrized) curve is an equivalence class of paths.

If φ_0, φ_1 are homeomorphisms (i.e. strictly increasing), then $\gamma_0 = \gamma_2 \circ \varphi$ with $\varphi = \varphi_1^{-1} \circ \varphi_0$.

2.2 Definition. A length structure on a Hausdorff space X is a family A of *admissible* paths together with a map $L : A \to [0, \infty]$, the length of paths in A such that the class A satisfies the following properties:

(i) A is closed under restrictions: If $\gamma : I \to X$ is an admissible path and $J \subset I$ an interval, then also $\gamma|_J$ is an admissible path.

(ii) A is closed under concatenations of paths: If $\gamma : [a, b] \to X$ is a path and $c \in [a, b]$ such that $\gamma_0 : [a, c] \to X$ and $\gamma_1 : [c, b] \to X$ are admissible paths, then also γ is an admissible path. γ the concatenation of γ_0 and γ_1.

2.3 Remark. Given two paths $\gamma_0 : [a, b] \to X, \gamma_1 : [c, d] \to X$ the concatenation between γ_0 and γ_1 is defined by

$$\gamma_0 * \gamma_1 : [a, d - c + b] \to X,\ \gamma_0 * \gamma_1(t) = \begin{cases} \gamma_0(t) & t \in [a, b], \\ \gamma_1(t - b + c) & t \in [b, d - c + b]. \end{cases}$$

(iii) A is closed under (at least) affine reparametrizations: For $\gamma : [a, b] \to X$ in A and $\varphi : [c, d] \to [a, b]$ with $\varphi(t) = \alpha t + \beta$, the composition $\gamma \circ \varphi : [c, d] \to X$ is also a path in A.

2.4 Remark. Every natural class of paths comes ususally with its own class of reparametrizations. For instance continuous paths and homeomorphisms, or C^1-curves and diffeomorphisms. (iii) therefore requires that this class of natural reparametrizations includes all linear maps.

The length L has to satisfy the following properties:

1. Additivity: $L(\gamma|_{[a,b]}) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}$ for any $c \in [a, b]$.

2. Given a path $\gamma : [a, b] \to X$ of finite length we define $L(\gamma, a, t) = L(\gamma|_{[a,t]})$. We require that $L(\gamma, a, \cdot)$ is a continuous function.

3. The length is invariant under affine reparametrizations: $L(\gamma \circ \varphi) = L(\gamma)$ for any affine homomorphism φ.

4. The length structure agrees with the topology of X: for any neighborhood U_x of x we have
$$\inf\{L(\gamma) : \gamma(a) = x, \gamma(b) \in X \backslash U_x\} > 0.$$

2.5 Examples. 1. Let $(V, |\cdot|)$ be a finite dimensional normed vector space. Let A be the class of piecewise differentiable paths $\gamma : [a, b] \to V$. A length structure on V is given via
$$\gamma \in A \mapsto L(\gamma) = \int_a^b |\gamma'(t)| dt.$$
For instance $V = \mathbb{R}^2$ with $|\cdot|_{eucl}$.

2. Let $(V, |\cdot|)$, A and L as before. And let $W \subset V$ be a subset with the induced topology. Let $B \subset A$ the paths γ in A such that $\mathrm{Im}(\gamma) \subset W$ and let $L^W = L|_B$. This is a length structure on B.

   For instance, let $(V, |\cdot|) = (\mathbb{R}^n, |\cdot|_{eucl})$ and let $W = M$ be an m-dimensional submanifold.

3. *Driving in Manhatten.* Consider $(\mathbb{R}^n, |\cdot|_{eucl})$, A and L as before. We restrict L to the class of paths that are broken lines that are parallel to the coordinate axes.

4. Let (M, g) be a Riemannian manifold, i.e. $p \in M \mapsto g_p$ smooth with g_p is an Euclidean inner product on T_pM, and let A be the family of piecewise differentiable curves in M. Then
$$\gamma \in A \mapsto L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt.$$

2.6 Definition. Let X be a Hausdorff space and $L : A \to [0, \infty]$ a length structure on X. For two points $x, y \in X$ we set
$$d_L(x, y) = \inf\{L(\gamma) : \gamma : [a, b] \to X, \gamma \in A, \gamma(a) = x, \gamma(b) = y\}.$$

A metric d on a Hausdorff space X that is obtained by a length structure is called a length metric or intrinsic metric. A metric space (X, d) whose metric is a length (intrinsic) metric is called length (intrinsic) space.

2.7 Example. 1. Let $X = \mathbb{R}^2$ and let L be the length induced by the Euclidean norm on the family of piecewise differentiable curves. Then $d_L(x, y) = |x - y|_{eucl}$ $\forall x, y \in \mathbb{R}^2$.

2. (”Metric on an island”) Let $X \subset \mathbb{R}^2$ be conencted. Admissible paths are all piecwise differentiable paths with image in X and the length of paths is the Euclidean length. If X is convex, the induced distance of this length structure on X coincides with $|\cdot - \cdot|_{eucl}$. But in general this is not the case.

2.8 Remark. The pair (X, d_L) is a ∞-metric space (Exercise).

The metric d_L is not necessarily finite. For instance, if X is a union of two disconnected components, there is no continuous path from one component to the other. Hence, $d_L(x, y)$ for points in different components is infinite. On the othere hand there may be points such that continuous path between them exist, but all have infinite length. We say two points belong to the same accessibility component if they can be connected by a path of finite length.

2.9 Remark. The topology of d_L can only be finer than the topology of X: any open set of X is an open set w.r.t. (X, d_L).

2.10 Definition. A length structure (A, L) is said to be complete if for every two points x, y there exists an admissible path $\gamma \in A$ joining them such that $L(\gamma) = d_L(x, y)$.

An intrinsic metric that is associated to a complete length structure is called strictly intrinsic.

2.11 Example.

1. ("crossing the swamp") Consider $\mathbb{R}^2$ and let $f : \mathbb{R}^2 \to (0, \infty)$ be continuous. We define the length of a piecewise differentiable path $\gamma : [a, b] \to \mathbb{R}^2$ by

   $$L(\gamma) = \int_a^b f(\gamma(t))|\gamma'(t)|dt.$$

   We interpret L as a weighted Euclidean length. Where f assigns big values it is more difficult to traverse (for instance, a swamp or a mountain trail).

2. ("Finslerian length") We consider $f : \mathbb{R}^2 \times \mathbb{R}^2 \to (0, \infty)$ and the same class of admissible paths as before. and define length by

   $$L(\gamma) = \int_a^b f(\gamma(t), \gamma'(t)).$$

   In order for this expression to be invariant under reparameterizations of path one has to require that $f(x, kv) = |k|f(x, v)$ for $k \in \mathbb{R}$ and $\forall x \in \mathbb{R}^2$.

   Examples of this type of length is the length structure associated to normed spaces where $f(x, v) = |v|$.

06.11.2023

2.12 Definition. Let (X, d) be a metric space and let $\gamma : [a, b] \to X$ be a path. Consider a partition $Z = (t_0, \dots, t_N)$ of $[a, b]$, i.e. $a = t_0 \leq t_1 \leq \dots \leq t_N = b$. We define

$$L^Z(\gamma) = \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)).$$

We can see $L^Z(\gamma)$ as the length of the polygon induced by Z. The length of γ is

$$L(\gamma) := L_d(\gamma) = \sup\{L^Z(\gamma) : Z \text{ is a partition of } [a, b]\} \in [0, \infty].$$

A path $\gamma : [a, b] \to X$ is said to be rectifiable if $L_d(\gamma) < \infty$.
The class of paths $\gamma : [a, b] \to X$ for $a \leq b \in \mathbb{R}$ with $L_d : A \to [0, \infty]$ is a length structure. We call L_d the induced length.

Remark. Let Z, Z' partitions of $[a, b]$. Z' is called a refinement of Z if $Z' \subset Z$. By Δ-inequality it follows $L^{Z'}(\gamma) \geq L^Z(\gamma)$ for $\gamma : [a, b] \to X$.
We define the mesh size of a partition $Z = (t_0, \dots, t_N)$ of $[a, b]$ as

$$|Z| = \max \{|t_{i-1} - t_i| : i = 1, \dots, N\}.$$

2.13 Lemma. *$L^{Z_i}(\gamma) \uparrow L(\gamma)$ if $|Z_i| \downarrow 0$.*

Proof. Exercise. □

2.14 Lemma. *Let $(V, |\cdot|)$ be a finite dimensional normed vector space and $\gamma : [a, b] \to V$ is a differentiable map. Then*

$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

2.15 Proposition (Properties of the induced length)**.** *Let $\gamma : [a, b] \to X$ be a path. The length $L = L_d$ induced by a metric d possesses the following properties.*

(i) Generalized Δ-inequality: $L(\gamma) \geq d(\gamma(a), \gamma(b))$.

(ii) Additivity: if $a < c < b$, then $L(\gamma, a, c) + L(\gamma, c, b) = L(\gamma)$. In particular $L(\gamma, a, t)$ is a nondecreasing function in $t \in [a, b]$.

(iii) If $\gamma : [a, b] \to X$ is rectifiable, then $L(\gamma, c, d)$ is a continuous function in $c, d \in [a, b]$.

(iv) L is lower semi-continuous on the space of continuous paths $\gamma : [a, b] \to X$ with respect to point-wise convergence, and hence w.r.t. uniform convergence.

Remark. In general then induced length is not continuous.

Proof. (i) By Δ-inequality it is clear that $L^Z(\gamma) \geq d(\gamma(b), \gamma(a))$ for any partition Z of $[a, b]$. Hence $L(\gamma) = \sup_Z L^Z(\gamma) \geq d(\gamma(a), \gamma(b))$.
(ii) Given a partition $Y = (t_0, \dots, t_N)$ and $c \in (a, b)$, $\exists i \in \{1, \dots, N\}$ such that $c \in [t_{i-1}, t_i]$. Then $Y' = (t_0, \dots, t_{i-1}, c, t_i, \dots, t_N)$ is also a partition and $L^{Z'}(\gamma) \geq L^Z(\gamma)$ by

the Δ-inequality. Moreover $Z_0 = (t_0, \dots, t_i, c)$ and $Z_1 = (c, t_i, \dots, t_n)$ are partitions of $[a, c]$ and $[c, b]$ respectively and $L^{Z'}(\gamma) = L^{Z_0}(\gamma|_{[a,c]}) + L^{Z_1}(\gamma|_{[c,b]})$. If we choose Z such that $L^Z(\gamma) + \epsilon \geq L(\gamma)$, then it follows $L(\gamma) \leq L(\gamma, a, c) + L(\gamma, c, b)$.

On the other hand, given two partitions of Z_0, Z_1 of $[a, c]$ and $[c, b]$, then $Z = Z_0 \cup Z_1$ is a partition of $[a, b]$. Similarly we get then that $L(\gamma, a, c) + L(\gamma, c, b) \leq L(\gamma)$.

(iii) We prove continuity in $d \in (a, b]$ from the left. The other case works analogously. Since $L(\gamma)$ is finite, for $\epsilon > 0$ we may choose a partition $Z = (t_0, \dots, t_N)$ of $[a, b]$ such that $L(\gamma) \leq L^Z(\gamma) + \epsilon$. By adding another point in Z (which preserves the inequality) we can assume that $d = t_j$ for $j \in \{1, \dots, N\}$. Since $L(\gamma, t_{i-1}, t_i) \geq d(\gamma(t_{i-1}), \gamma(t_i))$ $\forall i$ by (i),

$$L(\gamma, t_{j-1}, d) - d(\gamma(t_{j-1}), \gamma(d)) \leq L(\gamma) - L^Z(\gamma) < \epsilon.$$

This estimate holds now for any Z with $t_{j-1} = c$ that is arbitrarily close to d. Hence

$$L(\gamma, a, d) - L(\gamma, a, c) = L(\gamma, c, d) < d(\gamma(c), \gamma(d)) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this yields continuity from the left.

(iv) Let γ_j be paths that converge pointwise to γ, both defined on $[a, b]$. Choose $\epsilon > 0$ and let Z be a partition of $[a, b]$ as in (iii) for γ. Consider now $L^Z(\gamma_j)$ and choose j large enough such that $d(\gamma_j(z), \gamma(z)) < \epsilon$ for all $z \in Z$. Then it follows

$$L(\gamma) \leq L^Z(\gamma) + \epsilon \leq L^Z(\gamma_i) + \epsilon + 2(N+1)\epsilon \leq L(\gamma_j) + (2N+3)\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get $L(\gamma) \leq L(\gamma_j)$ and hence $L(\gamma) \leq \liminf L(\gamma_j)$. □

2.16 Definition. Let (X, d) be a metric space and let L_d be the induced length structure on continuous paths. We call $\hat{d} := d_{L_d}$ the induced intrinsic metric.

2.17 Example.

1. Consider $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$ with the restricted metric of $\mathbb{R}^2$. The induced intrinsic metric is the angular metric $\hat{d}(v, w) = \arccos\langle v, w\rangle_{eucl}$.
2. Consider $X = \mathbb{R}^n$ with $d(x, y) = \sqrt{|x - y|}$. d is a finite metric on $\mathbb{R}^n$ (check). But $\hat{d}(x, y) = \infty$ for all $x, y \in \mathbb{R}^n$. Indeed, if $\gamma : [0, 1] \to$ is a path, then

$$\begin{aligned} L_d(\gamma) &\geq \sum_{i=1}^{N} d(\gamma(t_{i-1}, t_i) = n \sum_{i=1}^{N} \frac{1}{n} \sqrt{|\gamma(t_{i-1}) - \gamma(t_i)|} \\ &\overset{\text{Jensen Inequality}}{\geq} n \sqrt{\sum_i \frac{1}{n} |\gamma(t_{i-1}) - \gamma(t_i)|} \\ &= \sqrt{n} \sqrt{\sum_i |\gamma(t_{i-1}) - \gamma(t_i)|} \geq \sqrt{n} \sqrt{|\gamma(1) - \gamma(0)|} \to \infty. \end{aligned}$$

2.18 Proposition. *Let (X, d) be a metric space.*

(1) If γ is a rectifiable curve in (X, d), then $L_{\hat{d}}(\gamma) = L_d(\gamma)$.

(2) The intrinsic metric induced by $\hat{d}$ coincides with $\hat{d}$, i.e. $\hat{\hat{d}} = \hat{d}$.

Proof. (1) By definition of L_d it follows $L_d(\gamma) \geq d(\gamma(a), \gamma(b))$ for every curve $\gamma : [a, b] \to X$. This implies $\hat{d} \geq d$. It follows that $L_{\hat{d}} \geq L_d$.

For the inverse inequality we pick $\gamma : [a, b] \to X$ that is rectifiable and a partition $Z = (t_0, \ldots, t_N)$ of $[a, b]$. By definition of $\hat{d}$ it follows that $\hat{d}(\gamma(t_{i-1}), \gamma(t_i)) \leq L_d(\gamma, t_{i-1}, t_i)$ $\forall i$. Hence

$$L^Z_{\hat{d}}(\gamma) \leq L_d(\gamma).$$

Since Z was an arbitrary partition we get the desired inequality $L_{\hat{d}}(\gamma) \leq L_d(\gamma)$. This proves (1).

(2) This follows trivially from (1). □

Remark. The assumption that the curve γ is rectifiable w.r.t. L_d is essential because otherwise the curve γ may not be continuous w.r.t. $(X, \hat{d})$.

2.19 Proposition. *Let (A, L) be a length structure and (X, d_L) the associated length space. Let $\hat{d}$ be the intrinsic metric induced by $d = d_L$ (or more precisely by L_{d_L}). Then $\hat{d} = d_L$.*

Proof. We repeat the argument from the previous proposition. Let $\gamma : [a, b] \to X$ be an admissible curve and $Z = (t_0, \ldots, t_N)$ a partition of $[a, b]$. By definition of d_L we have $d_L(\gamma(t_{i-1}), \gamma(t_i)) \leq L(\gamma, t_{i-1}, t_i)$. As before it follows that $L_{d_L}(\gamma) \leq L(\gamma)$ and hence $\hat{d}(\gamma(a), \gamma(b)) \geq d_L(\gamma(a), \gamma(b))$.

On the other hand we have $d_L(\gamma(a), \gamma(b)) \leq L_{d_L}(\gamma)$ by construction of L_{d_L} and hence $d_L \leq \hat{d}$. □

Remark. Note that given a metric d such that $d = \hat{d}$ where $\hat{d}$ is the induced intrinsic metric automatically implies that d was already intrinsic. We now can say (X, d) is a length space if and only if $d = \hat{d}$.

Remark. If γ_0 and γ_1 are two equivalent paths (a curve), then it is easy to see they have the same induced length L_d.

2.20 Definition. We say a path $\gamma : I \to X$ is parametrized with constant speed $c \in [0, \infty)$ if $L(\gamma|_{[a,b]}) = c(b-a)$ $\forall [a,b] \subset I$. If $c = 1$ we say γ has unit speed or we say γ is parametrized by arc length. Equivalently, γ is parametrized by arc length if and only if $\forall$ $a \in I$ and $t \in I$ we have

$$\frac{d}{dt} L(\gamma, a, t) = 1.$$

2.21 Proposition. *Every rectifiable curve $\gamma : [a,b] \to X$ can be represented in the form $\gamma = \bar{\gamma} \circ \varphi$ where $\bar{\gamma} : [0, L(\gamma)] \to X$ is parametrized by arc length. and φ is a nondecreasing continuous map from $[a,b]$ to $[0, L(\gamma)]$.*

Proof. Define $\varphi(t) = L(\gamma, a, t)$ $\forall t \in [a,b]$ and $\psi(\tau) = \inf\{t \in [a,b] : \varphi(t) = \tau\}$ for $\tau \in [0, L(\gamma)]$. We then define $\bar{\gamma}(\tau) = \gamma \circ \psi(\tau)$.

First we check that $\bar{\gamma} \circ \varphi(t) = \gamma(t)$. Indeed, by definition we have that $\varphi \circ \psi \circ \varphi(t) = \varphi(t)$ and hence $L(\gamma, a, t) = L(\gamma, a, \psi \circ \varphi(t))$ that implies $\gamma(t) = \gamma \circ \psi \circ \varphi(t) = \bar{\gamma} \circ \varphi(t)$.

It remains to verify that $\bar{\gamma} : [0, L(\gamma)] \to X$ is continuous and parametrized by arc length. We pick $\tau_0, \tau_1 \in [0, L(\gamma)]$ and $t_0, t_1 \in [a,b]$ such that $\varphi(t_i) = \tau_i$. Then $L(\gamma, t_0, t_1) = \varphi(t_1) - \varphi(t_0) = \tau_1 - \tau_0$. Moreover $d(\bar{\gamma}(\tau_0), \bar{\gamma}(\tau_1)) \leq |\tau_1 - \tau_0|$. Hence $\bar{\gamma}$ is continuous. It also holds $L(\bar{\gamma}, \tau_0, \tau_1) = L(\gamma, t_0, t_1) = \tau_1 - \tau_0$. Thus $\bar{\gamma}$ is parametrized by arc length. □

2.2 Existence of shortest paths

2.22 Definition. A sequence of curves uniformly converges to a curve γ if they admit parametrizations with the same domain that uniformly converge to a parametrization of γ.

2.23 Theorem (Arzela-Ascoli Theorem)**.** *In a compact metric space any sequence of curves with uniformly bounded lengths has a uniformly converging subsequence.*

Proof. Let (γ_i) be the sequence in the theorem. Each γ_i admits a parametrization on $[0,1]$ with constant speed. Uniformly bounded lengths means that the speeds of these parametrizations are uniformily bounded. Hence

$$d(\gamma_i(t), \gamma_i(t')) \leq L(\gamma, t, t') \leq C|t - t'| \ \forall t, t' \in [0,1] \text{ and } \forall i \in \mathbb{N}. \quad (*)$$

Let $S = \{t_j\}$ be a countable dense subset of $[0,1]$. Using a diagonal argument one can construct a subsequence γ_{n_i} of γ_i such that $\gamma_{n_i}(t_j)$ converges for $i \to \infty$ and for every $j \in \mathbb{N}$.

We will show that (γ_{n_i}) converges pointwise. W.l.o.g. (or by renaming the subsequence) we assume $\gamma_{n_i} = \gamma_i$.

For this one shows that $\gamma_i(t)$ is a Cauchy sequence (Exercise) and we can define $\gamma(t) := \lim_{i\to\infty} \gamma_i(t)$ for $t \in [0,1]$. Then we can pass to the limit in $(*)$ and get that $\gamma : [0,1] \to X$ is a continuous map.

Finally we have to show that the convergence of γ_i to γ is uniform. Let $\epsilon > 0$, pick $N \geq \frac{4C}{\epsilon}$, $N \in \mathbb{N}$ and let $M > 0$, such that $d(\gamma(k/N), \gamma_i(k/N)) < \epsilon/2$ for all $k = 0, 1, \ldots, N$

for all $i \geq M$. Then it follows for $t \in [k/N, (k+1)/N]$ that

$$\begin{aligned} d(\gamma_i(t), \gamma(t)) &\leq d(\gamma_i(t), \gamma_i(k/N)) + d(\gamma_i(k/N), \gamma(k/N)) + d(\gamma(k/N), \gamma(t)) \\ &\leq C|t - k/N| + \epsilon/2 + C|t - k/N| \leq \frac{2C}{N} + \epsilon/2 \leq \epsilon. \end{aligned}$$

Since M and ϵ don't depend on t, we get uniform convergence. □

2.24 Definition. Let (X, d) be a metric space. A curve $\gamma : [a, b] \to X$ is a shortest path if its length is minimal w.r.t. all curves with the same endpoints.

If (X, d) is a length space a path $\gamma : [a, b] \to X$ is a shortes path if and only if $L(\gamma) = d(\gamma(a), \gamma(b))$.

2.25 Proposition. *Let (X, d) be a length space and let γ_i be shortest paths that converge to a path γ as $i \to \infty$. Then γ is also shortest path.*

Proof. Uniform convergence of $\gamma_i : [a, b] \to X$ to γ implies in particular that $\gamma_i(a), \gamma_i(b) \to \gamma(a), \gamma(b)$. Since X is a length space, we have $L(\gamma_i) = d(\gamma_i(a), \gamma_i(b))$, and consequently $L(\gamma_i) \to d(\gamma(a), \gamma(b))$. But by lower semi-continuity of L it then follows $L(\gamma) \leq d(\gamma(a), \gamma(b))$. Hence γ is a shortest path. □

2.26 Proposition. *Let (X, d) be a compact metric space and let $x, y \in X$ such that there exists at least one rectifiable curve that connects them. Then there exists a shortest path between x and y.*

Proof. Consider $\hat{d}(x, y)$ that is the infimum of lengths of rectifiable curves between x and y. Hence $\exists$ (γ_i) such that $L(\gamma_i) \to \hat{d}(x, y)$. According to the Arzela-Ascoli theorem there exists a subsequence of (γ_i) that converges to a curve γ. The path γ has the same endpoints x and y and by lower semicontinuity of L we have $L(\gamma) \leq \hat{d}(x, y)$. Thus $L(\gamma) = \hat{d}(x, y)$. □

2.27 Definition. A metric space (X, d) is called locally compact if every point $x \in X$ has a pre-compact neighborhood.

2.28 Proposition. *If (X, d) is a complete locally compact length space, then every closed ball in X is compact.*

Proof. Let $x \in X$ be arbitrary. If $\bar{B}_r(x) = \{y \in Y : |x, y| \leq r\}$ is compact for some $r > 0$, then $\bar{B}_\rho(x)$ is compact for any $\rho < r$. Define $R := \sup\{r > 0 : \bar{B}_r(x) \text{ is compact}\}$. Since x has pre-compact neighborhood, we have $R > 0$. We set $\bar{B}_R(x) =: B$.

We prove B is compact. B is closed set in a complete space. Hence, it suffices to prove that B is totally bounded, i.e. for any $\epsilon > 0$ B contains a finite ϵ-net (Theorem 1.13).

We may assume $\epsilon < R$. Let $B' := \bar{B}_{R-\epsilon/3}(x)$. This ball is compact and therefore contains a finite $\epsilon/3$-net. Let $y \in B$. Assume $y \notin B'$. Since X is a length space, we have $d(y, B') = d(x, y) - (r - \epsilon/3) < \epsilon/3$ (see Corollary 2.35 below). Hence $\exists y' \in B'$ such that $d(y, y') < \epsilon/2$. On the other hand $d(y', S) < \epsilon/2$, and hence $d(y, S) < \epsilon$. It follows that S is an ϵ-net for B, and B is therefore compact.

Each $y \in B$ has a precompact neighborhood U_y. We pick a finite collection $\{U_y\}_{y \in Y}$ of such neighborhoods that cover B. The union of $U := \bigcup_{y \in Y} U_y$ is precompact. There exists $\epsilon > 0$ such that $B_\epsilon(B) \subset U$ (argue by contradiction). Since X is a length space, we also have $B_\epsilon(B) = B_{R+\epsilon}(x)$ (indeed, if $z \in B_\epsilon(B)\backslash B$, then $\epsilon > d(z, B) = d(x, z) - R$ and hence $z \in B_{R+\epsilon}(x)$. $B_{R+\epsilon}(x) \subset B_\epsilon(B)$ holds in any case). Moreover the closure $\bar{B}_{R+\epsilon}(x) \subset \bar{U}$ is compact. This is a contradiction with the definition of $R > 0$ and $R < \infty$. Hence $R = \infty$. □

2.29 Corollary. *Let (X, d) be complete locally compact length space. Then (X, d) is strictly intrinsic.*

Proof. We have to show that that for every $x, y \in X$ such that $d(x, y) = R < \infty$ $\exists$ a shortes path γ connection x and y. By the previous proposition $\bar{B}_{2R}(x)$ is compact, and since X is a length space there exists a rectifiable curve $\widetilde{\gamma}$ between x, y with $L(\widetilde{\gamma}) \leq R + \epsilon$. The Proposition 2.26 yields the existence of a shortest path γ w.r.t. paths in $\bar{B}_{2R}(x)$. Since $\widetilde{\gamma}$ is such a path, since $L(\widetilde{\gamma}) < 2R$ and since paths that start in x and leave $B_{2R}(x)$ at least have length $2R$, it follows that γ satisfies $L(\gamma) = d(x, y)$. □

Question: Given an intrinsic metric d induced by a length structure L, what is the relation between L and L_d?

2.30 Theorem. *If L is a lower semi-continuous length structure, i.e. lower semi-continuous w.r.t. pointwise convergence of paths in X, L coincides with the length structure induced by its intrinsic metric $d_L = d$ on all curves that are admissible for L.*

Proof. The inequality $L_d(\gamma) \leq L(\gamma)$ for an admissible path γ holds for any length structure (see Proposition 2.19).

Consider $L(t) = L(\gamma, a, t)$ for an admissible path $\gamma : [a, b] \to X$ with finit length. By the second property 2. of length structures $L(t)$ is uniformily continuous. Hence for any $\epsilon > 0$ there exists a partition $Z = (t_0, \ldots, t_N)$ of $[a, b]$ such that $d_L(\gamma(t_{i-1}), \gamma(t_i)) < \epsilon \; \forall i = 1, \ldots, N$. According to the definition of d_L for each $i = 1, \ldots, N$ there exists an admissible curve $\sigma_i : [t_{i-1}, t_i] \to X$ with endpoints $\sigma_i(t_{i-1}) = \gamma(t_{i-1})$ and $\sigma_i(t_i) = \gamma(t_i)$ such that $L(\sigma_i) \leq d_L(\gamma(t_{i-1}), \gamma(t_i)) + \epsilon/N$. We can consider the concatenation $c_\epsilon : [a, b] \to X$ of the curves σ_i and c_ϵ we have

$$L(c_\epsilon) = \sum_{i=1}^{N} L(\sigma_i) \leq \sum_{i=1}^{N} d_L(\gamma(t_{i-1}), \gamma(t_i)) + \epsilon \leq L_d(\gamma) + \epsilon.$$

From the triangle inequality we see that $d_L(\gamma(t), c_\epsilon(t)) < 3\epsilon$ for all $t \in [a, b]$. Hence c_ϵ converges pointwise to γ for $\epsilon \downarrow 0$ w.r.t. d_L. But the topology of d_L is always finer than the original topology of X. So by lower semi-continuity of L we obtain

$$L(\gamma) \leq \liminf_{\epsilon \to 0} L(c_\epsilon) \leq L_d(\gamma)$$

which is the desired inequality. □

In the following we will only consider lower continuous length structures L (defined on admissible paths). Hence $d_L = \hat{d}_L =: d$ and $L = L_d$ (on admissible paths). In particular $\hat{\hat{d}}_L = \hat{d}_L = d_L$.

W.l.o.g. if we consider an intrinsic metric space (X, d) we mean that $\hat{d} = d$ and $L = L_d$ on continuous paths.

2.31 Example. Let (M, g) be a Riemannian manifold, L the induced length structure on piecewise differentiable curves, and $\hat{d} = d_L$ the induced Riemannian distance. $(M, \hat{d})$ is locally compact length space and L is also lower semi-continuous. Hence the induced length $L_{\hat{d}}$ coincides with L on piecewise differentiable curves. If $(M, \hat{d})$ is complete as metric space, then $\forall x, y \in M$ $\exists$ a minimal geodesic between x and y (Hopf-Rinow).

We consider a metric space (X, d) as before.

2.32 Definition. (i) A point $z \in X$ is called a midpoint between $x, y \in X$ if $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$.

(ii) Given $\epsilon > 0$ a point $z \in X$ is called an ϵ-midpoint between $x, y \in X$ if

$$\max\{d(x, z), d(z, y)\} \leq \frac{1}{2}d(x, y) + \epsilon.$$

2.33 Lemma. *1. If d is strictly intrinsic, then for every two points $x, y \in X$ there exists a midpoint z.*

2. If d is intrinsic, then $\forall \epsilon > 0$ and for every two points $x, y \in X$ there exists an ϵ-midpoint z between them.

Proof. We first prove 2. Let $\gamma : [a, b] \to X$ be a path between $x, y \in X$ such that $L(\gamma) = d(x, y) + 2\epsilon$. Since $t \in [a, b] \mapsto L(\gamma, a, t)$ is continuous, there exists $t_0 \in [a, b]$ such that $L(\gamma, a, t_0) = \frac{1}{2}L(\gamma)$. Hence $d(\gamma(a), \gamma(t_0)) \leq L(\gamma, a, t_0) \leq \frac{1}{2}d(x, y) + \epsilon$. Similarly one can prove 1. □

2.34 Theorem. *Let (X, d) be a complete metric space.*

1. If for every $x, y \in X$ there exists a midpoint, then d is strictly intrinsic.

2. If for every $x, y \in X$ and for every $\epsilon > 0$ there exists an ϵ-midpoint, then d is intrinsic.

Proof. We only prove 1. and 2. is left as an exercise.

We construct a path $\gamma : [0, 1] \to X$ between x, y such that $\gamma(0) = x, \gamma(1) = y$ and $L(\gamma) = d(\gamma(0), \gamma(1))$. First we assign the values of γ for all dyadic rationals $\frac{k}{2^m}$, $k \in \{1, \ldots, 2^m\}$ and $m \in \mathbb{N}$. This is done by successively picking midpoints. The map that we obtain by such a contruction satisfies

$$d(\gamma(t), \gamma(t')) = (t' - t)d(x, y) \tag{1}$$

where t', t are dyadic and $t < t'$. Hence γ is defined on a dense subset of $[0, 1]$ and 1-Lipschitz continuous. Since (X, d) we can extend γ to the entire interval $[0, 1]$ as a 1-Lipschitz map using Proposition 1.8. Thus we obtain a path between x and y, and (1) holds for all $s, t \in [0, 1]$ with $s < t$. It also follows that $L(\gamma) = d(x, y)$. □

2.35 Corollary. *Let (X, d) be a length space, $x, y \in X$ and $r \in (0, d(x, y))$. Then $d(y, B_r(x)) = d(x, y) - r$.*

2.36 Corollary. *A complete length space X is a length space iff given $\epsilon > 0$ and 2 points $x, y \in X$ there exists a finite sequence $x = x_1, \ldots, x_k = y$ such that every two neighboring points in this sequence are ϵ-close (i.e. $d(x_i, x_{i+1}) \leq \epsilon \; \forall i$) and $\sum_{i=1}^{k-1} d(x_i, x_{i+1}) < d(x, y) + \epsilon$.*

2.37 Definition. Let $I \subset \mathbb{R}$ be an interval. A path $\gamma : I \to X$ is a shortest path if $\gamma|_{[a,b]}$ is a shortest path for any interval $[a,b] \subset I$.

A path $\gamma : I \to X$ is called a geodesic if $\forall t \in I$ there exists $[a,b]$ such that $t \in (a,b) \subset I$ and $\gamma|_{[a,b]}$ is a shortest path.

Remark. In general geodesic are not shortest path. Moreover shortest paths between points are not unique in general.

2.38 Theorem (Hopf-Rinow-Cohn-Vossen). *Let (X,d) be a locally compact length space. The following statements are equivalent:*

(i) X is boundedly compact, i.e. every closed metric ball in X is compact.

(ii) X is complete.

(iii) Every geodesic $\gamma : [0,a) \to X$ can be extended to a continuous path $\bar{\gamma} : [0,a] \to X$.

(iv) There is a point $p \in X$ such that every shortest path $\gamma : [0,a) \to X$ with $\gamma(0) = p$ can be extended to a continuous path $\bar{\gamma} : [0,a] \to X$.

By Corollary 2.29 each of the conditions imply that all points in X can be connected by a shortest path.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are straightforwart and left as an exercise.

We will show that (iv) implies (i). The proof is similar to the one of Proposition 2.28. The difference is that for Proposition 2.28 we were allowed to use completeness where in this case we only can use property (iv).

We fix $p \in X$ and we define again $R = \sup\{r : \bar{B}_r(p) \text{ is compact}\}$ where $R > 0$ by local compactness. Our goal is to show that $R = \infty$.

We assume $R < \infty$ and first show that $B_R(p)$ is precompact. For this we pick a sequence $(x_i) \subset B_R(p)$ and set $d(p, x_i) = r_i$. We will show that (x_i) has a converging subsequence. We can assume $r_i \to R$. Otherwise $x_i \in B_r(p)$ $\forall i$ for some $r \in (0,R)$ and by compactness of $\bar{B}_r(p)$ we find a converging subsequence.

Let $\gamma_i : [0, r_i] \to X$ be a sequence of shortest paths between p and x_i that is parametrized by arc length. These paths exit because every x_i belongs to a compact ball centered at p. We can choose a subseqence such that the restriction $\gamma_i|_{[0,r_1]}$ converges to a curve $\gamma^1 : [0, r_1] \to X$. From this subsequence we can choose another subsequence such that $\gamma_i|_{[0,r_2]}$ converges to $\gamma^2 : [0, r_2] \to X$. We continue this iteration and then pick a diagonal sequence γ_i that converges on $[0, r_j]$ for every j.

We can then define $\gamma(t) = \lim \gamma_i(t)$ for $t \in [0,R)$. The curves γ_i where nonexpanding, i.e. 1-Lipschitz,

$$d(\gamma_i(t), \gamma_i(s)) \leq |t - s|$$

and hence by continuity of this inequality also γ is 1-Lipschitz. Moreover γ is also a shortest path on each subinterval $[0, r_i]$ by Proposition 2.25. By (iv) there exists a continuous extension $\bar{\gamma} : [0,R] \to X$. Moreover $\gamma_i(r_i) = x_i$ converges $\bar{\gamma}(R)$ (Exercise). Hence $B_R(p)$ is precompact.

Therefore $\bar{B}_R(p)$ is compact and we can proceed like in the proof of Proposition 2.28 the end up with a contradiction. ☐

2.3 Metric speed

2.39 Definition. Let (X, d) be a metric space and $\gamma : I \to X$ a curve. The speed of γ at $t \in I$ denoted by $v_\gamma(t)$ is defined by

$$v_\gamma(t) := |\dot{\gamma}(t)| := \lim_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|}$$

if the limit exists.

Remark. The notation $|\dot{\gamma}(t)|$ is justified by the following observation. Let $\gamma : [a, b] \to (\mathbb{R}^n, |\cdot|_{eucl})$ be a differentiabl curve, then $v_\gamma(t)$ exists $\forall t \in [a, b]$ and $v_\gamma(t) = |\gamma'(t)|_{eucl}$.

2.40 Theorem. *Let (X, d) be a metric space and let $\gamma : [a, b] \to X$ be a rectifiable curve. Then for $\mathcal{L}^1$-a.e. $t \in [a, b]$ it holds either*

$$\lim_{\epsilon, \epsilon' \downarrow 0} \frac{L(\gamma|_{[t-\epsilon, t+\epsilon']})}{\epsilon + \epsilon'} = 0$$

or

$$\lim_{\epsilon, \epsilon' \downarrow 0} \frac{d(\gamma(t-\epsilon), \gamma(t+\epsilon))}{L(\gamma|_{[t-\epsilon, t+\epsilon']})} = 1.$$

2.41 Theorem (Vitali's Covering Theorem for $\mathbb{R}^n$)**.** *Let $X \subset \mathbb{R}^n$ be bounded and let $\mathfrak{B}$ be a collection of closed balls in $\mathbb{R}^n$ such that for every $x \in X$ there exists $\epsilon > 0$ and $B \in \mathfrak{B}$ with $x \in B$ and $\operatorname{diam}_B < \epsilon$. The there exists a countable subcollection $\{B_i\}_{i \in I}$ in $\mathfrak{B}$ of disjoint balls the still cover X up to a set of Lebesgue measure 0.*

Proof. We argue by contradiction and suppose the contrary. For every $\alpha > 0$ let Z_α denote the set of all $t \in [a, b]$ such that

$$\liminf_{\epsilon, \epsilon' \downarrow 0} \frac{L(\gamma|_{[t-\epsilon, t+\epsilon']})}{\epsilon + \epsilon'} = \alpha \quad (\dagger)$$

and

$$\lim_{\epsilon, \epsilon' \downarrow 0} \frac{d(\gamma(t-\epsilon), \gamma(t+\epsilon))}{L(\gamma|_{[t-\epsilon, t+\epsilon']})} = 1 - \alpha \quad (\ddagger).$$

By assumption $\mathcal{L}^1(Z_\alpha) > 0$ for all $\alpha > 0$ sufficiently small. Otherwise $\bigcup_{\alpha > 0} Z_\alpha$ would have 0 measure which is equivalent to the statement of the theorem.

We fix $\alpha > 0$ with $\mathcal{L}^1(Z_\alpha) > 0$ and set $Z_\alpha =: Z$ as well $\mathcal{L}^1(Z) = \mu$.

We choose $\epsilon_0 > 0$ so small, such that for any partition $\{y_i\}_{i=0,\dots,N}$ of $[a, b]$ with $\max_{1=1,\dots,N}(y_i - y_{i-1}) < \epsilon_0$, one has

$$L(\gamma) - \sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i)) < \mu\alpha^2/2.$$

(We find such ϵ_0 by Problem 1 on problem sheet 4.)

Consider now the set $\mathfrak{B}$ of all intervals of the form $[t - \epsilon, t + \epsilon']$ such that $t \in Z$ and $\epsilon, \epsilon' > 0$ satisfy $\epsilon + \epsilon' < \epsilon_0$, and

$$\frac{L(\gamma|_{[t-\epsilon, t+\epsilon']})}{\epsilon + \epsilon'} > \alpha$$

and

$$\frac{d(\gamma(t-\epsilon),\gamma(t+\epsilon))}{L(\gamma|_{[t-\epsilon,t+\epsilon']})} < 1-\alpha.$$

By the definition of Z every $t \in Z$ is contained in arbitrarily short element of $\mathfrak{B}$. Applying Vitali's covering theorem (Theorem 2.41) we can find a countable subfamily $\{[t_i-\epsilon_i, t_i+\epsilon_i]\}_{i=1}^{\infty}$ in $\mathfrak{B}$ of dijoint intervals that cover Z up to a set of zero measure. In particular

$$\sum_{i=1}^{\infty}(\epsilon_i+\epsilon_i') \geq \mu.$$

Hence for $M \in \mathbb{N}$ sufficiently large we have

$$\sum_{i=1}^{M}(\epsilon_i+\epsilon_i') > \mu/2.$$

Since the intervals from this subfamily are disjoint, we can find a partition $\{y_j\}_{j=0,\dots,N}$ such that $\max_i(y_i-y_{i-1}) < \epsilon_0$ such that $\forall i$ we have that $[t_i-\epsilon_i, t_i+\epsilon_i'] \subset [y_{j-1}, y_j]$ for some j. In particular, since the length of $[y_{j-1}, y_j]$ is smaller than ϵ_0, if $t_i \in [y_{j-1}, y_j]$, then $[y_{j-1}, y_j] \in \mathfrak{B}$.

We denote $L_j = L(\gamma|_{[y_{j-1},y_j]}$ and $d_j = d(\gamma(y_{j-1}), \gamma(y_j))$. By the choice of ϵ_0, we have

$$\sum_{j=1}^{N}(L_j-d_j) = L(\gamma) - \sum_{j=1}^{N} d_j < \mu\alpha^2/2.$$

If $[y_{j-1}, y_j]$ is in $\mathfrak{B}$ (i.e. $[y_{j-1}, y_j] = [t_i-\widetilde{\epsilon}_i, t_i+\widetilde{\epsilon}_i']$ for some i, then

$$L_j - d_j \geq L_j - (1-\alpha)L_j = \alpha L_j \geq \alpha^2(y_{j-1}-y_j) = \alpha^2(\widetilde{\epsilon}_i+\widetilde{\epsilon}_i').$$

Hence

$$\sum_{j=1}^{N}(L_j-d_j) \geq \alpha^2\sum_{i=1}^{M}(\widetilde{\epsilon}_i+\widetilde{\epsilon}_i') \geq \alpha^2\sum_{i=1}^{M}(\epsilon_i+\epsilon_i') > \frac{1}{2}\mu\alpha^2.$$

This is a contradiction. □

Proof of Vitali's Theorem. We may assume that every ball in $\mathfrak{B}$ contains a least one $x \in X$ and that every ball in $\mathfrak{B}$ has radius not greater than 1. Then all the balls are contained in the 2-neighborhood $B_2(X)$ which is bounded and hence has finite volume. We construct a sequence of balls $\{B_i\}_{i\in\mathbb{N}}$ by induction. Assume $B_1, \dots, B_m$ are already constructed. Let $\mathfrak{B}_m$ be the collection of balls in $\mathfrak{B}$ that do not intersect with $B_1, \dots, B_m$. If $\mathfrak{B}_m$ is empty, then $\{B_i\}_{i=1,\dots,m}$ covers the entire set X and the proof is finished. If $\mathfrak{B}_m$ is not empty, we choose B_{m+1} to be any element of $\mathfrak{B}_m$ with

$$\operatorname{diam} B_{m+1} > \frac{1}{2}\sup\{\operatorname{diam} B : B \in \mathfrak{B}_m\}.$$

The sequence of balls $\{B_i\}_{i\in\mathbb{N}}$ is disjoint by construction. We have to show that they cover X up to a set of measure 0. We have

$$\sum_{i=1}^{\infty}\mathcal{L}^n(B_i) < \infty$$

since the union of these balls has finite volume. Hence $\exists m \in \mathbb{N}$ such that $\sum_{i=m+1}^{\infty} \mathcal{L}^n(B_i) < \epsilon$. Let $x \in X \backslash \bigcup_i B_i$ and let B be any ball in $\mathfrak{B}$ that contains x and does not intersect the balls $B_1, \ldots, B_m$. Note that B must intersect with $\bigcup_i B_i$ because otherwise $B \in \mathfrak{B}_m$ for all $m \in \mathbb{N}$ which contradicts that $\mathcal{L}^n(B_i) \to 0$ (since $2\mathcal{L}(B_i) > \operatorname{diam} B \ \forall B \in \mathfrak{B}_{i-1}$).

Let $k \in \mathbb{N}$ be minimal such that $B \cap B_k \neq \emptyset$. Then $B \in \mathfrak{B}_{k-1}$ and hence $\operatorname{diam} B_k > \frac{1}{2} \operatorname{diam} B$. It follows that the distance between x and the center of B_k is not greater as 5 times the radius of B_k. Hence x belongs to the ball with the same center as B_k and 5 times its radius. We denote this ball with $5B_k$.

We have proved that $x \in X \backslash \bigcup_i B_i$ is contained in $5B_k$ for some $k > m$ and therefore $X \backslash \bigcup_i B_i \subset \bigcup_{i=m+1}^{\infty} 5B_i$. Hence

$$\mathcal{L}^n(X \backslash \bigcup_i B_i) \leq \sum_{i=m+1}^{\infty} \mathcal{L}^n(5B_i) = 5^n \sum_{i=m+1}^{\infty} \mu_n(B_i) < 5^n \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we showed that $X \backslash \bigcup_i B_i$ has measure 0. $\square$

In the proof of Theorem 2.40 it is possible to set $\epsilon = 0$. Then we obtain the following corollary.

2.42 Corollary. *Let (X, d) be a metric space and let $\gamma : [a, b] \to X$ be a rectifiable curve. Then for $\mathcal{L}^1$-a.e. $t \in [a, b]$ it holds either*

$$\lim_{\epsilon \downarrow 0} \frac{L(\gamma|_{[t,t+\epsilon]})}{\epsilon} = 0 \quad or \quad \lim_{\epsilon \downarrow 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{L(\gamma|_{[t,t+\epsilon]})} = 1.$$

2.43 Corollary. *Let X be a metric space, $\gamma : [a, b] \to X$ a Lipschitz curve. Then the speed $v_\gamma(t)$ exists for $\mathcal{L}^1$-almost every $t \in [a, b]$ and $L(\gamma) = \int_a^b v_\gamma(t)dt$.*

Proof. Recall the following fact. If $f : [a, b] \to \mathbb{R}$ is Lipschitz, $f'(t)$ exists $\mathcal{L}^1$-almost everywhere, and $\int_a^b f'(t)dt = f(b) - f(a)$.

If we define $f(t) = L(\gamma|_{[a,t]})$ for $t \in [a, b]$, then f is a Lipschitz function. For a.e. $t \in [a, b]$ we write

$$f'(t) = \lim_{\epsilon \to 0} \frac{L(\gamma|_{[t,t+\epsilon]})}{|\epsilon|} = \lim_{\epsilon \to 0} \underbrace{\frac{L(\gamma|_{[t,t+\epsilon]})}{d(\gamma(t), \gamma(t+\epsilon))}}_{\geq 1} \cdot \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|} \geq \limsup_{\epsilon \to} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|}.$$

Then we have either $f'(t) = 0$, or the first term in the last product goes to 1. In the first case we have

$$0 = \lim_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|} = v_\gamma(t).$$

In the second cases we have

$$v_\gamma(t) = \lim_{\epsilon \downarrow 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|} = f'(t).$$

Thus $v_\gamma(t)$ exists and equals $f'(t)$ in both cases (for $\mathcal{L}^1$-almost every t).

The theorem follows by integrating $v_\gamma(t) = f'(t)$. □

The last corollary actually holds for bigger class of paths.

2.44 Definition. Consider a metric space (X, d). A path $\gamma : [a, b] \to X$ is called absolutely continuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any sequence of disjoint intervals $[a_i, b_i]$, $i = 1, \ldots, N$ with $\sum_{i=1}^N (b_i - a_i) \leq \delta$ we have $\sum_{i=1}^N d(\gamma(t_{i-1}), \gamma(t_i)) \leq \epsilon$.

Remark. Let $\gamma : [a, b] \to X$ be a path. If γ is Lipschitz, then it is absolutely continuous.

Recall the following. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then f is differentiable almost everywhere, f' is integrable and

$$\int_a^b f'(t)dt = f(b) - f(a).$$

2.45 Lemma. *If a path $\gamma : [a, b] \to X$ is absolutely continuous, then it is rectifiable.*

Proof. Let $\epsilon = 1$ and pick $\delta > 0$ like in the definition of an absolutely continuous path, and let $M \in \mathbb{N}$ be such that $\frac{b-a}{M} \leq \delta$.

Let $Z = (t_0, \dots, t_M)$ is be the partition of $[a, b]$ given by $t_i = a + i\frac{b-a}{M}$. Let Z' be any partition with mesh size smaller than $\frac{b-a}{M}$. W.l.o.g. we can assume that $Z \subset Z'$. Hence, for every i there exist $y_0, \dots, y_k \in Z'$ such that $a_{i-1} = y_0 \leq \dots \leq y_k = a_i$. Moreover, it holds $\sum_{j=1}^{k}(y_j - y_{j-1}) \leq \delta$. Hence

$$\sum_{i=1}^{k} d(\gamma(y_{i-1}), \gamma(y_i)) \leq 1.$$

Hence $L^{Z'}(\gamma) \leq M$. Since Z' was any partition with mesh size smaller than $\frac{b-a}{M}$, we have γ is rectifiable. □

2.46 Theorem. *Let (X, d) be a metric space and let $\gamma : [a, b] \to X$ be an absolutely continuous path. Then the speed $v_\gamma(t)$ exists $\mathcal{L}^1$-a.e., is integrable and*

$$L(\gamma) = \int_a^b v_\gamma(t)dt.$$

Proof. If we show that $f(t) = L(\gamma, a, t)$ is absolutely continuous, we can finish the proof exactly like for the case of Lipschitz curves.

Let $\epsilon > 0$ and pick $\delta > 0$ as before. Hence, if (a_i, b_i), $i = 1, \dots, k$, are intervals such that $\sum_{i=1}^{k}(b_i - a_i) \leq \delta$, then $\sum_{i=1}^{k} d(\gamma(a_i), \gamma(b_i)) \leq \epsilon$.

This estimate stays true, if pick any family of finer intervals that come from partitions of the intervals $[a_i, b_i]$. Hence, it follows $\sum_{i=1}^{k} L(\gamma, a_i, b_i) = \sum_{i=1}^{k} f(b_i) - f(a_i) \leq \epsilon$. □

2.4 Length and Hausdorff measure

Let (X, d) be a metric space. Recall the definition of the 1-dimensional Hausdorff measure. For a countable family of sets $\{S_i\}$ in X, the 1-weight is defined as

$$w_1(\{S_i\}) = \sum_i \operatorname{diam} S_i.$$

Given $A \subset X$ for any $\epsilon > 0$ we set

$$\mathcal{H}_{1,\epsilon}(A) = \inf\{w_1(\{S_i\}) : \operatorname{diam} S_i \leq \epsilon, \forall i \in I\}$$

where the infimum runs over all countable coverings of A with $+\infty$ if no such covering exists. The 1-dimensional Hausdorff measure is then

$$\mathcal{H}^1(A) = C \lim_{\epsilon \downarrow 0} \mathcal{H}_{r,\epsilon}(A)$$

for a constant $C > 0$.

2.47 Lemma. *For any connected metric space X, it holds $\mathcal{H}^1(X) \geq \operatorname{diam} X$.*

Proof. First we note the in the definition of $\mathcal{H}^1$ it suffices to consider only coverings with open sets. Indeed, if $\{S_i\} \subset X$, with $i \in \mathbb{N}$, we define

$$S_i' = B_\delta(S_i) = \{y \in X : d(S_i, y) \leq \delta/2^i\}.$$

Then $\operatorname{diam} S_i' \leq \operatorname{diam} S_i + 2\delta/2^i$ and therefore $w_1(\{S_i'\}) \leq w_1(\{S_i\}) + 2\delta$. Since $\delta > 0$ is arbitrarily small, the claim follows.

Let $\{S_i\}$ be an open cover of X. Let $x, y \in X$. There exist a finite sequence of $S_{i_1}, \ldots, S_{i_n}$ such that $x \in S_{i_1}$ and $y \in S_{i_n}$ and $S_{i_k} \cap S_{i_{k+1}} \neq \emptyset$ $\forall k = 1, \ldots, n-1$. Indeed, fixing $x \in X$ let Y be the set of all points y such that such a sequence exists. Then any open set $U \subset X$ must be either in Y or in $X \backslash Y$. Hence Y and $X \backslash Y$ are open. Since X is connected we have $Y = X$.

Now let $\{S_i\}$ be an arbitrary countable covering of X, $x, y \in X$ and $\{S_{i_k}\}$ a sequence as before. It follows there exist $x_k \in S_{i_k} \cap S_{i_{k+1}}$ for all $k = 1, \ldots, n-1$ such that $x_0 = x$ and $x_n = y$. Clearly then $d(x_k, x_{k+1}) \leq \operatorname{diam} S_{i_{k+1}}$. Therefore

$$\sum_i \operatorname{diam} S_i \geq \sum_k \operatorname{diam} S_{i_k} \geq \sum_k d(x_k, x_{k+1}) \geq d(x, y).$$

Since $\{S_i\}$ and $x, y \in X$ are arbitrary, it follwos $\mathcal{H}^1(X) \geq \operatorname{diam} X$. $\square$

2.48 Theorem. *Let (X, d) be a metric space and let $\gamma : [a, b] \to X$ be a rectifiable simple curve. Then $L(\gamma) = \mathcal{H}^1(\gamma([a, b]))$.*

Remark. A path γ is simple if it is an injective map.

Proof. Let $L = L(\gamma) < \infty$ and $S = \gamma([a, b])$. Assume γ is parametrized by arc length, in particular $[a, b] = [0, L]$ w.l.o.g.

Consider the partitioin $t_i = i\frac{L}{N}$, $i = 0, \ldots N$, of $[0, L]$. It follows that $d(\gamma(s), \gamma(t)) \leq L(\gamma, s, , t) \leq L(\gamma, t_{i-1}, t_i)$ for all $s, t \in [t_{i-1}, t_i]$. Hence $\operatorname{diam} \gamma([t_{i-1}, t_i]) \leq L(\gamma, t_{i-1}, t_i) = L/N$. Hence, the sum of these diameter is smaller or equal than L. Since these diameter also go to 0 as $N \to \infty$, it follows that $\mathcal{H}^1(S) \leq L$.

On the other hand let $a = t_0 \leq \ldots t_N = b$ be a partition of $[a, b]$, and set $S_i = \gamma([t_i, t_{i+1}])$, $i = 0, \ldots, N-1$. Since γ is simple, the S_i are disjoint, up to finitely many points $\gamma(t_i)$. The unition of these points has $\mathcal{H}^1$-measure 0. Thus $\mathcal{H}^1(S) = \sum_i \mathcal{H}^1(S_i)$. The previous lemma implies

$$\mathcal{H}^i(S_i) \geq \operatorname{diam} S_i \geq d(\gamma(t_i), \gamma(t_{i+1})).$$

Since the partition was arbitrary, it follows that $\mathcal{H}^1(S) \geq L(\gamma)$. $\square$

3 Constructions

3.1 Locality of length spaces

3.1 Lemma. *Let X be a topological space that is covered by a collection of open sets $\{X_\alpha\}_{\alpha\in\Lambda}$. Assume each X_α is equipped with a length structure L_α and such that the following holds: if γ is a path that maps to the intersection of X_α and X_β then $L_\alpha(\gamma) = L_\beta(\gamma)$.*

Then there exists a unique length structure L on X whose restriction to every X_α is L_α. Moreover, if X is connected and all intrinsic metrics induced by L_α on X_α are finite, then so is L.

Proof. Consider a path $\gamma : [a, b] \to X$. The inverse images $\gamma^{-1}(X_\alpha)$ are an open covering $[a, b]$. The compactness of $[a, b]$ implies that there is a finite partition $a = t_0 \le t_1 \le \cdots \le t_n = b$ such that every segment $[t_i, t_{i+1}]$ is contained in one of the sets X_α and the length of $\gamma|_{[t_i,t_{i+1}]}$ is given by L_α. By additivity of length, the length of γ must be equal to the sum of lengths of its restricted intervals $[t_i, t_{i+1}]$. This proves the uniquness part of the lemma. Moreover this gives a way to define a length L on X. To complete the proof one has to check that L defined this way is independent of the choice of a partition and satisfies the properties of a length structure. (Exercise).

To prove the statement about finiteness we fix a point $x \in X$ and define Y as the set of all points such that the length distance between x and a point $y \in Y$ is finite. Every set X_α is either contained in Y or in $X\backslash Y$ by the triangle inequality. It follows that Y and $X\backslash Y$ are both open. But since X is connected we have $Y = X$. □

3.2 Corollary. *Consider two intrinsic metrics d_1 and d_2 defined on the same set X and inducing the same topology. Assume every point $x \in X$ has a neighborhood U_x such that $d_1(p, q) = d_2(p, q)$ $\forall p, q \in U_x$. Then $d_1 = d_2$.*

Proof. Exercise . □

Give an example that demonstrates that the corollary fails without the assumption that the metrics in quesiton are intrinsic.

3.3 Proposition. *If a complete metric d on a set X is not intrinsic, then there exists another metric d_1 on X such that $d \ne d_1$ but every point has a neighborhood where d and d_1 coincide.*

Proof. For every $\epsilon > 0$ define $d_\epsilon(x, y) = \inf \sum_{i=0}^{k} d(p_i, p_{i+1})$ where the infimum is taken over all finite sequences of points $p_0, p_1, \ldots, p_{k+1}$ such that $p_0 = x$, $p_{k+1} = y$ and $d(p_i, p_{i+1}) \le \epsilon$ for all $i = 0, 1, \ldots, k$. Clearly $d_\epsilon(x, y) = d(x, y)$ if $d(x, y) \le \epsilon$ and thus d_ϵ and d coincide on every ball of radius $\epsilon/2$. On the other, if $d_\epsilon = d$ for all $\epsilon > 0$, then d is intrinsic. □

3.2 Glued spaces

3.4 Example. 1. Consider the strip $\mathbb{R} \times [0,1]$ and for every $x \in \mathbb{R}$ identify $(x,1)$ with $(x+100,0)$. This is a topological cylinder. What is the distance between $(0,\frac{1}{2})$ and $(1000,\frac{1}{2})$ in this quotient? We can estimate the distance as follows. A sequence of sgements that connects the two points is

$$\begin{aligned}(0,\frac{1}{2}) \to (0,1) = \frac{1}{2}, \quad (0,1) \to (100,0) = 0,\\ (100,0) \to (100,1) = 1, \quad (100,1) \to (200,0) = 0,\\ \dots\\ (900,0) \to (900,1) = 1, \quad (900,1) \to (1000,0) = 0,\\ (1000,0) \to (1000,\frac{1}{2}) = 1.\end{aligned}$$

Hence the distance should be less than 11.

2. Consider $\mathbb{R}^2$ and identify (x,y) with the point given by $(-y,2x)$. Then distance between the origin and any other point is 0, since (x,y) is identified with $(\frac{1}{2}y,-x)$ that is identified with $(-\frac{1}{2}x,-\frac{1}{2}y)$ that is identified with $(-\frac{1}{4}y,\frac{1}{2}x)$ etc. The distance between these points is set to 0 and the sequence converges to $(0,0)$.

These examples suggest a general strategy for defining a metric on a space that results from identifying certain points

3.5 Definition. Let (X,d) be an ∞-metric space and let R be an equivalence relation on X. The *quotient semi-metric* d_R is defined as

$$d_R(x,y) := \inf\left\{\sum_{i=1}^{N} d(p_i,q_i) : p_1 = x, q_N = y, N \in \mathbb{N} \text{ and } p_{i+1} \sim_R q_i \forall i = 1,\dots,N-1\right\}.$$

We associate to the semi-metric (X,d_R) a metric space $(\hat{X},d_R)$ where $\hat{X} := X/d_R$ is the quotient space that arises from the equivalence relation $p \sim_{d_R} q \Leftrightarrow d_R(p,q)$. $(\hat{X},d_R)$ is the quotient metric space associated to $\sim_R$. One also says that $(\hat{X},d_R)$ results from gluing (X,d) along R.

Remark. It is possible that the relation $d_R \equiv 0$ is stronger than R, i.e. more points get identified in X/d_R than in X/R. As example consider $[0,1]$ and glue together all the rational points.

3.6 Remark. Gluing a length space yields a length space. To see this we first observe that $d_R \leq d$, i.e. $d_R([x],[y]) \leq d(x,y)\ \forall x,y \in X$. Hence every d-continuous curve is also d_R-continuous and $L_{d_R} \leq L_d$. If (p_i) and (q_i) are points as in the definition of d_R we can construct a curve between x and y in $(\hat{X},d_R)$ whose length is almost equal to $d_R(x,y)$. For this we concatenate almost shortest path between p_i and q_i for $i = 1,\dots,N$. Since q_i and p_{1+1} are identified in $\hat{X}$ this curve is continuous w.r.t. d_R. Hence $(\hat{X},d_R)$ is length space.

3.7 Definition (Gluing along subsets)**.** Let (X_α, d_α) be a collection of length spaces and consider the disjoint union $X := \dot{\bigcup}_\alpha X_\alpha$. We introduce a length ∞-metric on X by the following rule

$$d(x,y) := \begin{cases} d_\alpha(x,y) & \text{if } x, y \in X_\alpha, \\ \infty & \text{otherwise.} \end{cases}$$

Assume $\alpha = 0, 1$ and let $\mathcal{I} : Y_0 \subset X_0 \to Y_1 \subset X_1$ be a bijection. We introduce the equivalence relation R on $X = X_0 \dot{\cup} X_1$ generated by the relation $x \sim y \Leftrightarrow f(x) = y$. We denote the resulting glued space $X_0 \cup_{\mathcal{I}} X_1$.

3.8 Examples.

1. Consider a segement $[0,1]$ with the metric $|\cdot - \cdot|$. We introduce the equivalence relation R generated by $0 \sim 1$. The glued space $([0,1]/R, d_R)$ is a circle of length 1.

2. Begin with a square $[0,1] \times [0,1] = Q$ with the Euclidean metric. An equivalence relation R is induced from $(0,x) \sim (1,x)$ and $(x,0) \sim (x,1)$. The quotient space is a torus.

3.9 Lemma. *Let $b : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ be an arbitrary function, and consider the class D of all semi-metrics d on X such that $d \le b$. Then D contains a unique maximal semi-metric d_{max} such that $d_{max} \ge d$ $\forall d \in D$.*

Proof. For $x, y \in X$ we define

$$d_{max}(x,y) = \sup\{d(x,y) : d \in D\}.$$

The function d_{max} is non-negative, symmetric and satisfies $d_{max} \le b$. We only need to prove the triangle inequality for d_{max}. Let $x, y, z \in X$. Then

$$\begin{aligned} d_{max}(x,y) &= \sup_{d\in D} d(x,y) \le \sup_{d\in D}\{d(x,z) + d(z,y)\} \\ &\le \sup_{d\in D} d(x,z) + \sup_{d\in D} d(z,y) = d_{max}(x,z) + d_{max}(z,y). \end{aligned}$$

□

3.10 Corollary. *Let X be a set that is covered by a collection of subsets $\{X_\alpha\}_\alpha$ and each X_α carries a semi-metric d_α. Consider the class D of all semi-metric $d \le d_\alpha$ whenever $x, y \in X_\alpha$. Then D contains a unique maximal semi-metric d_{max} such that $d_{max}(x,y) \ge d(x,y)$ $\forall d \in D$ and $x, y \in X$. If all d_α are intrinsic, then so is d_{max}.*

Proof. We assume d_α is defined on X by setting $d(x,y) = \infty$ if $x \notin X_\alpha$ or $y \notin X_\alpha$. Then d_{max} is defined as in the previous Lemma where $b(x,y) = \inf_\alpha d_\alpha(x,y)$.

Let $\hat{d}_{max}$ be the intrinsic metric induced by d_{max}. If d_α is intrinsic, it follows $\hat{d}_{max} \le d_\alpha$ on X_α. So $\hat{d}_{max}$ belongs to D and hence $\hat{d}_{max} = d_{max}$. □

3.11 Theorem. *Let (X,d) be a metric space and R an equivalence relation on X. Consider*

$$b_R(x,y) = \begin{cases} 0 & \text{if } x \text{ is } R\text{-equivalent to } y, \\ d(x,y) & \text{otherwise.} \end{cases}$$

Then the maximal semi-metric among those not exceeding b_R coincides with the semi-metric d_R obtained by gluing (X,d) along R.

Proof. Let D denote the class of semi-metric not exceeding b_R. Clearly we have $d_R \in D$. We show $d_R \ge d'$ for any semi-metric $d' \in D$.

If $x, y \in X$ and $\{p_i\}_{i=1}^k, \{q_i\}_{i=1}^k$ as in the definition of the gluing metric, then by the triangle inequality we have

$$d'(x,y) \le \sum_{i=1}^k d'(p_i,q_i) + \sum_{i=1}^k d'(q_i,p_{i+1}) \le \sum_{i=1}^k \underbrace{b_R(p_i,q_i)}_{\le d(p_i,q_i)} + \sum_{i=1}^k \underbrace{b_R(q_i,p_{i+1})}_{\le b_R(q_i,p_{i+1})=0}.$$

Hence $d'(x,y) \le d_R(x,y)$. □

3.3 Products and Cones

3.12 Definition (Direct product)**.** Let X and Y be length spaces. The product $Z = X \times Y$ is equipped with the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)} \tag{2}$$

where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. It is easy to check that d is metric. d is called product metric, and the metric space (Z, d) is called direct product of X and Y.

3.13 Proposition. *The direct product of (strictly) intrinsic metric spaces is a (strictly) intrinsic metric space.*

Proof. We prove the statement regarding (Z, d) being instrinsic. We fix $\epsilon > 0$, let $z_i = (x_i, y_i) \in Z$. Since X and Y are intrinsic, we can find curves α and β such that $d_X(x_1, x_2) \geq L(\alpha) - \epsilon$ and $d_Y(y_1, y_2) \geq L(\beta) - \epsilon$. In particular, α and β are rectifiable. Then, there are reparametrizations $\bar{\alpha}$ and $\bar{\beta}$ proportional to arc length, defined on $[0, 1]$ and with constant speed $L(\alpha)$ and $L(\beta)$ respectively. Moreover, $\bar{\alpha}$ and $\bar{\beta}$ are Lipschitz and $v_{\bar{\alpha}} = L(\alpha)$ and $v_{\bar{\beta}} = L(\beta)$ a.e.

We define $\gamma = (\bar{\alpha}, \bar{\beta})$ that is also Lipschitz. Hence v_γ exists a.e. and from the definition it follows that $v_\gamma(t) = \sqrt{v_\alpha(t)^2 + v_\beta(t)^2} = \sqrt{L(\alpha)^2 + L(\beta)^2}$. Henc

$$L(\gamma) = \int_0^1 v_\gamma(t)dt = \sqrt{L(\alpha)^2 + L(\beta)^2} \leq \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} + C\epsilon$$

for a constant $C > 0$ independent of α and β but dependent on $L(\alpha)$ and $L(\beta)$. Hence d is intrinsic. □

3.14 Remark. There are other possible definitions for a product metric on Z. For instance, we can define $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. More general, we can consider any norm $\|\cdot\|$ on $\mathbb{R}^2$ such that the restrictions to the rays $\{x_0, y > 0\}$ and $\{x > 0, y_0\}$ are monotone. Then $d(z_1, z_2) = \|d_X(x_1, x_2), d_Y(y_1, y_2)\|$ a product metric.

The formula (2) is motivated by the Pythagorean theorem.

3.15 Fact. *A constant speed path in Z is a shortest path (a geodesic) if and only if it is the product of two shortest paths (geodesics) in X and Y with constant speed parametrizations.*

3.16 Definition (Topological cone)**.** A cone over a topological space X is the quotient of the product $X \times [0, \infty)$ w.r.t. the equivalence relation $\sim$ that is given by $(r, x) \sim (s, y) \Leftrightarrow r = s = 0\ \forall x, y \in X$. I.e. we identify all points in $\{0\} \times X$ as a single point that is called the origin (or apex, or tip) of the cone.

Question: How should we equip a cone with a metric?

Consider a subset X of $\mathbb{S}^2 = \{v \in \mathbb{R}^3 : \|v\|_{eucl} = 1\}$ equipped wiht the angular metric $d(x, y) := \angle(x, y)$. To built a cone over X we take all the rays from the origin $0 \in \mathbb{R}^3$ through a point $x \in X$. A point v in the union of all these rays can be described as a tuple (r, x) with $x \in X$ and $r = d_{eucl}(0, v)$. By the cosine formula we can write the distance between two vectors $v = (r, x)$ and $w = (s, y)$ as

$$\|v - w\|_{eucl} = d_{eucl}(v, w) = \sqrt{r^2 + s^2 - 2rs \cos \angle(x, y)}.$$

This motivates the following definition.

3.17 Definition. Let X be a metric space with $\mathrm{diam}(X) \leq \pi$. The cone metric d_C on $[0, \infty) \times X$ is given by the formula

$$d_C((r, x), (s, y)) = \sqrt{r^2 + s^2 - 2rs \cos d(x, y)}$$

$\forall (r, x), (s, y) \in [0, \infty) \times X$.

3.18 Definition. A subset A in a metric space (X, d) is said to be convex if the restriction of d to A is strictly intrinsic and finite.

A is said locally convex if every point $x \in A$ has a neighborhood U such that U is convex.

Remark. A submanifold N in a Riemannian manifold M is locally convex if and only if it is totally geodesic (all geodesics of N are geodesic of M).

3.19 Lemma. *If (X, d) is strictly intrinsic and finite and $F : X \to Y$ distance preserving, then $F(X)$ is convex in Y.*

3.20 Proposition. *Let X and Y be length spaces, and $\alpha : [a, b] \to X$, $\beta : [c, d] \to Y$ shortest paths. Then the product of their images $R = \mathrm{Im}\alpha \times \mathrm{Im}\beta$ is convex in $X \times Y$ and isometric to an Euclidean rectangle.*

Proof. The map $F : [a, b] \times [c, d] \to X \times Y$ given by $F(t, s) = (\alpha(t), \beta(s))$. This map is distance preserving. Then we apply the previous Lemma. □

3.21 Proposition. *If X is a metric space X with $\mathrm{diam}_X \leq \pi$, then d_C is a (semi)-metric.*

Proof. Positiveness and symmetry are clear. We prove the triangle inequality.

Consider $p_i = (r_i, x_i), i = 1, 2, 3$ in $C(X)$ and let $\alpha = d(x_1, x_2)$ and $\beta = d(x_2, x_3)$. Now construct 3 points $\bar{p}_i$ in $\mathbb{R}^2$ such that $|\bar{p}_i|_{eucl} = r_i$ and $\angle(\bar{p}_1\bar{p}_2) = \alpha$ and $\angle(\bar{p}_2, \bar{p}_3) = \beta$, and also the rays going through 0 and $\bar{p}_1$ and $\bar{p}_3$ are in different half-planes w.r.t. the ray through 0 and $\bar{p}_2$. By definition of the cone metric and our choice of $\bar{p}_i$, $i = 1, 2, 3$, we get $|\bar{p}_1 - \bar{p}_2|_{eucl} = d_C(p_1, p_2)$ and $|\bar{p}_2 - \bar{p}_3| = d_C(p_2, p_3)$.

We have two cases: $\alpha + \beta \leq \pi$ and $\alpha + \beta > \pi$. In the first case we have

$$\angle(\bar{p}_1, \bar{p}_3) = \alpha + \beta \geq d(x_1, x_3)$$

by the triangle inequality in X. Hence, by the properties of cos (and since $\alpha + \beta \leq \pi$), we have $|\bar{p}_1 - \bar{p}_3|_{eucl} \geq d_C(p_1, p_3)$. Then the triangle inequality for d_C follows from the triangle inequality in $\mathbb{R}^2$:

$$d_C(p_1, p_2) + d_C(p_2, p_3) = |\bar{p}_1 - \bar{p}_2|_{eucl} + |\bar{p}_2 - \bar{p}_3|_{eucl} \geq |\bar{p}_1 - \bar{p}_3|_{eucl} \geq d_C(p_1, p_3).$$

In the second case we argue as follows. Since $\alpha + \beta > \pi$, the broken line between $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_3$ lies outside the sector formed by the ray through 0 and $\bar{p}_1$ and 0 and $\bar{p}_3$. Hence, this broken path is longer than the path from $\bar{p}_1$ to $\bar{p}_3$ through 0. Hence

$$d_C(p_1, p_2) + d_C(p_2, p_3) \geq r_1 + r_3 \geq \sqrt{(r_1 + r_3)^2} = \sqrt{r_1^2 + r_3^2 - 2r_1 r_3 \cos d(x_1, x_3)}.$$

□

3.22 Lemma. *If X is a length space with $\mathrm{diam}_X \leq \pi$, and γ is a shortest segment in X, then the cone over the image of γ is a convex flat surface in the cone $C(X)$ over X.*

Proof. Let $\gamma : [0, L] \to X$ be a shortest path in X parametrized by arclength. We introduce polar coordinates (r, φ) on the Euclidean plane. More precisely, if (x, y) are the standard coordinates of $\mathbb{R}^2$, then $(x(r, \varphi), y(r, \varphi)) = (r \cos \varphi, r \sin \varphi)$. We denote by Q the set of points in the plane whose φ coordinate is between 0 and L. Recall the Euclidean distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is

$$d_{eucl}((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(|\varphi_2 - \varphi_1|)}.$$

The map $F : Q \to C(X)$ is given via $F(r, \varphi) = (r, \gamma(\varphi))$. The image of F is the cone over γ. The map F is also distance preserving. Indeed

$$\begin{aligned} d_C^2(F(r, \varphi), F(r', \varphi')) &= r^2 + (r')^2 - 2rr' \cos d_X(\gamma(\varphi), \gamma(\varphi')) \\ &= r^2 - (r')^2 - 2rr' \cos(\varphi - \varphi') = d_{eucl}^2((r, \varphi), (r', \varphi')). \end{aligned}$$

This implies $F(Q)$ is flat and convex. □

3.23 Example. Let $X = \mathbb{S}^2$ with the angular metric. Then $C(X) = \mathbb{R}^2$. A shortest path γ in X then is an arc contained in a great circle of X, and so the cone over γ is a planar sector. Any point in this sector has cone coordinates $(r, \gamma(t))$. If γ is parametrized by arc length, r and t are precisely the polar coordinates in this planar sector.

Remark. The cone metric on $[0, \infty) \times X$ is first a semi-metric. The equivalence relation $d_C((r, x), (s, y)) = 0$ coincides with the relation $\sim$ from before. $[0, \infty) \times X/d_C = [0, \infty) \times X/\sim$ and d_C is a metric on $[0, \infty) \times X/\sim$. We write $C(X)$ for the metric cone over (X, d).

3.24 Remark. Let $\bar{\gamma} : [a, b] \to C(X)$ be a shortest path not passing through 0. We can write $\bar{\gamma}(t) = (r(t), \gamma(t))$ where $r(t)$ is a curve in $(0, \infty)$ and $\gamma(t)$ is a curve in X. From the proof of the triangle inequality in $C(X)$ we see that the triangle inequality between any three points $\gamma(t_1), \gamma(t_2)$ and $\gamma(t_3)$ in X for $t_1 < t_2 < t_3$ turns into an equality. This implies $L(\gamma) = d(\gamma(a), \gamma(b)$, hence γ is a shortest path.

Hence there is an injective correspondence between shortest paths in X of lenght strictly less than π and shortest paths in $C(X)$ not passing through 0. As for shortest paths passing through the origin, it is easy to see the following. Every point $(x, r) \in C(X)$ is connected to the origin by a unique shortest path $(x, t)_{t \in [0,r]}$. The concatenation of two such segments with endpoints (x_1, r_1) and (x_2, r_2) is a shortest path if and only if $d(x_1, x_2) = \pi$.

How can we define the cone over larger spaces. The previous formula does not work since, for instance, the triangle inequality may fail. We have the following guidelines: the previous formula shall hold for small distances $\leq \pi$ and the resulting cone shall be a length space. Existence and uniqueness of such a metric is guaranteed by Lemma 3.1.

3.25 Definition (Cone over a larger space)**.** Let X be a metric space. The cone distance $d_C(a, b)$ between points $a = (t, x)$ and $b = (s, y)$ in $C(X)$ is defined as

$$d_C(a, b) = \begin{cases} \sqrt{t^2 + s^2 - 2ts \cos d_X(x, y)} & d_X(x, y) \leq \pi, \\ t + s & d_X(x, y) \geq \pi. \end{cases}$$

Remark. Alternatively, one can define $\bar{d}(x, y) = \min\{d(x, y), \pi\}$ that is a metric on X, and define $C(X) := C(X, \bar{d})$ where the metric of $C(X, \bar{d})$ is given by the previous formula.

3.26 Theorem. *The metric d_C on $C(X)$ is intrinsic (resp. striclty intrinsic) if and only if the metric d is intrinsic (resp. strictly intrinsic) at distances less than π. The latter means for any two points $x, y \in X$ such that $d(x, y) < \pi$ there is a curve in X connection x and y whose length is arbitrarily close (resp. equal) to $d(x, y)$.*

Proof. Assume first $d < \pi$. Let $x, y \in X$ and $a = (t, x), b = (s, y) \in C(X)$. If γ is a shortest path between x, y then we apply the previous Lemma, and we know that the cone over γ is a flat surface that embeds into $C(X)$. It follows there exists a curve of length $d_C(a, b)$ connecting a and b. If $d(x, y) \geq \pi$, then $d_C(a, b) = t + s$ and there are two segments connecting a and b with the origin and the union of this segments is shortest path between a and b. Thus d_C is strictly intrinsic.

Conversely, if d_C is striclty intrinsic, for any two points $x, y \in X$ with $d(x, y) < \pi$, we apply the result of Problem 3 on Problem Sheet 7 to a shortes path $\widetilde{\gamma}$ between $a = (1, x)$ and $b = (1, y)$. Since $L(\widetilde{\gamma}) = d_C(a, b) < 2$, $\widetilde{\gamma}$ does not pass through the origin and hence has a well defined projection γ in X that is a geodesic in X by the previous Remark. □

3.27 Remark (Warped products and spherical suspensions).

1. Let (X, d) be a metric space with $\mathrm{diam}_X \leq \pi$ and consider $[0, \pi] \times X$. We can identify the points $(0, x)$ $\forall x \in X$, and the points (π, y) $\forall y \in X$. On the corresponding quotient $X/\sim$ space we introduce a metric d_Σ as follows:

$$\cos d_\Sigma((s, x), (t, y)) = \cos s \cos t + \sin s \sin t \cos d(x, y).$$

   The space $(X/\sim, d_\Sigma) = \Sigma(X)$ is called spherical suspension of X.

2. Let X and Y be two complete length spaces and $f : X \to [0, \infty)$ continuous. For a Lipschitz curve $\gamma = (\alpha, \beta) : [a, b] \to X \times Y$ in $X \times Y$ we can define a length by the following formula

$$L(\gamma) = \int_a^b \sqrt{v_\alpha(t)^2 + (f \circ \alpha)^2(t) v_\beta(t)^2} dt.$$

   The intrinsic metric induced by this length structure on $X \times Y$ is called warped product metric. The warped product is denoted with $X \times_f Y$.

   If $X = [0, \infty)$ and $f(r) = r$, then $[0, \infty) \times_r Y = C(Y)$ (Exercise).

3.4 Angles

Question: How can we measure angles in a metric space?

For this first we consider two rays $\alpha, \beta : [0, \infty) \to \mathbb{R}^2$ emanating from the same point $a = \alpha(0) = \beta(0)$. We can pick $t, s > 0$ and apply to the cosine rule to the triangle $\Delta(a, \alpha(t), \beta(s))$. That is

$$\arccos \frac{|a - \alpha(t)|_2^2 + |a - \beta(s)|_2^2 - |\alpha(t) - \beta(s)|_2^2}{2|a - \alpha(t)||a - \beta(s)|} = \angle \alpha(t) b \beta(s).$$

This expression gives the angle between α and β at a.

We can now mimic this definition in a general length space. Let α, β be geodesic rays in a metric space (X, d) and replace $|\cdot - \cdot|_2$ with d. But in general the expression above will depend on t and s.

3.28 Definition. Let (X, d) be a metric space and let $x, y, z \in X$ be 3 distinct points. We define the comparison angle $\tilde{\angle} xyz$ of xyz at y by

$$\tilde{\angle} xyz = \arccos \frac{d(x,y)^2 + d(y,z)^2 - d(x,z)^2}{2d(x,y)d(y,z)}.$$

The geometric meaning of this definition is as follows. We pick a triangle $\Delta\bar{x}\bar{y}\bar{z}$ in $\mathbb{R}^2$ whose sides correspond to $d(x,y)$, $d(y,z)$ and $d(z,x)$. Then $\tilde{\angle} xyz = \angle\bar{x}\bar{y}\bar{z}$.

3.29 Definition. Let $\alpha, \beta : [0, \epsilon) \to X$ be two paths in a length space (X, d) with $\alpha(0) = p = \beta(0)$. We define the angle $\angle(\alpha, \beta)$ between α and β as

$$\angle(\alpha, \beta) = \lim_{s,t \to 0} \tilde{\angle}\alpha(s)p\beta(t)$$

if the limit exists.

If α and β are shortest paths parametrized by arc length, then $d(p, \alpha(s)) = s$ and $d(p, \beta(t)) = t$, and

$$\angle\alpha(s)p\beta(t) = \arccos \frac{s^2 + t^2 - d(\alpha(s), \beta(t))^2}{2st}.$$

3.30 Fact.

1. *Every shortest path forms zero angle with itself.*
2. *Let $\alpha : [a, b] \to X$ and $\beta : [b, c] \to X$ be two shortest segments with $\alpha(b) = p = \beta(b)$, such that their concatenation is also a shortest path, then the angle between α and β at p is π.*

Remark. We will mainly deal with length spaces that admit curvature bounds. In such spaces the angle between shortest paths is always defined. For more general metric spaces one may also consider so-called upper angles, defined as

$$\angle_U(\alpha, \beta) = \limsup_{s,t \to 0} \angle\alpha(s)p\beta(t).$$

3.31 Theorem. *Let (X, d) be a metric space. Consider 3 paths γ_1, γ_2 and γ_3 starting at the same point $p \in X$. Assume the angle $\angle(\gamma_1, \gamma_2) = \alpha_3, \angle(\gamma_2, \gamma_3) = \alpha_1$ and $\angle(\gamma_1, \gamma_3) = \alpha_2$ exist. Then*

$$\alpha_3 \le \alpha_1 + \alpha_2.$$

Proof. The statement is trivial if $\alpha_1 + \alpha_2 \ge \pi$. So suppose this is not the case.

Given $\epsilon > 0$ there exist $a = \gamma(s)$, $b = \gamma_2(t)$ and $c = \gamma_3(r)$ such that for each angle we have

$$|\alpha_1 - \theta(b, c)|, |\alpha_2 - \theta(a, c)|, |\alpha_3 - \theta(a, b)| \le \epsilon$$

where in this situation we define

$$\theta(a, b) = \frac{d(p,a)^2 + d(p,b)^2 - d(a,b)^2}{2d(p,a)d(p,b)}.$$

We pick 4 points $\bar{p}, \bar{a}, \bar{b}, \bar{c}$ in the Euclidean plan $\mathbb{R}^2$ such that $\Delta\bar{a}\bar{p}\bar{c}$ and $\Delta\bar{c}\bar{p}\bar{b}$ are comparison triangles for Δapc and Δcpb.

We fix a, b and move c towards p. Formally this means we fix s and t and decrease $r > 0$. For c close enough to p, we have that $\bar{p}$ and $\bar{c}$ are situated on one side of the line that goes through $\bar{a}$ and $\bar{b}$. On the other hand fixing c and moving a and b towards p we obtain a configuration with $\bar{p}$ and $\bar{c}$ on opposite sides of the line through $\bar{a}$ and $\bar{b}$.

By continuity we can finde $s, t, r > 0$ such that $\bar{c}$ belongs to the segment $[\bar{a}, \bar{b}]$. Here $[\bar{a}, \bar{b}]$ is a notation for the set $\{(1-t)\bar{a} + t\bar{b} \in \mathbb{R}^2 : t \in [0,1]\}$.

For this configuration ($\bar{c} \in [\bar{a}, \bar{b}]$) we have

$$|\bar{a} - \bar{b}| = |\bar{a} - \bar{c}| + |\bar{c} - \bar{b}| = d(a,c) + d(c,b) \geq d(a,b).$$

We now add a point $\widetilde{b}$ to this configuration in $\mathbb{R}^2$ such that

$$|\bar{p} - \bar{b}| = |\bar{p} - \widetilde{b}| = d(p,b), \quad |\bar{a} - \widetilde{b}| = d(a,b)$$

and such that $\widetilde{b}$ lies on the same side of the line through $\bar{p}$ and $\bar{a}$ as $\bar{b}$.

Recall $\theta(a,b)$ is equal to the angle of the triangle $\Delta\bar{a}\bar{p}\widetilde{b}$ in p, that is $\angle\bar{a}\bar{p}\widetilde{b}$. Similarly $\theta(a,c) = \angle\bar{a}\bar{b}\bar{c}$ and $\theta(b,c) = \angle\bar{b}\bar{p}\bar{c}$. Hence

$$\theta(a,c) + \theta(c,a) = \angle\bar{a}\bar{b}\bar{b}.$$

Comparing the triangles $\Delta(\bar{b}\bar{p}\bar{a})$ and $\Delta\widetilde{b}\bar{p}\bar{a}$, we see that they have two equal sides, and $|\bar{a} - \bar{b}| \geq |\bar{a} - \widetilde{b}|$. Thus $\angle\bar{a}\bar{p}\bar{b} \geq \angle\bar{a}\bar{p}\widetilde{b}$. It follows

$$\theta(a,c) + \theta(b,c) \geq \theta(a,b).$$

Combining this with the estimate at the beginning we get

$$\alpha_3 \leq \alpha_1 + \alpha_2 + 3\epsilon \quad \forall\epsilon > 0.$$

This finishes the proof. □

3.32 Definition. A curve γ (starting in p) has a direction a p if the angle $\angle(\gamma, \gamma)$ exists. We say two curves α and β starting in p have the same direction at p if $\angle(\alpha, \beta)$ exists and is 0.

The equivalence class of curves starting in p with the same direction is called a *direction in p*.

4 Metric spaces with curvature bounds

4.1 Definition. Let (X, d) be a length space and let $p \in X$. The distance to p is the real valued function d_p on X defined by

$$d_p(x) = d(p, x).$$

Let $\gamma : [0, L] \to X$ be a shortest path parametrized by arc lenght between points x, y. We also write $\mathrm{Im}\gamma =: [x, y]$. Then the 1-dimensional distance function is defined by

$$g(t) = d(p, \gamma(t)) = d_p(\gamma(t)).$$

4.2 Remark (Comparison configuration in the Euclidean plane). Let $x, y \in X$, γ and g be as in the previous definition.

Consider $\mathbb{R}^2$ with $d = d_{eucl} = |\cdot - \cdot|$. We choose $\bar{x}, \bar{y} \in \mathbb{R}^2$ such that $|\bar{x} - \bar{y}| = d(x, y) = L$ and let $\bar{\gamma} : [0, L] \to \mathbb{R}^2$ the segment that is the shortest path between $\bar{x}, \bar{y} \in \mathbb{R}^2$ parametrized by arclength. More precisely $\gamma(t) = \bar{x} + t\frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|}$.

Next we choose a reference point $\bar{p} \in \mathbb{R}$ such that $|\bar{p} - \bar{x}| = d_p(x) = d(p, x)$ and $|\bar{p} - \bar{y}| = d_p(y)the = d(b, y)$. (Why is this possible?)

This comparison configuration is unique up to rigid motions.

We call $\bar{g}(t) := |\bar{p} - \bar{\gamma}(t)|$ the comparison function for g.

We are going to define spaces with nonpositive (nonnegative) curvature by saying that distance functions g are more convex (concave) than the corresponding comparison function, i.e. $g_0(t) \geq g(t)$ $(g_0(t) \leq g(t))$. Since we also want our definition to be local we formulate it as follows.

4.3 Definition (Distance condition)**.** We say that a length space (X, d) is nonpositively curved (nonnegatively curved) if every point $x \in X$ has a neighborhood $U = U_x$ such that the following holds: $\forall p \in U$ and $\forall \gamma$ that is a shortest path in U the comparison function g_0 for the corresponding $g = d_p \circ \gamma$ satisfies

$$g_0(t) \geq g(t) \quad (g_0(t) \leq g(t)) \ \forall t \in [0, L].$$

We will use the name Alexandrov space for spaces with curvature bounded from above or below, and in particular for spaces with nonpositive or nonnegative curvature.

4.4 Example. The space $(\mathbb{R}, |\cdot|)$ has nonpositive and nonnegative curvature, because $\mathbb{R}$ embeds distance preserving into $\mathbb{R}^2$.

4.5 Example. We glue together 3 copies of $[0, \infty) \subset \mathbb{R}$ by gluing at the point 0. The resulting glued space R_3 has nonpositive curvature.

Indeed, we can argue as follows. Denote O the common point of the three rays. Every shortest path in R_3 is eigher a segment in one the 3 rays, or a concatenation of two segments in two different rays that meet at O.

Let $\gamma : [0, L] \to R_3$ be any shortest path and let $p \in R_3$. If two of the three points $\gamma(0), \gamma(L)$ and p belong to the sam ray, then the statement is trivial because γ and p are contained in union of two rays, and such subset of R_3 is isometric to $\mathbb{R}$.

So we consider the case when all 3 points $\gamma(0) = a, \gamma(L) = b$ and p belong two different rays. For every $x \in [O, a]$ one has $d(p, x) = d(p, a) - d(a, x)$. For the function g this means

$$g(t) = d_p(\gamma(t)) = d(p, \gamma(t)) = g(0) - t \quad \text{if } \gamma(t) \in [O, a].$$

On the other, for the function g_0 one has

$$g_0(t) = |p - \bar{\gamma}(t)| \geq |p - a| - |a - \bar{\gamma}(t)| = g_0(0) - t = g(0) - t = g(t).$$

The case $\gamma(t) \in [O, b]$ works similar.

Consider a metric space X from now on we often write $|xy|$ for the distance between two points in X and $[xy]$ for a shortes segment between x, y. Note that $[xy]$ may not be unique. A triangle in X is collection of 3 points $x, y, z \in X$ connected by shortest segments. $\angle abc$ denotes the angle between $[ba]$ and $[bc]$ (if the angle is welldefined).

4.6 Example. Let K be the cone over a circle of length $L > 0$. Then K is a space of nonnegative curvature if $L \leq 2\pi$ and K is a space of nonpositive curvature if $L \geq 2\pi$.

Indeed we can argue as follows. First note that the cone over a circle is (locally) flat outside of the vertex: Every subcone over segment with length $\alpha \leq \max\{L/2, \pi\}$ is convex and isometric to a planar sector with angle α.

Pick a shortest path $\gamma : [0, L] \to K$ and a point $p \in K$, and consider the triangle Δ composed of the three shortest paths between $p, \gamma(0)$ and $\gamma(L)$. There are two possibilities:

1. Δ bounds a region not containing O or one of the points a, b, p coincide with O.

2. Δ bounds a region containing O, or some of its sides pass through O.

More precisely, the first case means that one of the points a, b, p is contained the planar sector that is the cone over Imβ where $\gamma = (\alpha, \beta)$ is the shortest path that connects the other two points. In this case the triangle composed of the 3 points is isometric to a flat triangle in Euclidean space and therefore the distance function g coincides with g_0.

The second case we treat for $L < 2\pi$ and $L > 2\pi$ separately. The case $L = 2\pi$ is trivial, because then K is isometric to $\mathbb{R}^2$.

$L < 2\pi$. We cut $\mathbb{R}^2$ along the segments $[0, a]$ and $[0, b]$ and $[0, p]$. Each of the ensuing sectors is isometric to a planar sector since $L < 2\pi$. Since the sum of the angles is bounded from above by 2π we may put together these sectors in $\mathbb{R}^3$ to form a wedge with vertices $\bar{a}, \bar{b}, \bar{p}$ and 0. The surface of this wedge with the gluing metric is isometric to K.

The triangle $\Delta\bar{a}\bar{b}\bar{p}$ lies in the plane spanned by $\bar{a}, \bar{b}$ and $\bar{p}$ in $\mathbb{R}^3$, and is also a comparison configuration for a, b and p. Since the intrinsic distances in K are bigger than in the ambient space $\mathbb{R}^3$, we have $g_0(t) \leq g(t)$ where the later comes from the intrinsic distance in K. Hence K is nonnegatively curved.

Now suppose $L > 2\pi$. The triangles $\Delta abO, \Delta apO$ and ΔbpO are flat, i.e. isometric to Euclidean ones. Consider the first and last of these triangles and place isometric copies $\Delta\bar{a}\bar{b}\bar{O}$ and $\Delta\bar{b}\bar{p}\bar{O}$ in the plane at different sides of the common side $\bar{O}\bar{b}$ (if a shortest path $[ab]$ and $[bp]$ passes through O its isometric copy degenerates to a segment.) Observe $\angle\bar{b}\bar{O}\bar{a} + \angle\bar{b}\bar{O}\bar{p} \geq \pi$ and $\angle\bar{a}\bar{O}\bar{p} \leq \angle aOp$ (since $L > 2\pi$); hence $|\bar{a}\bar{p}| \leq |ap|$. Let us rotate the trianglel $\Delta\bar{b}\bar{p}\bar{O}$ around $\bar{b}$ until $|\bar{a}\bar{p}|$ is equal to $|ap|$. This shows that isometric copies of triangles ΔabO and ΔbpO lie without overlapping in a planar triangle $\Delta\bar{a}\bar{b}\bar{p}$ whose sides are equal to Δabp. This argument works for any pair of triangles. Hence isomtric copies of all triangles lie without overlapping in the planar triangle $\Delta\bar{a}\bar{b}\bar{p}$. Hence all distances between points in the sides of Δabp are less or equal to distances between corresponding points on the sides of the comparison triangle $\Delta\bar{a}\bar{b}\bar{p}$, i.e. $g \leq g_0$.

Remark. From the above proof one can see that the converse statement is also true, i.e. a cone over a circle is nonnegatively (nonpositively) curved only if the length of the circle is not greater (not smaller) than 2π.

Given a triangle Δabc in a metric space X a triangle $\Delta\bar{a}\bar{b}\bar{c}$ in $\mathbb{R}^2$ with

$$|ab| = |\bar{a}\bar{b}|, \ |bc| = |\bar{b}\bar{c}|, \ |ac| = |\bar{a}\bar{c}|$$

is called a comparison triangle for Δabc.

Remark. It is clear that a comparison triangle in $\mathbb{R}^2$ is unique up to rigid motions.

4.7 Definition (Triangle comparison condition)**.** A length space X is a space of nonpositive (nonnegative) curvature if in some neighborhood of each point the following holds: For every Δabc and every point $d \in [ac]$, on has $|db| \leq |\bar{d}\bar{b}|$ $(\geq)$ where $\bar{d}$ is the point on the side $[\bar{a}\bar{c}]$ of a comparison triangle $\Delta\bar{a}\bar{b}\bar{c}$ such that $|\bar{a}\bar{d}| = |ad|$.

Such a neighborhood is called a normal region.

One can always choose a normal region U_x so small such that all shortest path with endpoints in U_x are still contained in a possibly larger normal region V_x. For instance, one can pick a normal region V_x and then let U_x be a ball around x in V_x sufficiently small.

4.8 Remark. For nonpositively curved spaces, if one can choose the whole space X as a normal neighborhood, one calls X a $CAT(0)$ space. Here CAT stands for comparison of Cartan-Alexandrov-Toponogov and (0) indicates that we compare with flat space, i.e., that we impose a zero upper curvature bound.

Spaces with nonnegative or nonpositive curvature are also called Alexandrov spaces.

Roughly speaking, all sufficiently small triangles in a space of nonposi- tive (resp. nonnegative) curvature are not thicker (resp. not thinner) than corresponding Euclidean triangles.

11.12.2023

Let X be a length space. Recall that a triangle in a length space is a collection of 3 points and a choice of shortest paths between these points. Let a, b and c points and consider shortest paths α, β and γ between b and a, b and c, and c and a. We assume $\alpha(0) = \beta(0) = b$. The definition of angle between α and β at b is

$$\angle(\alpha\beta) = \lim_{s,t\to 0} \arccos\left(\frac{s^2 + t^2 - |\alpha(s)\beta(t)|}{2ts}\right)$$

if the limit exists. In this case we also write $\angle(\alpha\beta) = \angle abc$, the angle at b of the triangle formed by a, b and c and the shortest paths α, β and γ.

Recall that $\tilde{\angle} abc$ denotes the comparison angle, i.e. $\tilde{\angle} abc = \angle \bar{a}\bar{b}\bar{c}$ for a comparison triangle $\Delta\bar{a}\bar{b}\bar{c}$ in $\mathbb{R}^2$ associated to Δabc.

Also note that, given two points $a, b \in X$, after we picked a shortest path γ between a and b we write $[ab] = \mathrm{Im}\gamma$.

4.9 Definition (Angle comparison condition)**.** A length space X is a space of nonpositive curvature if every point of X has a neighborhood such that for every triangle Δabc contained in this neighborhood, the angles $\angle abc$, $\angle bac$ and $\angle cba$ are well defined and satisfy the inequalities

$$\angle abc \le \tilde{\angle} abc, \quad \angle bca \le \tilde{\angle} bca, \quad \angle cab \le \tilde{\angle} cab.$$

For nonnegative curvature the inequalities $\le$ are replaced with $\ge$, and we add the following condition: For any two shortest paths $[pq]$ and $[xy]$ where x is an inner point of $[qp]$, one has that $\angle pxy + \angle yxq = \pi$.

Let α, β be two shortest paths parametrized by arclength starting at the same point p. We refer to such a configuration as "a hinge". We introduce $\theta(s,t) = \tilde{\angle}\alpha(s)p\beta(t)$, i.e. $\theta(s,t)$ is the angle at $\bar{p}$ in a comparison triangle $\Delta\bar{\alpha(s)}\bar{p}\bar{\beta(t)}$.

4.10 Definition (Monotonicity condition)**.** A length space X is a space of nonpositive (nonnegative) curvature if it can be covered by neighborhoods such that for any two shortest segments α and β contained in such a neighborhood and starting from the same point p, the correspondingn function $\theta(s,t)$ is nondecreasing, i.e. "$\theta(s,t) \uparrow$ if $s,t \uparrow$" (nonincreasing) in each variable s and t when the other one is fixed.

From the definition directly follows that:

4.11 Proposition. *If X has nonpositive (nonnegative) curvature then the angle between any two shortest paths in X is well-defined.*

4.1 Equivalence of definitions

We first prove an elementary fact in Euclidean geometry. We consider $\mathbb{R}^2$ with the Euclidean distance $|xy| = |x-y|_{eucl}$, $x, y \in \mathbb{R}^2$.

4.12 Lemma (Alexandrov's Lemma)**.** *Let a, b, c and d be points in $\mathbb{R}^2$ such that a and c are in different halfplanes w.r.t. the line that goes through b and d. Consider a triangle $\Delta a'b'c'$ in $\mathbb{R}^2$ such that*

$$|ab| = |a'b'|, \quad |bc| = |b'c'|, \quad |ad| + |dc| = |a'c'|$$

and let d' be a point in the side $[a'c']$ such that $|ad| = |a'd'|$. Note that in particular $|ad| + |dc| = |a'c'| \leq |ab| + |bc|$ by the Δ-inequality.

Then $\angle adb + \angle bdc < \pi$ if and only if $|b'd'| < |bd|$. In this case, one also has $\angle b'a'd' < \angle bad$ and $\angle b'c'd' < \angle bcd$.

And $\angle adb + \angle bdc > \pi$ if and only if $|b'd'| > |bd|$. In this case, one also has $\angle b'a'd' > \angle bad$ and $\angle b'c'd' > \angle bcd$.

Proof. We use the following fact. If two sides of a planar triangle are fixed, then the angle between these two sides is a monotone increasing function of the third side. More precisely, if $\Delta xyz, \Delta x'y'z'$ are Euclidean triangles such that $|xy| = |x'y'|$ and $|yz| = |y'z'|$, then $\angle xyz > \angle x'y'z'$ if and only if $|xz| > |x'z'|$, and vice versa.

Take a point c_1 on the ray formed by a and d such that d is between a and c_1, and such that $|dc| = |dc_1|$. Suppose $\angle adb + \angle bdc > \pi$; then $\angle bdc_1 < \angle bdc$. Hence $|bc_1| < |bc| = |b'c'|$.

Now we apply the observation for the triangles Δabc_1 and $\Delta a'b'c'$ for which $|ab| = |a'b'|$ and $|ac_1| = |a'c'|$. Since $|bc_1| < |b'c'|$, it follows $\angle bac_1 < \angle b'a'c'$.

Hence, considering the triangle Δbad and $\Delta b'a'd'$, we get $|bd| < |b'd'$.

The case $\angle adb + \angle bdc < \pi$ works the same way, up to reversing the inequalities. □

4.13 Theorem. *All the definitions for nonpositive (nonnegative) curvature are equivalent.*

Proof. The proofs for spaces of nonpositive and nonnegative curvature are similar up to reversing the inequalities. We prove the case of nonpositively curved spaces and indicate necessary modifications for the case of nonnegative curvature.

1. The distance condition and the triangle condition are equivalent.

It is clear that the distance condition implies the triangle condition. Assume the triangle condition. Given $p \in X$ let U_p be the corresponding normal region. We can choose $\epsilon > 0$ small enough such that $B_\epsilon(p) \subset U_p$ and $\forall x, y, z$ we have that every $\Delta xyz \subset U_p$. Then it follows that $B_\epsilon(p)$ satisfies the required properties.

2. Assume the triangle condition. Let us show the monotonicity condition.

Consider a hinge of two shortes paths α, β with $\mathrm{Im}\alpha = [pa]$ and $\mathrm{Im}\beta = [pb]$, and a point a_1 on $[p, a]$. Let $\Delta\bar{p}\bar{a}\bar{b}$ and $\Delta\bar{p}\bar{a}_1\bar{b}$ be comparison triangles for Δpab and Δpa_1b. Let $\widetilde{a}$ be a point on $[\bar{p}\bar{a}]$ such that $|\bar{p}\widetilde{a}| = [pa_1|$. Then the triangle condition implies that $|\bar{b}\widetilde{a}| \geq |ba_1| = |\bar{b}\bar{a}_1|$. This means that $\angle\bar{a}\bar{p}\bar{b} \geq \angle\bar{a}_1\bar{p}\bar{b}$. This is the angle monotonicity.

3. The monotonicity condition implies the angle condition.

Let δabc be a triangle. The side $[ba]$ and $[bc]$ are given by shortest paths α and β with $\alpha(0) = \beta(0) = b$. By the monotonicity of angles we have

$$\angle abc \equiv \angle(\alpha\beta) = \lim_{t\to 0} \theta(t, t) \leq \theta(|ab|, |bc|)$$

where θ as before. Since $\theta(|ab|, |bc|) = \angle\bar{a}\bar{b}\bar{c}$, this is the angle condition.

4. The Angle condition implies the triangle condition.

Consider a triangle Δabc and a point d in the side $[ac]$. Note that

$$\angle bda + \angle bdc \geq \angle adc = \pi \tag{3}$$

by the angle triangle inequality. We place comparison triangle $\Delta\bar{a}\bar{b}\bar{d}$ and $\Delta\bar{c}\bar{b}\bar{d}$ in different half planes w.r.t. the line $\bar{b}\bar{d}$ in $\mathbb{R}^2$. By the angle condition it follows for the comparison angles

$$\angle\bar{a}\bar{d}\bar{b} + \angle\bar{c}\bar{d}\bar{b} \geq \pi.$$

Now we apply Alexandrov's Lemma. Let $\Delta\bar{a}_1\bar{b}_1\bar{c}_1$ be a comparison triangle for Δabc and let $\bar{d}_1$ be the point on $[\bar{a}_1\bar{c}_1]$ such that $|\bar{a}_1\bar{d}_1| = |ad|$. Alexandrov's Lemma yields $|bd| = |\bar{b}\bar{d}| \leq |\bar{b}_1\bar{d}_1|$. This is the triangle condition for δabc and $d \in [ac]$.

This finishes the proof of the equivalences. □

Remark. For the proof in the case of nonnegative curvature we can just reverse all the inequalities.

However we use the triangle inequality for angle for the inequality (3). This inequality cannot be reversed.

However this property was included in the definition of the angle condition for nonnegative curvature.

Hence, to finish the proof on the equivalence of the definitions we only need to show the following lemma.

4.14 Lemma. *If a space X has nonnegative curvature in the sense of the monotonicity condition, then for any shortest path, the sum of adjacent angles is equal to π. In other words, if d_0 is an inner point of a shortest path $[a_0b_0]$ and $[d_0c_0]$ is a shortes path, then $\angle a_0d_0c_0 + \angle c_0d_0b_0 = \pi$.*

13.12.2023

Proof. By the triangle inequality for angles we always have $\angle a_0 d_0 c_0 + \angle c_0 d_0 b_0 \geq \pi$.

To prove the opposite inequality let a, b and c be arbitrary points in the shortest paths $[a_0 d_0]$, $[d_0 b_0]$ and $[d_0 c_0]$ respectively. We place comparison triangles $\Delta \bar{a}\bar{d}_0\bar{c}$ and $\Delta \bar{c}\bar{d}_0\bar{b}$ on different sides of the line $[\bar{c}\bar{d}_0]$ in $\mathbb{R}^2$. Let $\Delta \bar{a}_1\bar{c}_1\bar{b}_1$ be a comparison triangle for the triangle Δabc and let $\bar{d}_1$ be on $[\bar{a}_1\bar{b}_1]$ such that $|ad_0| = |\bar{a}_1\bar{d}_1|$. The monotonicity condition implies that $\angle \bar{c}\bar{a}\bar{d}_0 \geq \angle \bar{c}_1\bar{a}_1\bar{b}_1$. Hence $|\bar{d}_0\bar{c}| \geq |\bar{d}_1\bar{c}_1|$. By Alexandrov's Lemma it follows that $\angle \bar{a}\bar{d}_0\bar{c} + \angle \bar{c}\bar{d}_0\bar{b} \leq \pi$. Passing to the limit in this inequality as a, b and c approach d_0, we get $\angle a_0 d_0 c_0 + \angle c_0 d_0 b_0 \leq \pi$. ☐

4.2 Analysis of the distance function

Recall the distance condition. A length space (X, d) is nonpositively curved (nonnegatively curved) if every point $x \in X$ has a neighborhood $U = U_x$ such that the following holds: $\forall p \in U$ and $\forall \gamma$ that is a shortest path in U the comparison function g_0 for the corresponding $g = d_p \circ \gamma$ satisfies

$$g_0(t) \geq g(t) \quad (g_0(t) \leq g(t)) \ \forall t \in [0, L].$$

The comparison function is given by $g_0(t) = |\bar{p} - \bar{\gamma}(t)|$ where $\bar{\gamma}$ is a shortest segment in $\mathbb{R}^2$ and $\Delta \bar{p}\bar{\gamma}(0)\bar{\gamma}(1)$ is a comparison triangle for $\Delta p\gamma(0)\gamma(1)$. Notice that even without curvature restrictions not every continuous functions can arise as a 1-dimensional distance function. g must be nonnegative and nonexpanding, since d_p is 1-Lipschitz because of the Δ-inequality.

We want a complete list of all possible functions g_0 that can arise as comparsion function. If $\bar{\gamma}$ is a straight line in $\mathbb{R}^2$ and $\bar{p} \in \mathbb{R}^2$ is a point, then $g_0(t) = |\bar{p} - \bar{\gamma}(t)| = \sqrt{(t+c)^2 + h^2}$ where c is the parameter such that $\bar{\gamma}(-c)$ is the orthogonal projection of $\bar{p}$ to $\bar{\gamma}$ and $h = |\bar{p} - \bar{\gamma}(-c)|$.

Observe now that $\frac{d^2}{dt^2}(g_0(t))^2 = 2$. If $\sigma(t) = tL$, then $f(t) = (g_0(\sigma(t)))^2$ satisfies

$$f'' = 2L^2(, \quad f(0) = |\bar{p} - \bar{\gamma}(0)|^2 = a^2, \ f(1) = |\bar{p} - \bar{\gamma}(1)|^2 = b^2. \tag{4}$$

Hence

$$f(t) = (1-t)a^2 + tb^2 + 2(1-t)tL^2.$$

Note that $a = |p\gamma(0)|$, $b = |p\gamma(L)|$ and $L = |\gamma(0)\gamma(L)|$.

The function f indeed has this form, since $(1-t)a^2 + tb^2 + 2(1-t)tL^2$ is the unique solution of (4).

It follows

$$\begin{aligned} & g(t) \geq g_0(t) \ (\leq) \ \forall t \in [0, L] \\ \Leftrightarrow \ & g(sL)^2 \geq f(s) \ (\leq) \ \forall t \in [0, 1] \\ \Leftrightarrow \ & d(p, \hat{\gamma}(s))^2 \geq (1-s)d(p, \hat{\gamma}(0))^2 + sd(p, \hat{\gamma}(1))^2 - (1-s)sL(\hat{\gamma}) \ (\leq) \ \forall s \in [0, 1] \end{aligned}$$

where $\hat{\gamma} : [0, 1] \to X$ is the constant speed reparameterization of γ, i.e. $\hat{\gamma}(s) = \gamma \circ \sigma(s)$ with $\sigma(s) = sL$.

Similar, we can choose $\sigma(s) = (1-s)t_0 + st_1$ and consider $\hat{\gamma} = \gamma \circ \sigma$ and $(g_0 \circ \sigma)^2$. $\hat{\gamma}$ is then the constant speed reparameterization of $\gamma|_{[t_0,t_1]}$. Then we also get

$$d_p(\hat{\gamma}(s))^2 \geq (\leq)(1-s)d_p(\hat{\gamma}(0))^2 + sd_p(\hat{\gamma}(1))^2 - (1-s)s(t_1-t_0)^2L^2.$$

It follows that $u(t) = g(t)^2$ satisfies

$$\begin{aligned} & u'' \geq 2 \ (\leq) \ \text{ in the distributional sense on } (0,L) \\ \Leftrightarrow \ & \int_0^L \left(u\varphi'' + 2\varphi\right) dt \geq 0(\leq)\forall \varphi \in C^2((0,L)) \\ \Leftrightarrow \ & u(t) - t^2 \text{ is convex (concave)} \\ & \text{if } X \text{ is a Riemannian manifold} \underset{\Leftrightarrow}{} \ \nabla^2 d_p^2 \leq 2g_X \text{ where } g_X \text{ is the Riemannian metric.} \end{aligned}$$

4.15 Corollary. *A length space X has nonpositive (nonnegative) curvature iff $\forall x \in X$ $\exists U$ neighborhood of x s.t. $\forall p \in U$ and $\forall \gamma : [0,L] \to U$ shortest path $u \circ d_p(t) - t^2$ is convex (concave).*

4.16 Example. Consider $X = (\mathbb{R}^2, \|\cdot\|_1)$ where $\|x\|_1 = |x_1| + |x_2|$ (1-Norm).

X is a complete, locally compact length space. Straight lines are shortest paths because $L^{\|\cdot\|}((1-t)x+ty) = \|x-y\|_1$. But there are more shortest paths other than straight lines.

Claim: X is not a space of nonpositive, or nonnegative curvature.

Consider $\gamma(t) = (1-t, t)$, $t \in [0,1]$. Then $g(t) = 1 \ \forall t \in [0,1]$. We have $L^{\|\cdot\|_1}(\gamma) = 2$.

We have

$$f(t) = (1-t)1^2 + t1^2 - (1-t)t2^2 = 1 - 4(1-t)t = 1 - 4t + 4t^2.$$

(What is the comparison distance function g_0?)

Henc $g(t2)^2 = 1 \geq f(t) = g_0(t2)^2$. Hence X is not of nonpositive curvature.

On the other hand, we consider $\gamma(t) = ((1-t)\frac{1}{2} + t\frac{1}{2}, (1-t)\frac{1}{2} - t\frac{1}{2}) = (\frac{1}{2}, \frac{1}{2} - t)$ for $t \in [0,1]$.

It follows that $L^{\|\cdot\|_1}(\gamma) = 1$ and $g(t)^2 = d_0(\gamma(t)) = \|\gamma(t)\|_1^2 = \left(\frac{1}{2} + |\frac{1}{2} - t|\right)^2$. We have $g(0)^2 = 1 = g(1)^2$ and $g(\frac{1}{2})^2 = \frac{1}{4}$.

The distance comparison function is

$$g_0(t)^2 = (1-t)1^2 + t1^2 - (1-t)t1^2 \ \Rightarrow \ g_0(\frac{1}{2})^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

Hence $g_0(\frac{1}{2}) = \frac{\sqrt{3}}{2} \geq \frac{1}{2} = g(\frac{1}{2})$.

Therefore X is not nonnegatively curved.

4.17 Theorem. *Let X be a length space of nonpositive (nonnegative) curvature. Suppose sequences of shortest paths $[a_ib_i]_{i\in\mathbb{N}}$ and $[a_ic_i]_{i\in\mathbb{N}}$ converge uniformly to shortest paths $[ab]$ and $[ac]$ respectively. Then*

1. *$\angle bac \geq \limsup_{i\to\infty} \angle b_ia_ic_i$ for nonpositive curvature,*
2. *$\angle bac \leq \liminf_{i\to\infty} \angle b_ia_ic_i$ for nonnegative curvature.*

Proof. For $r > 0$ let $b' \in [ab]$, $c' \in [ac]$ and $b'_i \in [a_i b_i], c'_i \in [a_i c_i]$ points at distance $r > 0$ from a and a_i respectively.

We denote with $\theta(r)$ and $\theta_i(r)$ the comparison angles $\tilde{\angle} b' a c'$ and $\tilde{\angle} b'_i a_i c_i]$. We have $|a_i b'_i| = |ab'| = |a_i c'_i| = |ac'| = r$ and $|b'_i c'_i| \to |bc|$ (where r is fixed). Hence

$$\lim_{i\to\infty} \theta_i(r) = \theta(r) \ \forall r > 0.$$

If the curvature is nonpositive, then θ, θ_i are nondecreasing functions. Hence $\theta_i(r) \geq \alpha_i$ and consequently

$$\theta(r) \geq \limsup_{i\to\infty} \alpha_i \ \forall r > 0.$$

Thus $\alpha \geq \limsup_{i\to\infty} \alpha_i$.

If the curvature is nonnegative, then θ, θ_i is nonincreasing. Hence $\theta_i(r) \leq \alpha_i$. So

$$\theta(r) \leq \liminf_{i\to\infty} \alpha_i \ \forall r > 0.$$

Thus $\alpha \leq \liminf_{i\to\infty} \alpha_i$. □

4.3 First variation formula

Consider $\bar{\gamma} : [0, L] \to \mathbb{R}^2$ and $p \in \mathbb{R}^2$, and let $l(t) = |p - \bar{\gamma}(t)|$. Then the following first variation formula holds

$$\frac{dl}{dt} = -\cos \angle(p - \bar{\gamma}(t), \gamma'(t)) = -\langle p - \bar{\gamma}(t), \gamma'(t) \rangle.$$

We show that a similar formula holds in spaces of nonpositive or nonnegative curvature.

Let X be a length space, $\gamma : [0, T] \to X$ a unit speed shortest path, $a = \gamma(0)$ and $\gamma(T) = d$, and $p \in X \backslash \{a\}$. For every $t \in [0, T]$ we set $l(t) = |p\gamma(t)|$ and we fix a shortest path σ_t between $\gamma(t)$ and p.

4.18 Proposition. *If the angle $\alpha = \angle pad$ between γ and $[ap] = \sigma_0$ exists, then*

$$\limsup_{t \downarrow 0} \frac{l(t) - l(0)}{t} \leq -\cos \alpha.$$

4.19 Remark. The left hand side of the previous inequality does not depend on σ_0. Hence we get

$$\limsup_{t \downarrow 0} \frac{l(t) - l(0)}{t} \leq -\cos \alpha_{min}$$

where α_{min} is the infimum of angles between γ and all possible shortest paths from $a = \gamma(0)$ to p.

4.20 Lemma. *Let Δabc be a triangle in $\mathbb{R}^2$, $\alpha = \angle bac$, $t = |ac|$. Then*

$$\left| \cos \alpha - \frac{|ab| - |bc|}{t} \right| \leq \frac{t}{|ab|}.$$

Proof. Denote $|ab| = y$ and $|bc| = z$. The cosine rule gives

$$\cos \alpha = \frac{t^2 + y^2 - z^2}{2ty} = \frac{y^2 - z^2}{2ty} + \frac{t}{2y} = \frac{y - z}{t} \frac{y + z}{2y} + \frac{t}{2y}.$$

Then

$$\begin{aligned} \left| \cos \alpha - \frac{y - z}{t} \right| &= \left| \frac{y - z}{t} \frac{y + z}{2y} + \frac{t}{2y} - \frac{y - z}{t} \right| \\ &\leq \left| \frac{y - z}{t} \right| \left| \frac{y + z}{2y} - 1 \right| + \frac{t}{2y} \leq 1 \cdot \frac{t}{2y} + \frac{t}{2y} \leq \frac{t}{y}. \end{aligned}$$

The last inequality follows by the triangle inequality. Indeed $\left| \frac{y-z}{t} \right| \leq 1$ and $\left| \frac{y+z}{2y} - 1 \right| = \frac{|z-y|}{2y} \leq \frac{t}{2y}$. □

Proof of the proposition. We consider two variable points b on $[ap] = \sigma_0$ and c on $\mathrm{Im}\gamma$, i.e. $c = \gamma(t)$. The triangle inequality implies

$$|ab| - |bc| = |ap| - (|bp| + |bc|) \leq l(0) - l(t).$$

Then we apply the previous lemma to the comparison triangle Δabc. This yields

$$\cos \tilde{\angle} bac \leq \frac{|ab| - |bc|}{t} + \frac{t}{|ab|} \leq -\frac{l(t) - l(0)}{t} + \frac{t}{|ab|}.$$

We let the points b and c converge to a so fast such that $\frac{t}{|ab|} \to 0$. The the statement follows by passing to the limit in the last inequality. □

4.21 Theorem. *Let X be a space of nonpositive (nonnegative) curvature, let γ, σ_t and $l(t)$ be as above ,and assume that a sequence σ_t converges to σ_0 for some sequence $\{t_i\}_{i\in\mathbb{N}} \to 0$ as $i \to \infty$. Then there exists a limit*

$$\lim_{t_i \to 0} \frac{l(t_i) - l(0)}{t_i} = -\cos\alpha$$

where α is the angle at a between σ_0 and γ.

Proof. We only need to show that

$$\liminf_{i\to\infty} \frac{l(t_i) - l(0)}{t_i} \geq -\cos\alpha.$$

We fix $r > 0$ such that $|ap| > 5r$ and $B_{5r}(a)$ is a normal region for the triangle condition for nonpositive or nonnegative curvature. We may also assume that $\gamma(t_i) \in B_r(a)$ for all $i \in \mathbb{N}$. We set $c_i = \gamma(t_i)$ $\forall i$, and let b_i be the point on the shortest path $[c_i p] = \sigma_{t_i}$ such that $|b_i c_i| = r$. We will prove that

$$\limsup_{i\to\infty} \tilde{\angle} a c_i b_i \leq \pi - \alpha.$$

This implies the theorem. Indeed, applying the previous lemma it holds

$$l(0) = |pa| \leq |pb_i| + |b_i a| \leq |pb_i| + |b_i c_i| - t_i \cos\tilde{\angle} a c_i b_i + \frac{t_i^2}{|b_i c_i|}.$$

Since $|pb_i| + |b_i c_i| = l(t_i)$, it follows that

$$\frac{l(t_i) - l(0)}{t_i} \geq \cos\tilde{\angle} a c_i b_i - \frac{t_i}{|b_i c_i|} = \cos\tilde{\angle} a c_i b_i - \frac{t_i}{r}.$$

It follows that

$$\liminf_{i\to\infty} \frac{l(t_i) - l(0)}{t_i} \geq \liminf_{i\to\infty} \cos\tilde{\angle} a c_i b_i \geq \cos(\pi - \alpha) = -\cos\alpha.$$

Hence the theorem follows.

The proof of the missing inequality is different for nonpositively and nonnegatively curved spaces.

1. Let X be a space of nonnegative curvature. Then

$$\tilde{\angle} a c_i b_i \leq \angle a c_i b_i = \pi - \angle b_i c_i d$$

by the angle condition. Then, by semi-continuity of angles we have

$$\liminf_{i\to\infty} \angle b_i c_i d \geq \alpha.$$

2. Let X be of nonpositive curvature. Denote b the point in $[ap] = \sigma_0$ such that $|ab| = r$. Then $\angle bab_i \leq \tilde{\angle} bab_i$ and $\tilde{\angle} bab_i \to 0$ as $i \to \infty$ because $|b_ib| \to 0$ while $|ab|$ and $|ab_i|$ stay bounded away from zero. Hence, we also have $\angle bab_i \to 0$ for $i \to \infty$.

   Byt the triangle inequality ofr angles it follows

$$|\angle c_iab_i - \angle c_iab| \leq \angle bab_i \to 0 \text{ for } i \to \infty.$$

   Hence $\angle c_iab_i \to \alpha$ as $i \to \infty$. Then

$$\liminf_{i\to\infty} \tilde{\angle} c_iab_i \geq \liminf_{i\to\infty} \angle c_iab_i = \alpha$$

   by the angle condition.

   On the other hand $\tilde{\angle} c_iab_i + \tilde{\angle} ac_ib_i \to \pi$ as $i \to \infty$ because $\tilde{\angle} c_iab_i + \tilde{\angle} ac_ib_i + \tilde{\angle} ab_ic_i = \pi$ and $\tilde{\angle} ab_ic_i \to 0$. Thus

$$\limsup_{i\to\infty} \tilde{\angle} ac_ib_i = \pi - \liminf_{i\to\infty} \tilde{\angle} c_iab_i \leq \pi - \alpha.$$

   This finishes the proof.

□

4.22 Corollary. *Let X be nonpositively or nonnegatively curved complete locally compact space, $\gamma : [0, T] \to X$ a geodesic p.b.a.l. $p \in X$ with $p \neq \gamma(0)$. Then the function $t \mapsto l(t)$ has the right derivative and*

$$\lim_{t\downarrow 0} \frac{l(t) - l(0)}{t} = -\cos \alpha_{min}$$

where α_{min} is the minimum of angles between γ and shortest paths connecting $\gamma(0)$ and p.

Proof. Choose a sequence $\{t_i\}$ such that

$$\frac{l(t_i) - l(0)}{t_i} \to \liminf_{t\downarrow 0} \frac{l(t) - l(0)}{t}.$$

Fix shortest paths σ_{t_i} between p and $\gamma(t_i)$. By the Arzela-Ascoli Theorem σ_{t_i} subconverges to a shortest path σ_0. Then by the previous Theorem we have

$$\lim_{i\to\infty} \frac{l(t_i) - l(0)}{t_i} = -\cos \alpha$$

where α is the angle between γ and σ_0. Thus

$$\liminf_{t\downarrow 0} \frac{l(t) - l(0)}{t} = -\cos \alpha \geq -\cos \alpha_{min}.$$

Note that in this last inequality we actually have equality, so $\alpha = \alpha_{min}$ and this minimal angle is indeed attained (by σ_0). □

4.4 Nonzero curvature bounds

4.23 Definition. Let $k \in \mathbb{R}$. The k-plane $\mathbb{M}_k$ is one of the following spaces.

1. $\mathbb{R}^2$, if $k = 0$;
2. $\mathbb{S}^2_{\frac{1}{\sqrt{k}}}$, the Euclidean sphere of radius $\frac{1}{\sqrt{k}}$, if $k > 0$;
3. $\mathbb{H}^2_{\frac{1}{\sqrt{-k}}}$, the hyperbolic plane of curvature $k < 0$.

Remark. Recall that $\mathbb{S}^2_{\frac{1}{\sqrt{k}}} = \{v \in \mathbb{R}^3 : |v|^2_{eucl} = (v^1)^2 + (v^2)^2 + (v^3)2 = \frac{1}{\kappa}\}$, the sphere of radius $\frac{1}{\sqrt{k}}$ in $\mathbb{R}^3$. The shortest paths are segment of great circles

$$\cos(t)v + \sin(t)w = \gamma(t), v \perp w \in \mathbb{S}^2_{\frac{1}{\sqrt{k}}}.$$

The -1-plane can be defined as the -1-ball in the 3-dimensional Minkowski space $(\mathbb{R}^3, \langle\cdot,\cdot\rangle_1)$ where $\langle v, w\rangle_1 = -(v^1)^2 + (v^2)^2 + (v^3)^2 = -1$. Other models of the hyperbolic plane are the half-space and the Poincaré model. The k-plane for $k < 0$ is then obtained by rescaling $\mathbb{M}_{-1}$ with $-k$.

Remark. The k-plane is bounded if and only if $k > 0$. We denote the diameter of the k-plane by π_k, i.e.

$$\pi_k = \begin{cases} \pi/\sqrt{k} & \text{if } k > 0; \\ \infty & \text{if } k \leq 0; \end{cases}$$

We need the following elementary property of the k-plane. For $a, b, c > 0$ such that $a+b+c < 2\pi_k$ there exists a unique triangle in the k-plane with the sides a, b and c, up to rigid motions, i.e. an isometry of the k-plane to itself. Hence for every sufficiently small triangle in a length space, there is a unique (up to rigid motions) comparison triangle in the k-plane. For $k \leq 0$ we can drop the word "sufficiently small".

4.24 Remark (Comparison configuration in the k-plane)*.* Let X be a length space and let $p, x, y \in X$, $\gamma : [0, L] \to X$ a shortest path between x, y and $g(t) = d_p \circ \gamma(t)$. We assume that $|px| + |py| + |xy| < 2\pi_k$.

Consider the k-plane $\mathbb{M}_k$ with the induced Riemannian distance $|\cdot,\cdot|$. We choose $\bar{x}, \bar{y} \in \mathbb{M}_k$ such that $|\bar{x}\bar{y}| = L$ and let $\bar{\gamma} : [0, L] \to \mathbb{M}_k$ be a shortest path between $\bar{x}, \bar{y} \in \mathbb{M}_k$ with constant speed.

We choose a reference point $\bar{p} \in \mathbb{M}_k$ such that $|\bar{p}\bar{x}| = |px|$ and $|\bar{p}\bar{y}| = |py|$. This comparison configuration is unique up to isometries of $\mathbb{M}_k$.

We call $g_k(t) := |\bar{p}\bar{\gamma}(t)|$ the comparison function for g in the k-plane.

Similarly, for a triangle $\Delta xyz \subset X$ with $|xy| + |yz| + |zx| < 2\pi_k$ we can choose points $\bar{x}, \bar{y}$ and $\bar{z}$ in $\mathbb{M}_k$ such that $|\bar{x}\bar{y}| = |xy|$, etc. and shortest paths $[\bar{x}\bar{y}]$, etc. The collection of these shortest paths in $\mathbb{M}_k$ is called a comparison triangle $\Delta\bar{x}\bar{y}\bar{z}$ in k-plane for Δxyz.

4.25 Definition. Let X be a length space and $k \in \mathbb{R}$. The following statements are equivalent.

(i) Triangle condition.
For every point $x \in X$ $\exists$ a neighborhood $U_x \subset X$ such that for any triangle Δabc with $|ab|+|bc|+|ca| \leq 2\pi_k$ contained in U_x and for any point $d \in [ac]$ the inequality $|bd| \geq |\bar{b}\bar{d}|$ ($|bd| \leq |\bar{b}\bar{d}|$) holds where $\Delta\bar{a}\bar{b}\bar{c}$ is a comparison triangle in the k-plane and $\bar{d} \in [\bar{a}\bar{c}]$ is the point such that $|\bar{a}\bar{d}| = |ad|$.

(ii) Distance condition.
Every point $q \in X$ has a neighborhood U such that the following holds: $\forall p \in U$ and $\forall \gamma : [0, L] \to X$ that is a shortest path in U such that $|p\gamma(0)|+|p\gamma(L)|+|\gamma(0)\gamma(L)| < 2\pi_k$ the comparison function g_k in k-plane for the corresponding $g = d_p \circ \gamma$ satisfies

$$g_k(t) \geq g(t) \quad (g_k(t) \leq g(t)) \ \forall t \in [0, L].$$

4.26 Definition (Generalized trigonometric functions)**.** We define $\cos_k, \sin_k : [0, \infty) \to \mathbb{R}$ for $k \in \mathbb{R}$ as the solutions of

$$\begin{cases} u'' + ku = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases} \qquad \begin{cases} u'' + ku = 0 \\ u(0) = 0 \\ u'(0) = 1 \end{cases}$$

More precisely

$$\cos_k(t) = \begin{cases} \cos(\sqrt{k}t) \\ 1 \\ \cosh(\sqrt{-k}t) \end{cases} \qquad \sin_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}t) & k > 0 \\ t & k = 0 \\ \frac{1}{\sqrt{-k}} \sin(\sqrt{-k}t) & k < 0 \end{cases}$$

4.27 Remark. Consider a triangle Δxyz in $\mathbb{M}_k$ for $k > 0$ with sides $a = |yz|$, $b = |xz|$ and $c = |xy|$. Then the following cosine rule hold. For $k > 0$ we have

$$\cos_k c = \cos_k a \cos_k b + k \sin_k a \sin_k b \cos \angle_k xzy$$

where $\angle_k xzy = \angle \sigma_0 \sigma_1$ is the angle at z of two shortest paths σ_0 and σ_1 between z and x, and z and y. In particular, if $\angle_k xzy = \pi/2$ the spherical Theorem of Pythagoras holds

$$\cos_k c = \cos_k a \cos_k b.$$

4.28 Remark. Alexandrov's lemma is still true in $\mathbb{M}_k$ (and has the same proof) if the 0-plane is replaced with the k-plane.

4.29 Lemma. *Consider the k-plane $\mathbb{M}_k$, $k > 0$, and $g_k(t) = |\bar{p}\bar{\gamma}(t)|$ for a shortest path $\bar{\gamma}$ in $\mathbb{M}_k$. Then $\cos_k g_k(t)) = u(t)$ satisfies*

$$u'' + ku = 0, u(0) = \cos_k g_k(0), \ u(L) = \cos_k g_k(L).$$

Proof. By the law of Pythagoras we have $\cos_k(t - t_0) = \cos_k a \cos_k g_k(t)$ where $\bar{\gamma}(t_0) = q$ is the point on $\bar{\gamma}$ closest to p, and $a = g_k(t_0)$. Note that for $\gamma(t_0)$ that is closest to p, by the first variation formula in $\mathbb{M}_k$, we have $0 = \frac{d}{dt}_{t_0} |\bar{p}\bar{\gamma}(t)| = -\cos \angle_k p\bar{\gamma}(t_0)\bar{\gamma}(L)$. Hence u satisfies the desired ODE with $u(0) = \cos_k g(0)$ and $u(L) = \cos_k g(L)$. $\square$

4.30 Corollary. *The inequality $g_k \geq g$ $(\leq)$ in the distance conditions holds if and only if*

$$\cos_k g_k \leq \cos_k g \ (\geq).$$

By similar arguments as for the case $k = 0$ this holds if and only if $\cos_k g = v$ satisfies

$$v'' + kg \leq 0 \quad (\geq), \quad v(0) = \cos_k g(0), \ v(L) = \cos_k g(L) \text{ in the distributional sense.}$$

4.31 Definition. We define $\mathrm{md}_k : [0, \infty) \to \mathbb{R}$ for $k \in \mathbb{R}$ as the solutions of

$$\begin{cases} u'' + ku = 1 \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

More precisely

$$\mathrm{md}_k(t) = \begin{cases} \frac{1}{k}(1 - \cos_k(t)) & k > 0 \\ \frac{1}{2}t^2 & k = 0 \\ \frac{1}{k}(\cos_k(t) - 1) & k < 0 \end{cases}$$

4.32 Fact. *A length space X has curvature bounded from above (below) by k iff the following holds: Every point $q \in X$ has a neighborhood U such that the following holds: $\forall p \in U$ and $\forall \gamma : [0, L] \to X$ that is a shortest path in U such that $|p\gamma(0)| + |p\gamma(L)| + |\gamma(0)\gamma(L)| < 2\pi_k$ the function $g = d_p \circ \gamma$ satisfies*

$$\mathrm{md}_k \circ g + k \, \mathrm{md} \circ g \geq 0 \quad (\leq 0).$$

4.5 Globalisation theorems

4.33 Theorem.

1. *Globalisation for nonpositive curvature: Every complete simply connected space of curvature $\leq k \leq 0$ is a space of curvature $\leq k$ in the large.*
2. *Toponogov's globalization theorem. For any $k \in \mathbb{R}$, every complete space of curvature $\geq k$ is a space of curvature $\geq k$ in the large.*

5 The Gromov-Hausdorff topology

5.1 Uniform convergence

Recall that a sequence of real-valued functions $(f_n)_{n\in\mathbb{N}}$ on a set X is said to converge uniformly to a function f if

$$\sup_{x\in X} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

A metric on X can be considered a a real-valued function on $X \times X$. Hence, we say that a sequence of metrics d_n on X converges uniformily to a metric d on X if

$$\underbrace{\sup_{x,x'\in X} |d_n(x,x') - d(x,x')|}_{=:\|d_n - d\|_\infty} \to 0 \text{ as } n \to \infty.$$

Let X be a topological space and let (Y, d_Y) be a metric space. If $f : X \to Y$ is a homoeomorphism, then he pull-back metric f^*d_Y on X is defined via

$$(f^*d_Y)(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

f^*d_Y is indeed a metric on X and the induced toplogy coincides with the topology of X. Indeed, f is an isometry w.r.t. f^*d_Y and d_Y. In particular, f is an homeomorphism w.r.t. to the induced topologies.

5.1 Definition. A sequence of metric spaces (X_n, d_n), $n \in \mathbb{N}$, converges uniformly to a metric space (X, d) if

$$\sup_{f_n: X_n \to X \text{ homeomorphism}} \sup_{x,y\in X_n} |f_n^* d(x,y) - d_n(x,y)| \to 0 \text{ as } n \to \infty.$$

5.2 Hausdorff distance

Let X be a metric space. Consider two subsets $A, B \subset X$.

Question: How can we compare A and B? What is a distance between A and B?

Consider $S \subset X$. The r-neighborhood of S in X is defined as

$$B_r(S) = \bigcup_{x\in S} B_r(x) = \{y \in Y : d(y, S) < r\}$$

where $d(z, S) := \inf_{x\in X} |zx|$.

5.2 Definition. The Hausdorff distance between subsets A and B in X, denoted $d_H(A, B)$, is defined by

$$d^H(A, B) = \inf\{r > 0 : A \subset B_r(B) \text{ and } B \subset B_r(A)\}.$$

We call $r \in (0, \infty]$ in the infimum on the RHS an Hausdorff bound for A and B.

5.3 Fact. *Let A and B be subsets of a metric space and $r > 0$. The following holds.*

1. $d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$.

2. *$d_H(A,B) \le r$ if and only if $d(a,B) \le r$ $\forall a \in A$ and $d(b,A) \le r$ $\forall b \in B$. This fails if one replaces $\le$ with $<$.*

Proof. Exercise. □

5.4 Proposition. *Let X be a metric space. Then*

1. *d_H is a semi-metric on 2^X, the set of all subsets in X.*

2. *$d_H(A, \overline{A}) = 0$ for any $A \subset X$ where $\overline{A}$ is the closure of A.*

3. *If A and B are closed subsets in X, then $d_H(A,B) = 0$ if and only if $A = B$.*

Proof. 1. The triangle inequality follows from the following observation: for $A \subset X$ and $r_1, r_2 > 0$ one has that $B_{r_1}(B_{r_2}(A)) \subset B_{r_1+r_2}(A)$ by the triangle inequality in X.

2. $d(x, \overline{A}) = 0$ if $x \in A$ since $A \subset \overline{A}$. For $x \in \overline{A}$ we have $d(x, A) = 0$ by the definition of closure. Hence $d_H(A, \overline{A}) = 0$.

3. Assume this is not true. Then $\exists x \in A \backslash B$. Since B is closed there is $r > 0$ such that $B_r(x)$ does not intersect with B. Hence $x \notin B_r(B)$ and therefore $d_H(A,B) \ge r > 0$. □

We denote with $\mathfrak{M}(X)$ the set of closed subsets in X equipped with the Hausdorff distance. Hence $(\mathfrak{M}(X), d_H)$ is a ∞-metric space.

5.5 Proposition. *If X is a complete metric space, then $\mathfrak{M}(X)$ (equipped with d_H) is complete.*

Proof. Let $\{S_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathfrak{M}(X)$. Let S be the set of all points $x \in X$ such that for any neighborhood U of x one has $U \cap S_n \neq \emptyset$ for infinitely many n.

Claim: $S_n \overset{H}{\to} S$.

We fix $\epsilon > 0$ and let n_0 be such that $d_H(S_n, S_m) < \epsilon$ $\forall n, m \ge n_0$. If suffices to show that $d_H(S, S_n) < 2\epsilon$ for any $n \ge n_0$.

1. We have $d(x, S_n) < 2\epsilon$ $\forall x \in S$ and $\forall n \ge n_0$. Indeed: $\exists m \ge n_0$ such that $B_\epsilon(x) \cap S_m \neq \emptyset$. In other words $\exists y \in S_m$ such that $|xy| < \epsilon$. Since $d_H(S_m, S_n) < \epsilon$, one has $d(y, S_n) < \epsilon$ and therefore $d(x, S_n) \le |xy| + d(y, S_n) < 2\epsilon$.

2. Moreover $d(x, S) < 2\epsilon$ $\forall x \in S_n$. Let $n_1 = n$ and for any $k > 1$ choose n_k such that $n_k > n_{k+1}$ and $d_H(S_p, S_q) < \frac{\epsilon}{2^k}$ for all $p, q \ge n_k$. Then define a sequence (x_k) where $x_k \in S_{n_k}$ as follows. Let $x_1 = x$ and x_{k+1} is a point in $S_{n_{k+1}}$ such that $|x_k x_{k+1}| < \epsilon/2^k$ for all k. Such a point can be found because $d_H(S_{n_k}, S_{n_{k+1}}) < \epsilon/2^k$. Since $\sum_{k=1}^\infty |x_k x_{k+1}| < \epsilon < \infty$ the sequence (x_k) is a Cauchy sequence and hence it converges to a point $y \in X$. Then $|xy| = \lim |xx_n| \le \sum_{k=1}^n |x_k x_{k+1}| < \epsilon$. Since $y \in S$ by construction, it follows that $|xS| < 2\epsilon$.

With **1.** and **2.** it follows that $d_H(S, S_n) < 2\epsilon$ $\forall n \ge n_0$. □

5.6 Theorem (Blaschke)**.** *If X is compact, then $\mathfrak{M}(X)$ is compact.*

Proof. Since we already know that $\mathfrak{M}(X)$ is complete, it suffices to show that $\mathfrak{M}(X)$ is totally bounded. Let S be a finite ϵ-net in X. We will show that 2^S is an ϵ-net in $\mathfrak{M}(X)$. Let $A \in \mathfrak{M}(X)$. Consider

$$S_A = \{x \in S : d(x, A) \le \epsilon\}.$$

Since S is an ϵ-net in X, for every $y \in A$ there exists $x \in S$ such that $|xy| \le \epsilon$. Since $d(x, A) \le |xy| \le \epsilon$, this point $x \in S$ belongs to S_A, hence $d(y, S_A) \le \epsilon$ for $y \in A$. Since $d(x, A) \le \epsilon$ for any $x \in S_A$ (by the definition of S_A), it follows that $d_H(A, S_A) \le \epsilon$. Since A is arbitrary, this proves 2^S is an ϵ-net in $\mathfrak{M}(x)$. □

Remark. The set of compact convex subsets in any fixed ball in $\mathbb{R}^n$ is compact w.r.t. d_H.

5.3 Gromov-Hausdorff distance

Idea: We want to compare metric spaces by introducing a distance d_{GH} on the family of all metric spaces. We require

1. If X and Y are metric spaces that are are subsets in a metric space Z, then $d_{GH}(X, Y) \le d_H^Z(X, Y)$.
2. If X and Y are isometric metric spaces, then $d_{GH}(X, Y) = 0$.

The Gromov-Hausdorff distance d_{GH} is defined as the maximal metric on the class of metric spaces satisfying these properties.

5.7 Definition. Let X and Y be metric spaces. The Gromov-Hausdorff distance between X and Y, denoted by $d_{GH}(X, Y)$, is defined as follows. For $r > 0$, we require that $d_{GH}(X, Y) < r$ iff there exists a metric space Z and subspaces X' and Y' of Z that are isometric, w.r.t. to the restriction of the metric of the ambient space Z, to X and Y such that $d_H^Z(X', Y') < r$.

In other words, $d_{GH}(X, Y)$ is the infimum of all $r > 0$ such that there exists a metric space Z and distance preserving maps $\iota_X : X \to Z$, $\iota_Y : Y \to Z$ such that $d_H^Z(\iota_X(X), \iota_Y(Y)) < r$.

Remark. If X and Y are isometric, then $d_{GH}(X, Y) = 0$. Indeed, let f be an isometry. Choose $Z = X$ and $\iota_X = \mathrm{id}_X$ and $\iota_Y = f$.

5.8 Remark. Note that X' and Y' are not equipped with the induced **intrinsic** metric in Z, but with the induced metric. If X is a sphere with the Standard Riemannian metric, one cannot choose $Z = \mathbb{R}^3$ and $X' \simeq \mathbb{S}^2 \subset \mathbb{R}^3$ in the definition of d_{GH}. X', with the restricted metric of $\mathbb{R}^3$, is not isometric to X!

Remark. In general, it is a hard problem to compute the GH distance between explicitly given metric spaces.

5.9 Example. Recall: an ϵ-net Y in a metric space X is defined by the property that $\forall x \in X\ \exists y \in Y$ such that $|xy| < \epsilon$. Hence $d_H^X(X, Y) \le \epsilon$ and therefore $d_{GH}(X, Y) \le \epsilon$.

5.10 Remark. The definition of d_{GH} deals with a huge classs of metric spaces, namely all Z that contain isometric copies of X and Y. It is possible to reduce this class. It is enough to consider the infimum of $r > 0$ such that there exists a semi-metric d on the disjoint union $X\dot{\cup}Y$ such that $d|_{X\times X} = d_X$ and $d_{Y\times Y} = d_Y$ and $d_H(X,Y) < r$ in the space $(X\dot{\cup}Y, d)$.

Proof. Let Z, X', Y' be an admissible triple in the infimum of the definition of $d_{GH}(X,Y)$. We fix isometries $\iota_X : X \to X'$ and $\iota_Y : Y \to Y'$. Then we define a distance d' on $X\dot{\cup}Y$ as follows. For $x, \widetilde{x} \in X$ define $d'(x,\widetilde{x}) = d_X(x,\widetilde{x})$. Analogously for $y, \widetilde{y} \in Y$. If $x \in X$ and $y \in Y$, we set $d'(x,y) = d_Z(\iota_X(x), \iota_Y(y))$. This yields a semi-metric on $X\dot{\cup}Y$ such that $d_H(X,Y) < r$ (if $X' \cap Y' \neq \emptyset$, it may happen that $d'(x,y) = 0$). The quotient metric space $X\dot{\cup}Y/d'$ is isometric to $X' \cup Y'$ (in Z) equipped with $d_Z|_{X\cup Y\times X\cup Y}$.

To obtain a metric on $X\dot{\cup}Y$, define $d(x,y) = d_Z(\iota_X(x), \iota_Y(y)) + \delta$ where $\delta > 0$ is arbitrary. Then $d_H(X,Y) < r + \delta$ in $(X\dot{\cup}Y, d)$. □

5.11 Proposition. *d_{GH} satisfies the triangle inequality, i.e.*

$$d_{GH}(X_1, X_3) \leq d_{GH}(X_1, X_2) + d_{GH}(X_2, X_3)$$

for any metric spaces X_1, X_2, X_3.

Proof. Let d_{12} and d_{23} be metrics on $X_1\dot{\cup}X_2$ and on $X_2\dot{\cup}X_3$ respectively, extending the metrics on X_1, X_2 and X_3. We define a distance between $x_1 \in X_1$ and $x_3 \in X_3$ by

$$d_{13}(x_1, x_3) = \inf_{x_2 \in X_2} \{d_{12}(x_1, x_2) + d_{23}(x_2, x_3)\}.$$

One can check that d_{13} is a metric on $X_1\dot{\cup}X_3$ that extends the metrics on X_1 and X_3.

Consider r_1, r_2 such that $d_H(X_1, X_2) \leq r_1$ and $d_H(X_2, X_3) \leq r_2$. Hence, $\forall x_1 \in X_1$ $\exists x_2 \in X_2$ such that $d_{12}(x_1, x_2) \leq r_1$ and $\exists x_3 \in X_3$ such that $d_{23}(x_2, x_3) \leq r_2$. Together with the definition of d_{13} it follows that $\forall x_1 \in X_1$ $\exists x_3 \in X_3$ such that $d_{13}(x_1, x_3) \leq r_1 + r_2$ and the same statement for X_1 and X_3 in reversed roles. From the definition of the Hausdorff distance we get $d_H(X_1, X_3) \leq r_1 + r_2$, and since r_1 and r_2 are arbitrary numbers larger than $d_H(X_1, X_2)$ and $d_H(X_2, X_3)$, it follows

$$d_H(X_1, X_3) \leq d_H(X_1, X_2) + d_H(X_2, X_3).$$

If we take the infimum over all d_{12} and d_{23} we obtain the desired inequality. □

5.12 Definition. Let X and Y be two sets. A correspondence between X and Y is a set $\mathfrak{R} \subset X \times Y$ satisfying the following. $\forall x \in X$ $\exists y \in Y$ such that $(x,y) \in \mathfrak{R}$, and $\forall y \in Y$ $\exists x \in X$ such that $(x,y) \in \mathfrak{R}$.

5.13 Example. A surjective map $f : X \to Y$ defines a correspondence $\mathfrak{R}$ between X and Y via

$$\mathfrak{R} = \{(x, f(x)) : x \in X\}.$$

Remark. Not every correspondence is associated to a map.

5.14 Definition.

1. Let X and Y be metric spaces and $f : X \to Y$ a map. The distortion of f is defined by
$$\operatorname{dist} f = \sup_{x,x' \in X} \left| d_Y(f(x), f(x')) - d_X(x, x') \right|.$$

2. Let $\mathfrak{R}$ be a correspondence between metric spaces X and Y. The distortion of $\mathfrak{R}$ is defined by
$$\operatorname{dist} \mathfrak{R} = \sup\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in \mathfrak{R}\}.$$

5.15 Remark.

- The definition of distortion of map resembles the one dilatation of a Lipschitz map. The difference is that the latter measures relative changes while the former measures absolute changes.
- For the correspondence $\mathfrak{R}$ associated to a map f, one has $\operatorname{dist} \mathfrak{R} = \operatorname{dist} f$.
- If $\mathfrak{R}$ is a distortion between metric spaces X and Y, then $\operatorname{dist} \mathfrak{R} = 0$ if and only if $\mathfrak{R}$ is induced by an isometry from X to Y.

5.16 Theorem. *For any two metric spaces X and Y it holds that*
$$d_{GH}(X, Y) = \frac{1}{2} \inf_{\mathfrak{R}} \operatorname{dist} \mathfrak{R}$$
where the infimum is taken over all correspondences $\mathfrak{R}$ between X and Y.

Proof. **1.** We show: $\forall r > 0$ with $d_{GH}(X, Y) < r$ $\exists \mathfrak{R}$ correspondence with $\operatorname{dist} \mathfrak{R} < 2r$.
Since $d_{GH}(X, Y) < r$, we may assume that X and Y are subspaces of some metric space (Z, d) and $d_H(X, Y) < r$ in Z. Define
$$\mathfrak{R} = \{(x, y) \in X \times Y : d(x, y) < r\}.$$
That $\mathfrak{R}$ is a correspondence follows from $d_H(X, Y) < r$. Then we have for $(x, y), (x', y') \in \mathfrak{R}$, that
$$|d(x, x') - d(y, y')| \leq d(x, y) + d(x', y') < 2r.$$
2. We show that $d_{GH}(X, Y) \leq \frac{1}{2} \operatorname{dist} \mathfrak{R}$ for any correspondence $\mathfrak{R}$.
Pick a correspondence $\mathfrak{R}$ and let $\operatorname{dist} \mathfrak{R} = 2r$. We show there exists a semimetric d on $X \dot{\cup} Y$ such that $d|_{X \times X} = d_X$ and $d|_{Y \times Y} = d_Y$, and $d_H(X, Y) \leq r$ in $(X \dot{\cup} Y, d)$. For this we define for $x \in X$ and for $y \in Y$
$$d(x, y) = \inf\{d_X(x, x') + d(y', y)\} + r$$
where the infimum is w.r.t. all $(x', y') \in \mathfrak{R}$. On X and Y we set the distance d as d_X and d_Y, respectively. d is a semi-metric. In particular the triangle inequality is easy to check. Note that the choice of the constant r in the definition of d as $\frac{1}{2}$ of the distortion of $\mathfrak{R}$ is necessary for the triangle inequality.

For $x \in X$ there exists $y \in Y$ such that $(x, y) \in \mathfrak{R}$. Hence $d(x, y) = r$ and it follows that $d_H(X, Y) \leq r$ w.r.t. d. □

...

5.17 Definition. Let X and Y be metric spaces and $\epsilon > 0$. A map $f : X \to Y$ is called ϵ-isometry if $\operatorname{dist} f \leq \epsilon$ and $f(X) \subset Y$ is an ϵ-net in Y.

Remark. 1. Recall that an isometry between metric spaces X and Y is a map that is distance preserving ($\operatorname{dist} f = 0$) and surjective. In this sense we have that an ϵ-isometry generalizes the concept of isometry.

2. Recall that an ϵ-net S in Y is a family of points such that $d_Y(y, S) \leq \epsilon$ forall $y \in Y$.

5.18 Corollary. *Let X and Y be metric spaces and $\epsilon > 0$. Then*

1. *If $d_{GH}(X, Y) < \epsilon$, then there exists a 2ϵ-isometry from X to Y.*

2. *If there exists an ϵ-isometry from X to Y, then $d_{GH}(X, Y) < 2\epsilon$.*

Proof. 1. Let $\mathfrak{R}$ be a correspondence between X and Y with $\operatorname{dist} \mathfrak{R} < 2\epsilon$. $\forall x \in X$ we choose $f(x) \in Y$ such that $(x, f(x)) \in \mathfrak{R}$. This defines a map $f : X \to Y$. Then $\operatorname{dist} f \leq \operatorname{dist} \mathfrak{R} < 2\epsilon$. Hence we only need to show that $f(X)$ is an 2ϵ-net. For $y \in Y$ we consider $x \in X$ such that $(x, y) \in \mathfrak{R}$. Since both y and $f(x)$ are in correspondence with x it follows

$$d_Y(y, f(x)) \leq d(x, x) + \operatorname{dist} \mathfrak{R} < 2\epsilon.$$

Hence $d(y, f(X)) < 2\epsilon$.

2. Let f be an ϵ-isometry. We define $\mathfrak{R} \subset X \times Y$ by

$$\mathfrak{R} = \{(x, y) \in X \times Y : d(y, f(x)) \leq \epsilon\}.$$

Then $\mathfrak{R}$ is a correspondence because $f(X)$ is an ϵ-net in Y. Moreover, if $(x, y) \in \mathfrak{R}$ and $(x', y') \in \mathfrak{R}$, one has

$$\begin{aligned}\left|d_Y(y, y') - d_X(x, x')\right| &\leq \left|d_Y(f(x), f(x')) - d_X(x, x')\right| + \left|d_Y(y, y') - d_Y(f(x), f(x'))\right| \\ &\leq \operatorname{dist} f + d_Y(y, f(x)) + d_Y(y', f(x')) \leq 3\epsilon.\end{aligned}$$

Hence $\operatorname{dist} \mathfrak{R} \leq 3\epsilon$ and therefore $d_{GH}(X, Y) \leq \frac{3}{2} r < 2r$.

□

5.19 Theorem. *The Gromov-Hausdorff distance defines a finite metric on the space of isometry classes of compact metric spaces, i.e. it is nonnegative, symmetric, satisfies the triangle inequality and $d_{GH}(X, Y) = 0$ if and only if X and Y are isometric.*

Proof. We only need the check the very last statement. Let X and Y be compact metric spaces such that $d_{GH}(X, Y) = 0$. Hence, given a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ with $\epsilon_n \downarrow 0$ for every $n \in \mathbb{N}$ there exists an ϵ_n-isometry $f_n : X \to Y$. Let $S \subset X$ be a countable dense set in X. Using the Cantor diagonal procedure, we can choose a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of (f_n) such that $f_{n_k}(x)$ converges in Y as $k \to \infty$ $\forall x \in S$. W.l.o.g. we assume that this holds already for (f_n). Then we can define a limit map $f : S \to Y$ by setting $f(x) = \lim_{n \to \infty} f_n(x)$ $\forall x \in S$. Since

$$|d_Y(f_n(x), f_n(y)) - d_X(x, y)| \leq \operatorname{dist} f_n \leq \epsilon_n \to 0 \quad \forall x, y \in X$$

it follows that $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in S$. Hence $f : S \to Y$ is distance preserving from the entire X to Y. Similar, we can construct a distance preserving map from Y to X. Hence, we have that the composition $f \circ g$ is distance preserving from Y to itself. Since Y is compact, the map $f \circ g$ is bijective. Hence f is surjective and therefore an isometry. □

5.20 Remark. A sequence $\{X_n\}_{n\in\mathbb{N}}$ of compact metric spaces converges to a compact metric space X in Gromov-Hausdorff sense if $d_{GH}(X_n, X) \to 0$ as $n \to \infty$. Since d_{GH} is a metric the limit X is unique. We write $X_n \overset{GH}{\to} X$.

We have $X_n \overset{GH}{\to} X$ if there are numbers $\{\epsilon_n\}$ and ϵ_n-isometries $f_n : X \to X_n$ such that $\epsilon_n \downarrow 0$.

5.21 Example. The Hausdorff convergence of compact subsets A_n to A in a metric space X implies Gromov-Hausdorff convergence of these subsets A_n, A equipped with the induced metric they inherit from X. The converse is in general false.

If a sequence $\{X_n\}_{n\in\mathbb{N}}$ converges uniformly to X then $X_n \overset{GH}{\to} X$. Indeed, by uniform convergence there exist homoemorphisms $f_n : X_n \to X$ such that

$$\sup_{x,y\in X_n} |f^* d_X(x,y) - d_{X_n}(x,y)| = \operatorname{dist} f_n \to 0 \text{ as } n \to \infty.$$

Since f_n is a surjective map, it gives a correspondence such that the distortion vanishes.

5.22 Example. Let $\{d_n\}$ be a sequence of metrics on a fixed set X which converges uniformly to some function $d : X \times X \to \mathbb{R}$. Then d is obviously a semi-metric, and the quotient metric space X/d is the Gromov-Hausdorff limit of the spaces (X, d_n). Not that if X is a finite set, it suffices to require $d_n(x,y) \to d(x,y)$ for every pair $x, y \in X$.

5.23 Example. Every compact metric space X is a limit of finite spaces. To see this, take a sequence $\epsilon_n \downarrow 0$ of positive numbers and choose a finite ϵ_n-net S_n in X for every n. Since $\forall y \in X\ \exists x \in S_n$ such that $d(x,y) \le \epsilon_n$, it follows that $d_H^X(X, S_n) \le \epsilon_n$ and hence $d_{GH}(X, S_n) \le \epsilon$ where S_n is equipped with the induced metric.

Remark. Recall that X is compact iff X is complete and totally bounded, i.e. $\forall \epsilon > 0$ there exists a finite ϵ-net.

By taking appropriate ϵ-nets one can essentially reduce GH convergence of compact metric spaces to convergence of finite subsets.

5.24 Definition. Let X, Y be two compact metric spacees, and let $\epsilon, \delta > 0$. X and Y are (ϵ, δ)-approximations of each other if there exist finite collections of points $\{x_i\}_{i=1,\dots,N}$ and $\{y_i\}_{i=1,\dots,N}$ in X and Y, respectively, such that

1. The set $\{x_i : 1 \le i \le N\}$ is an ϵ-net in X, and $\{y_i : 1 \le i \le N\}$ is an ϵ-net in Y.
2. $|d_X(x_i, y_j) - d(y_i, y_j)| < \delta\ \forall i, j = 1, \dots, N$.

If $\delta = \epsilon$, we call an (ϵ, δ)-approximations and ϵ-approximation.

5.25 Proposition. *Let X, Y be compact metric spaces.*

1. *If Y is an (ϵ, δ)-approximations of X, then $d_{GH}(X, Y) < 2\epsilon + \delta$.*
2. *If $d_{GH}(X, Y) < \epsilon$, then Y is a 5ϵ-approximation of X.*

Proof. 1. Let $X_0 = \{x_i\}_{i=1,\dots,N}$ and $Y_0 = \{y_i\}_{i=1,\dots,N}$ be as in the previous definition. The second point in the definition means that the correspondence $\{(x_i, y_i) : i = 1, \dots, N\}$ between these two finte sets has distortion less than δ. If follows that $d_{GH}(X_0, Y_0) \le \delta/2$. Moreover since X_0 and Y_0 are ϵ-nets in X and Y, respectively, we have that $d_{GH}(X, X_0)$

and $d_{GH}(Y, Y_0)$ are less than ϵ. The claim follows from the triangle inequality.

2. By assumption there is a 2ϵ-isometry f between X and Y. Let $X_0 = \{x_i\}_{i=1,\dots,N}$ be a finite ϵ-net in X and define $y_i = f(x_i), i = 1, \dots, N$. Then it follows that

$$|d(x_i, x_j) - d(y_i, y_j)| < 2\epsilon < 5\epsilon$$

for all i, j. We have to show that $Y_0 = \{y_i\}_{i=1,\dots,N}$ is a 5ϵ-net in Y. We have $f(X)$ is 2ϵ-net in Y. Hence, for $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leq 2\epsilon$. Since X_0 is an ϵ-net, there exists $x_i \in X_0$ such that $d_X(x, x_i) \leq \epsilon$. Then

$$d_Y(y, f(x_i)) \leq d(y, f(x)) + d(f(x), f(x_i)) \leq 2\epsilon + d(x, x_i) + \operatorname{dist} f \leq 5\epsilon.$$

Hence $d_Y(y, Y_0) \leq 5\epsilon$. □

5.26 Corollary. *Let X, X_n be compact metric spaces. $X_n \overset{GH}{\to} X$ if and only if the following holds. $\forall \epsilon > 0$ there exists a finite ϵ-net S in X and ϵ-net S_n in each X_n such that $S_n \overset{GH}{\to} S$. Moreover, we can choose S_n and S to have the same cardinality.*

Proof. If such ϵ-nets exists, then X_n is an ϵ-approximation of X for n sufficiently small. Then $X_n \overset{GH}{\to} X$ by the previous Proposition. For the other implication we pick a $\epsilon/2$-net S in X and construct corresponding nets S_n in X_n. For this we pick a sequence of ϵ_n-approximations $f_n : X \to X_n$ with $\epsilon_n \downarrow 0$ and define $S_n = f_n(S)$. Then $S_n \overset{GH}{\to} S$ and S_n are ϵ-nets as in the previous proof. □

Remark. Let finite metric spaces S_n converge in GH sense to a finite metric space S, i.e. the distances between points in S_n converge to distances between points in S. It follows that all geometric characteristics of the set S_n converge to those of S, for instance diameter.

5.4 Compactness Theorem

Since the GH topology is very weak, we expect that there are many compact sets.

The results of the previous propositions imply that if a sequence of compact metric spaces X_n converges in GH sense, the spaces must contain ϵ-nets of uniformily bounded cardinality for every $\epsilon > 0$. It follows that if a family of compact metric spaces $\mathfrak{X}$ is precompact w.r.t. GH convergence, then the sizie of a minimal ϵ-net is uniformily bounded over all elements of $\mathfrak{X}$. In fact, together with a uniform diameter bound for elements in $\mathfrak{X}$ this is sufficients for precompactness.

5.27 Definition. We say a family of $\mathfrak{X}$ of compact metric spaces is uniformly totally bounded if

1. There is a constant $D > 0$ such that $\operatorname{diam}_X \leq D$ for all $X \in \mathfrak{X}$.
2. For every $\epsilon > 0$ there exists a natural number $N = N(\epsilon)$ such that every $X \in \mathfrak{X}$ contains an ϵ-net consisting of no more than N points.

5.28 Theorem. *Any class $\mathfrak{X}$ of uniformly totally bounded compact metric spaces is precompact in the GH topology. That is, any sequence of elements of $\mathfrak{X}$ contains a converging subsequence.*

Proof. Let D and $N(\epsilon)$ as in the previous definition. Define inductively $N_k = N_{k-1}+N(\frac{1}{k})$ for all $k \geq 2$ and $N_1 = N(1)$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of metric space in $\mathfrak{X}$. In every X_n we consider the union of all $\frac{1}{k}$-nets with $k \in \mathbb{N}$. This is a countable, dense subset $S_n := \{x_{n,i}\}_{i\in\mathbb{N}}$ in X_n such that $\forall k \in \mathbb{N}$ the first N_k points of S_n form an $\frac{1}{k}$-net in X_n. The distances $|x_{n,i}x_{n,j}|$ do not exceed D, i.e. belong to the interval $[0, D]$. By the Cantor diagonal procedure we can extract a subsequence of n such that $|x_{n,i}x_{n,j}|$ converge for all $i, j \in \mathbb{N}$. We assume $n \in \mathbb{N}$ is already this subsequence.

We constrcut a limit space $\bar{X}$ for $\{X_n\}$ as follows. Consider $\mathbb{N} =: X$ and define

$$d(i, j) = \lim_{n\to\infty} |x_{n,i}x_{n,j}| \quad \forall i, j \in X.$$

d is a semi-metric on X and the quotient X/d is a metric space. Let $\bar{i} \in X/d$ be the point obtained from i. This quotient space may not be complete, so let $\overline{X}$ be the completion of X/d.

We have to show that $\{X_n\}$ converges in GH sense to $\bar{X}$, and that $\bar{X}$ is compact. For this consider the set $S^{(k)} = \{\bar{i} : 1 \leq i \leq N_k\} \subset \bar{X}$.

Claim: $S^{(k)}$ is an $\frac{1}{k}$-net in $\overline{X}$. Indeed $S^{(k)}_n = \{x_{n,i} : 1 \leq i \leq N_k\}$ is an $\frac{1}{k}$-net in the respective space X_n. Hence, for every $x_{n,i}$ there exists $j \leq N_k$ such that $|x_{n,i}x_{n,j}| \leq \frac{1}{k}$. Since N_k is finite and does not depend on n, for every $i \in \mathbb{N}$ fixed there is $j \leq N_k$ such that $|x_{i,n}x_{j,n}| \leq \frac{1}{k}$ for infinitely many indices n. Hence, since $|x_{n,i}x_{n,j}|$ converges to $d(\bar{i}, \bar{j})$, we have $d(\bar{i}, \bar{j}) \leq \frac{1}{k}$. Thus $S^{(k)}$ is an $\frac{1}{k}$-net in X/d and hence in $\bar{X}$. Since $\bar{X}$ is complete and has an $\frac{1}{k}$-net for every $k \in \mathbb{N}$, $\bar{X}$ is compact.

Futhermore, by construction $S^{(k)}$ is the uniform (hence GH) limit of the finite sets $S^{(k)}_n$ as $n \to \infty$. Thus, for every $k \in \mathbb{N}$ we have an $\frac{1}{k}$-net in $\bar{X}$ which is a Gromov-Hausdorff limit of some $\frac{1}{k}$-nets in X_n. Hence $X_n \overset{GH}{\to} \bar{X}$. □

5.5 Convergence of length spaces

5.29 Theorem. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of length spaces, X a complete metric space, and $X_n \overset{GH}{\to} X$. Then X is a length space.*

Remark. Compare the theorem with the fact that convexity is preserved w.r.t. Hausdorff convergence.

Proof. By Theorem 2.34 it suffices to prove that any two points $x, y \in X$ possess an ϵ-midpoint $\forall \epsilon > 0$.

Let $n \in \mathbb{N}$ be such that $d_{GH}(X_n, X) < \frac{\epsilon}{10}$. Then there exists a correspondence $\mathfrak{R}$ between X and X_n such that $\operatorname{dist} \mathbb{R} \leq \frac{\epsilon}{5}$. Take points $\widetilde{x}$ and $\widetilde{y}$ in X_n that correspond to x and y. Since X_n is a length space there exists and $\frac{\epsilon}{5}$-midpoint $\widetilde{z}$ for $\widetilde{x}$ and $\widetilde{y}$. Let $z \in X$ be a point that corresponds to $\widetilde{z}$. then

$$|xz| - \frac{1}{2}|xy| \leq |\widetilde{x}\widetilde{z}| - \frac{1}{2}|\widetilde{x}\widetilde{y}| + 2 \operatorname{dist} \mathfrak{R} \leq \frac{\epsilon}{5} + \frac{2\epsilon}{5} < \epsilon.$$

Similar for y in place of x. Hence $\max\{|xz|, |zy|\} \leq \frac{1}{2}|xy| + \epsilon$. Hence z is a 2ϵ-midpoint for x, y. □

5.30 Examples.

- Let X_n be the sphere $\mathbb{S}^2$ with a geodesic ball of radius $\frac{1}{n}$ removed and equipped with the induced intrinsic metric. The sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to $\mathbb{S}^2$.
- Let X_n be obtained the same way from the circle $\mathbb{S}^1$. Then $\{X_n\}_{n\in\mathbb{N}}$ does not converge to $\mathbb{S}^1$.
- Let X be a straight line segment in $\mathbb{R}^2$, such as $[0,1] \times \{0\} \times \{0\}$, and let X_n be the boundary of its $\frac{1}{n}$-neighborhood equipped with the induced length metric from $\mathbb{R}^3$. Then $X_n \overset{GH}{\to} X$ as $n \to \infty$.
- Let X be a planar disc in $\mathbb{R}^3$, i.e. $X \simeq B_1(0) \times \{0\}$, and again let X_n be the boundary of its $\frac{1}{n}$-neighborhood as before. The sequence $\{X_n\}_{n\in\mathbb{N}}$ converges in GH sense, but the limit is not X.

Metric Graphs A metric segment of length a is a metric space isometric to the segment $[0, a] \subset \mathbb{R}$.

5.31 Definition. A metric graph is the result of gluing a disjoint collection of metric segments $\{E_i\}$ and points $\{v_j\}$ (both regarded with the length metric of the disjoint union) along an equivalence relation R defined on the union of the set $\{v_j\}$ and the set of endpoints of the segments. The segments $\{E_j\}$ are called edges and the equivalence classe of the endpoints are called vertices of the graph. The length of an edge is the length of the corresponding sement.

5.32 Proposition. *Every compact length space can be obtained as a GH limit of finite graphs.*

Proof. Pick $\epsilon > 0$ and $\delta > 0$ small, such that $\delta \ll \epsilon$. Let S be a finite δ-net in X. We define a graph G a follows. The set of vertices of G is S, and two points $x, y \in S$ are connected by an edge if and only if $|xy| < \epsilon$. The length of this edge is $|xy|$.

We show that the graph G is an ϵ-approximation of X if $\delta > 0$ is small enough, say $\delta < \frac{1}{4}\frac{\epsilon^2}{\operatorname{diam} X}$. We consider S as a subset of S and of X. Obviously S is an ϵ-net in both spaces, and $|xy|_G \geq |xy|$ for all $x, y \in S$ where $|\cdot|_G$ denotes the distance in G.

It remains to show that $|xy|_G \leq |xy| + \epsilon$.

Let γ be a shortest path in X between x and y. We choose n points $x_1, \dots, x_n$ with $n \leq 2L(\gamma)/\epsilon$, deviding γ into n intervals of lengths not greater than $\frac{\epsilon}{2}$. Let $x = x_0, y = x_{n+1}$. For every $i = 1, \dots, n$ there exists $y_i \in S$ such that $|x_i y_i| \leq \delta$. Moreover we set $x = y_0$ and $y = y_{n+1}$. Note that $|y_i y_{i+1}| \leq |x_i x_{i+1}| + 2\delta < \epsilon$ for all $i = 0, \dots, n$. In particular, y_i and y_{i+1} are connected by edge in G. Then

$$|xy|_G \leq \sum_{i=0}^{n} |y_i y_{i+1}| \leq \sum_{i=0}^{n} |x_i x_{i+1}| + 2\delta n = |xy| + 2\delta n.$$

Recall $n \leq 2L(\gamma)/\epsilon \leq 2 \operatorname{diam} X/\epsilon$. Hence

$$|xy|_G \leq |xy| + \delta \frac{4 \operatorname{diam} X}{\epsilon} < |xy| + \epsilon$$

if $\delta < \frac{1}{4}\epsilon^2 / \operatorname{diam} X$.

Thus we have a finite graph that is an ϵ-approximation of X. Passing ϵ to zero yields a sequence of graphs converging to X in GH sense. □

5.6 Pointed Gromov-Hausdorff convergence

5.33 Definition. A pointed metric space is a pair (X, o) consisting of a metric space X and a point $o \in X$.

A sequence $\{(X_n, o_n)\}_{n \in \mathbb{N}}$ of pointed metric spaces converges in the Gromov-Hausdorff sense to a pointed metric space (X, o) if the following holds. $\forall r > 0$ and $\forall \epsilon > 0$ there exists $n(r, \epsilon) \in \mathbb{N}$ such that for every natural $n > n(r, \epsilon)$ there is a map $f := f_n^{r,\epsilon} : B_r(o_n) \to X$ (not necessarily continuous or measurable) such that the following hold:

1. $f(o_n) = o$,
2. $\operatorname{dist} f < \epsilon$,
3. $B_\epsilon(f(B_r(o_n)) \supset B_{r-\epsilon}(p)$.

We write $(X_n, o_n) \stackrel{GH}{\to} (X, o)$.

5.34 Remark. The first two requirements in the definition imply that $f(B_r(o_n)) \subset B_{r+\epsilon}(o)$. Hence, together with third condition, one has that the ball $B_r(o_n)$ in X_n lies within GH distance of order $\epsilon > 0$ from a subset $K \subset X$ such that $B_{r-\epsilon}(o) \subset K \subset B_{r+\epsilon}(o)$. If X is a length space, this is true for $K = B_r(o)$. In other words, $(X_n, o_n) \stackrel{GH}{\to} (X, o)$ then $B_r(o_n) \stackrel{GH}{\to} B_r(o)$ $\forall r > 0$, provided X is a length sapce (exercise). This may fail if X is not a length space. To see this, construct a sequence of compact metric spaces $\{X_n\}_{n \in \mathbb{N}}$ that converges to X, but no sequence of closed unit balls in X_n converges to a closed ball in X.

5.35 Remark. The property that the balls $B_r(o_n)$ converge to $B_r(o)$ does not yet imply pointed GH convergence. The first requirement puts the points o_n and o into a special position. To illustrate this consider the following that is left as an exercise: Construct a compact metric space X with two points $p, q \in X$ such that for every $r > 0$ the balls $B_r(p)$ and $B_r(q)$ are isometric in X (hence $B_r(p)$ converges to $B_r(q)$ $\forall r > 0$), but there is no isometry from X to itself that maps p to q. The latter statement means that (X, p) does not converge in pointed GH sense to (X, q).

5.36 Fact. *Let X_n, X be compact metric spaces. Then*

- $(X_n, o_n) \overset{GH}{\to} (X, o)$ *implies* $X_n \overset{GH}{\to} X$.
- *If* $X_n \overset{GH}{\to} X$ *and* $o \in X$, *then one can choose* $o_n \in X_n$ *such that* $(X_n, o_n) \overset{GH}{\to} (X, o)$.

Remark. If $(X_n, p_n) \overset{pGH}{\to} (X, p)$, then the same sequence also converges in pointed GH sense to the completion.

5.37 Definition. We say that 2 pointed compact metric spaces (X, p) and (X', p') are isometric if there is an isometry $f : X \to X'$ such that $f(p) = p'$. Such a map is called a pointed isometry from (X, p) to (X, p').

5.38 Theorem. *Let $(X, p), (X', p')$ be two complete pointed Gromov-Hausdorff limits of a sequence $\{(X_n, p_n)\}_{n=1}^{\infty}$ and assume that X is boundedly compact (i.e. every closed and bounded set is compact). Then (X, p) and (X', p) are isometric.*

Proof. Let $r > 0$ and $\epsilon > 0$. From the definition of pointed GH convergence we can construct a correspondence between sets $Y_{r,\epsilon} \subset X$ and $Y'_{r,\epsilon} \subset X'$ such that $Y_{r,\epsilon}$ and $Y'_{r,\epsilon}$ contain the balls of radius $r - \epsilon$ and are contained in the balls of radius $r + \epsilon$ centered at p and p' respectively. We have that p and p' are in correspondence to each other and the distortion $\operatorname{dist} \mathfrak{R}_{r,\epsilon} < \epsilon$.

Choosing one point corresponding to a point in $Y_{r,\epsilon}$ yields a map $f_{r,\epsilon} : Y_{r,\epsilon} \to Y'_{r,\epsilon}$ that maps p to p' and has distortion $< \epsilon$. By a Cantor diagonal argument, first for $\epsilon \downarrow 0$, then for $r \uparrow \infty$, we can construct a distance preserving map f from a dense subset S in X to X' with $f(p) = p'$. Hence f extends to a distance preserving map on X. It maps every ball of radius $B_r(p)$ in X to the corresponding ball in $B_r(p')$.

Because of compactness of the closure of the balls we can procede with the same argument as in Theorem 5.19 and we obtain f is surjective, hence an isometry. $\square$

5.39 Remark. Assume $\{(X_n, p_n)\}_{n\in\mathbb{N}}$ is boundedly compact and $(X_n, p_n) \overset{GH}{\to} (X, p)$ where (X, p) is complete. Then (X, p) is boundedly compact.

In the following we usually consider spaces that are boundedly compact.

Like for GH convergence one can prove the following theorems.

5.40 Theorem. *Let $(X_n, p_n) \overset{GH}{\to} (X, p)$ where X_n are length spaces and X is complete. Then X is a length space.*

5.41 Theorem. *Let $\mathfrak{X}$ be the class of pointed metric spaces with the following property. $\forall r > 0$ and $\forall \epsilon > 0$ there exists $N(r, \epsilon)$ such that for $(X, p) \in \mathfrak{X}$ the ball $B_r(p)$ in X admits an ϵ-net of no more than $N(r, \epsilon)$ points. Then the class $\mathfrak{X}$ is precompact in the sense that any sequence of spaces in $\mathfrak{X}$ contains a converging subsequence in pointed GH sense.*

6 Alexandrov spaces with curvature bounded from below

Throughout this section Alexandrov spaces are complete, connected, strictly intrinsic spaces with curvature bounded from below by k for $k \in \mathbb{R}$.

In the case of positive k we exclude the following exceptional 1-dimensional spaces: $\mathbb{R}$, $[0, \infty)$ as well as segments and circles with diameter larger than $\pi/\sqrt{k}$.

Recall Toponogov's Globalization Theorem:

Theorem. Any Alexandrov space with curvature bounded from below by k is an Alexandrov space with curvature bounded from below by k in the large.

Remark. "In the large" means that the curvature comparison conditions are satisfied for all triangles for which comparison triangles **exists and is unique** (uniqueness is understood up to rigid motions of the k-plane). The latter requirement is essential only for the case of curvature $\geq k > 0$, i.e. we have the following cases

- if the the perimeter of the triangle Δabc is strictly smaller than $2\pi/\sqrt{k}$, then there exists a unique comparison triangle.
- if the perimeter is equal to $2\pi/\sqrt{k}$ and each side is strictly smaller than $\pi/\sqrt{k}$, then there exists a unique comparison triangle. More precisely the comparison triangle is given by a great circle in the k-plane.

  Otherwise, we have $|ac| = \pi/\sqrt{k}$ or $|ab| + |bc| = \pi/\sqrt{k}$ and in this case comparison triangles are not unique in the k-plane.

Recall: If a, b, c are different points in a length space, then we denote with $\Delta\bar{a}\bar{b}\bar{c}$ the comparison triangle in the k-plane if it exists and is unique. The comparison angle $\tilde{\angle}_k abc$ is the angle $\angle_k \bar{a}\bar{b}\bar{c}$ of the comparison triangle $\Delta\bar{a}\bar{b}\bar{c}$ in $\bar{b}$ in the k-plane. The angle $\angle_k \bar{a}\bar{b}\bar{c}$ is a function of the 3 distances $|ab|$, $|bc|$ and $|ca|$. It is well defined if $|ab| + |bc| + |ca| < \pi_k$ where π_k is the diameter of the k-plane.

6.1 Quadruple Condition

6.1 Proposition. *A locally compact length space X is a space of curvatre $\geq k$ iff $\forall x \in X$ $\exists U \subset X$ a neighborhood of x such that every collection of 4 different points $a, b, c, d \in U$ the following condition is satisfied:*

$$\tilde{\angle}_k bac + \tilde{\angle}_k cad + \tilde{\angle}_k dab \leq 2\pi.$$

We call this inequality the quadruple condition for the quadruple $(a; b, c, d)$. Note that a is in a special position.

Remark. Note that this condition does not rely on the existence of shortest paths, so it can be used unmodified for not strictly intrinsic metric spaces.

Proof. **1.** Assume the quadruple condition holds for all quadruples. Pick a triangle Δabc and $d \in X$ on the shortest path $[ac]$ between a and c. Apply the quadruple condition for $(d; a, b, c)$. Since $\tilde{\angle}_k adc = \pi$, it follows $\tilde{\angle}_k bdc + \tilde{\angle}_k bda \leq \pi$. By Alexandrov's Lemma (for the k-plane) it follows that

$$|bd| \leq |\bar{b}\bar{d}|$$

where $\bar{a}, \bar{b}, \bar{c}$ is a comparison triangle in the k-plane and $\bar{d}$ is point on $[\bar{a}\bar{c}]$ with $|\bar{a}\bar{d}| = |ad|$. This is exaclty the distance condition for curvature $\geq k$.

2. Let X be a space with curvature $\geq k$, let $(a; b, c, d)$ be a quadruple and let a' be a point in a shortest path $[ab]$ between a and b. Then

$$\begin{aligned}\tilde{\angle}_k ba'd + \tilde{\angle}_k da'c + \tilde{\angle}_k ca'b &\leq \angle ba'd + \angle da'c + \angle ca'b \\ &\leq (\angle_k ba'd + \angle_k da'a) + (\angle_k aa'c + \angle_k ca'b) = 2\pi\end{aligned}$$

where we used the angle condition, the triangle inequality for angles and the fact that the sum for adjacent angle equals π in spaces with curvature $\geq k$.

Now let a' converge to a. Continuity of comparison angles implies the statement. □

6.2 Corollary. *Shortest paths in a space of curvature bounded from below by k do not branch. Namely, if two shortest paths $\gamma : [0, L] \to X$ and $\gamma' : [0, L'] \to X$ with $L' \leq L$ and $\gamma(0) = \gamma'(0)$ satisfy $\gamma|_{[0,\epsilon]} = \gamma'|_{[0,\epsilon]}$ then $\gamma'([0, L']) \subset \gamma([0, L])$.*

Proof. Assume w.l.o.g. that γ and γ' are parametrized by arc length and $L = L'$. Assume further that $\gamma(L) = b \neq \gamma'(L) = c$, $\gamma(0) = \gamma'(0) = d$ and $\gamma|_{[0,\epsilon]} = \gamma'|_{[0,\epsilon]}$ with $\gamma(\epsilon) = \gamma'(\epsilon) = a$ for some $\epsilon > 0$. From the quadruple condition it follows that $\tilde{\angle}_k bac = 0$. Hence $|bc| = 0$. □

6.3 Theorem (Bonnet's Theorem for Alexandrov spaces)**.** *Let X be an Alexandrov space with curvature $\geq k > 0$. Then* $\mathrm{diam}_X \leq \pi/\sqrt{k}$.

Proof. Assume $\exists a, b \in X$ such that $|ab| > \pi/\sqrt{k}$. We may assume there exists $\epsilon \in (0, \pi/4)$ such that $|ab| = (\pi + \epsilon)/\sqrt{k}$. Let c be the midpoint on the shortest path $[a, b]$ and let $U = B_{\epsilon/(8\sqrt{k})}(c)$.

1. *Claim.* U contains a point which does not belong to $[ab]$.

Otherwise: $\forall x \in X$ let γ be a shortes path from x to c. Hence the image of γ has nonempty intersection with U. Hence, a subsegment of γ coincides with $[ab]$ close to c. Since geodesics do not branch, it follows that x belongs to the unique geodesics that contains the segment $[ab]$. Hence, all of X is covered by two shortes paths starting in c and passing through a and b. We conclude that X would be one of the one-dimensional exceptional spaces.

2. Choose $x \in U \backslash [ab]$ and let y be the nearest point to x on $[ab]$. Then we have $|ay| > \pi/2\sqrt{k}$ and $|by| > \pi/2\sqrt{k}$. For this note that $|xy| < \epsilon/(8\sqrt{k})$ (since already $|xc| < \epsilon/(8\sqrt{k})$) and

$$|ay| \geq |ac| - |cy| \geq |ac| - (|cx| + |xy|) \geq (\pi + \epsilon)/(2\sqrt{k}) - \epsilon/4\sqrt{k} = (\pi/2 + \epsilon/4)/\sqrt{k}.$$

Let γ be a parametrization of $[ab]$ (by arclength) and let $l(t) = |x\gamma(t)|$. We showed that l is differentiable for $t \in [0, L]$ and $\operatorname{argmin} l(t) = y$. Let $\gamma(t_0) = y$. Hence $l'(t_0) = 0$ and by the first variation formula we have that

$$\angle xya = \angle xyb = \pi/2.$$

We consider a comparison triangle $\Delta\bar{x}\bar{y}\bar{a}$ for Δxya in the k-plane. By Toponogov's theorem we have $\angle\bar{x}\bar{y}\bar{a} \leq \angle xya = \pi/2$. Since $|\bar{y}\bar{a}| > \pi/(2\sqrt{k})$, it follows that $|\bar{x}\bar{a}| < |\bar{y}\bar{a}|$. Thus $|xa| < |ya|$. Similarly we prove $|xb| < |yb|$. Hence

$$|ya| + |yb| > |xa| + |xb| \geq |ab|.$$

But since y belongs to the geodesic path $[ab]$, we have a contradiction. □

6.4 Corollary. *Let X be an Alexandrov space with curvature $\geq k > 0$. Then every triangle in X has perimeter not greater than $2\pi/\sqrt{k}$.*

Proof. Assume first that $\mathrm{diam}_X < \pi/\sqrt{k}$, and assume there are points x, y, z such that $|xy| + |yz| + |yz| > 2\pi/\sqrt{k}$. Fix shortest paths between these points. By continuity of the distance function we find points $y' \in [xy]$ and $z' \in [xz]$ such that $|xy'| + |xz'| + |y'z'| = 2\pi/\sqrt{k}$. Consider a triangle $\Delta xy'z'$ (with shortest paths between the points). Since $\mathrm{diam}_X < \pi/\sqrt{k}$, each side is strictly shorter than $\pi/\sqrt{k}$. Hence a comparison triangle $\Delta\bar{x}\bar{y}'\bar{z}'$ in the k-plane is defined and unique, and equal to a great circle. It follows that all comparison angles are equal to π. Since y' and z' are points on the shortest path from x to y, and x to z respectively, it follows by the triangle comparison condition for curvature $\geq k$ that $\angle y'z'z = \angle z'y'y = 0$. Since shortest paths are nonbranching it follows that $y, z \in [y'z']$ and in particular $|yz| \leq |y'z'|$, as well as the segment $[y, z]$ is contained in $[y'z']$. It follows that the perimeter of Δxyz is equal to $2\pi/\sqrt{k}$. This is a contradiction.

In the general case, we pick $\epsilon \in (0, k)$. Hence $\mathrm{diam}_X \leq \pi/\sqrt{k} < \pi/\sqrt{k-\epsilon}$. Moreover, by monotonictity of condition "curvature bounded from below by k" in $k \in \mathbb{R}$ we have that X has curvature $\geq k - \epsilon$. Then we can apply the previous step and obtain the every triangle has perimeter bounded from above by $2\pi/\sqrt{k-\epsilon}$. Sending ϵ to 0 yields the result. □

6.5 Remark. The (local) quadruple condition is equivalent to the following modified quadrupel condition:

For every $x \in X$ there exists an open neighborhood U such that for any quadruple $(a; b, c, d)$ in U there exists a quadruple $(\bar{a}; \bar{b}, \bar{c}, \bar{d})$ in the k-plane such that the segments $[\bar{a}, \bar{b}]$, $[\bar{a}, \bar{c}]$ and $[\bar{a}, \bar{d}]$ devide the full angle at $\bar{a}$ into 3 angles less than or equal to π where $|\bar{a}\bar{b}| = |ab|$, $|\bar{a}\bar{c}| = |ac|$, $|\bar{a}\bar{d}| = |ad|$, and $|\bar{b}\bar{c}| \geq |bc|$, $|\bar{c}\bar{d}| \geq |cd|$ and $|\bar{d}\bar{b}| \geq |db|$.

6.6 Proposition. *Let Γ be group that acts by isometries on a metric space (X, d) such that the orbits $O(p) = \{f(p) : f \in \Gamma\}$ are closed. We consider the quotient $Q = X/\Gamma$ equipped with the quotient topology. Let $\pi : X \to Q$ be the projection map that is continuous. A metric ρ on Q given by $\rho(\pi(p), \pi(q)) = \inf\{d(p, r) : r \in o(q)\}$.*

If (X, d) is a locally compact length space (hence strictly intrinsic) with curvature $\geq k$, then (Q, ρ) is also a space with curvature $\geq k$.

Proof. Let $p \in X$ and $U = B_r(p)$ a region where the quadruple condition is satisfied for every quadruple. We show the same is true for $U_0 = B_{r/2}(\pi(p)) \subset Q$.

Let $(a_0; b_0, c_0, d_0)$ be a quadruple in U_0. Since X is locally compact and the orbit $\pi^{-1}(a_0)$ is closed, there exists a point $a \in \pi^{-1}(a_0)$ nearest to p. In particular $|pa| = |\pi(p)a_0|$. Similarly we can now find points $b, c, d \in U$ such that $|ab| = |a_0b_0|$, $|ac| = |a_0c_0|$ and $|ad| = |a_0d_0|$. Note that $a, b, c, d \in U$ (by triangle inequality). By the definition of the quotient metric we also have $|bc| \geq |b_0c_0|$, $|c, d| \geq |c_0d_0|$ and $|db| \geq |d_0b_0|$. Hence the quadruple condition in the form of the previous remark on X implies the quadruple condition on Q (in the form of the remark). □

6.7 Example. Let $\mathbb{Z}_2 = \{e, g\}$ act on $\mathbb{R}^3$ by symmetries: $g(x) = -x$ for all $x \in \mathbb{R}^3$. Then $Q = \mathbb{R}^3/\mathbb{Z}_2$ is a space of nonnegative curvature. Q is isometric to the Euclidean cone $C(\mathbb{P}^2)$ over the projective space $\mathbb{P}^2$ equipped with the metric induced from the canonical Riemannian metric of constant curvature 1.

Example: Euclidean cones Recall the Definition 3.17 for the Euclidean cone over a metric space X.

6.8 Theorem. *Let X be a locally compact, connected length space. The following two statements are equivalent.*

1. *X has curvature ≥ 1,*
2. *$C(X)$ is not a straight line and is a space curvature ≥ 0.*

Proof. First we note that if $C(X)$ is a straight line, then X consists of 2 points at distance π and hence X would not be connected.

Consider a triangle Δabc in $C(X)$ whose sides do not pass through the origin o. Its projection to X is a triangle $\Delta a'b'c'$ in X with side lengths less than π. The sides of Δabc are contained in convex flat sectors, the sub-cones over the sides of $\Delta a'b'c'$.

1. *Claim:* Δabc in K satisfies the triangle condition for curvature ≥ 0 iff $\Delta a'b'c'$ in X does so for curvature ≥ 1, provided the perimeter of $\Delta a'b'c'$ is not greater than 2π.

Let $\Delta \bar{a}'\bar{b}'\bar{c}'$ be a comparison triangle for $\Delta a'b'c'$ in the 1-plane $\mathbb{S}^2 \subset \mathbb{R}^3$. Place points $\bar{a}, \bar{b}, \bar{c}$ in the rays through 0 and $\bar{a}'$, $\bar{b}'$ and $\bar{c}' \in \mathbb{R}^3$ such that $|0\bar{a}| = |oa|$, $|0\bar{b}| = |ob|$ and $|0\bar{c}| = |oc|$. The resulting triangle $\Delta\bar{a}\bar{b}\bar{c}$ is a comparison triangle for Δabc by the definition of the cone metric.

Pick a point d in $[ac]$ and let $d' \in [a'c']$ be the projection of d to X. Let $\bar{d}$ and $\bar{d}'$ be the corresponding points in the Euclidean segment $[\bar{a}\bar{c}]$ and $[\bar{a}', \bar{c}']$ respectively. The subcone over $[a'c']$ in $C(X)$ is isometric to the planar sector in $\mathbb{R}^3$ spanned by the spherical segment $[\bar{a}'\bar{c}']$. An isometry from this sub-cone to this sector sends $[ac]$ isometrically to $[\bar{a}\bar{c}]$, in particular it sends d to $\bar{d}$. Furthermore this isometry sends the segment $[a'c']$ in X isometrically to the spherical segment in $[\bar{a}'\bar{c}']$, d' is sent so $\bar{d}'$. It follows that $\bar{d}$ belongs to the ray through 0 and $\bar{d}'$ and $|0\bar{d}| = |od|$.

Assuming the distances $|ob|$ and $|od|$ fixed, the distance $|bd|$ in $C(X)$ is an increasing funnction of $|b'd'|$. If $|b'd'| = |\bar{b}'\bar{d}'|$, then $|bd| = |\bar{b}\bar{d}|$, because $|0\bar{b}| = |ob|$ and $|0\bar{d}| = |od|$ (for this recall the formula of distances in the Euclidean metric cone). Hence $|b'd'| \geq |\bar{b}'\bar{d}'|$ if and only if $|bd| \geq |\bar{b}\bar{d}|$. This proves the desired statement about the distance conditions for Δabc and $\Delta a'b'c'$, i.e. the claim.

2. Here we finish the proof.

Every triangle with side lengths less than π in X is a projection of some triangle in $C(X)$. Thus, from the claim we get that, if $C(X)$ has curvature ≥ 0, then X has curvature ≥ 1.

Similar, a projection of a sufficiently small region in $C(X)\backslash\{0\}$ for the triangle condition (normal region) is a region for the triangle condition in X. Conversely, a sub-cone over sufficiently small normal region in X is a normal region in $C(X)\backslash\{0\}$.

Let us consider triangles Δabc in X such that no side passes through o, but whose projection to X, $\Delta a'b'c'$ has perimeter $L > 2\pi$. Then sub-cone over $\Delta a'b'c'$ in $C(X)$ is an image under an arcwise isometry of the cone $C(\mathcal{S})$ over the circle $\mathcal{S}$ of length L. Moreover Δabc is the image of a triangle in $C(\mathcal{S})$. Since an arcwise isometry is a nonexpanding map and $C(\mathcal{S})$ has nonpositive curvature, the triangle Δabc satisfies the condition for curvature ≤ 0.

On the other hand, this triangle in $C(\mathcal{S})$ corresponding to Δabc does not satisfy the condition for curvature ≥ 0 (because the origin of $C(\mathcal{S})$ is contained in one of the sides of this triangle, because $L > 2\pi$). Hence neither does Δabc. Consequently, if we assume $C(X)$ has curvature ≥ 0, then X cannot contain a triangle with perimeter $> 2\pi$.

The case when $\Delta a'b'c'$ in X has sides greater than π are considered similarly. Such triangle correspond to triangle in $C(X)$ with sides passing through o, and the later is the image under an arcwise isometry of a triangle in a nonpositively but not nonnegatively curved cone over a segment $L > \pi$.

As before we conclude that, assuming $C(X)$ has curvature ≥ 0, then X cannot contain triangles with side length greater or equal to π. □

Example: k-cones Let $k \in \mathbb{R}$ and X be a metric space with $\operatorname{diam}_X \leq \pi$. The k-cone over X, denoted by $C_k(X)$ consist of the origin o and pairs (r, x) where $x \in X$ and $r > 0$ ($r \leq \pi/\sqrt{k}$ if $k > 0$). The distance from (r, x) to the origin is r and the distance between $(r_0, x_0) = a_0$ and $(r_1, x_1) = a_1$ is defined as the side $|\bar{a}_0\bar{a}_1|$ of a triangle $\Delta\bar{a}_0\bar{o}\bar{a}_1$ in the k-plane with $|\bar{o}\bar{a}_i| = r_i$, $i = 0, 1$, and $\angle\bar{a}_0 o a_1 = |x_0 x_1|$.

If $k > 0$, the point $(x, \pi/\sqrt{k})$ should be identified because their distance is 0. In this case the k-cone is called k-spherical cone or k-suspension.

By the definition one has that $C_k(\mathbb{S}^1)$ is isometric to k-plane, and similarly $C_k(\mathbb{S}^n)$ is isometric to the standard $(n+1)$-dimensional space of constant curvature k.

Theorem. *Let X be a locally compact, connected length space and $k \in \mathbb{R}$. The following statements are equivalent.*

1. *X has curvature ≥ 1,*

2. *$C_k(X)$ is not a straight line or a circle and has curvature $\geq k$.*

6.2 Strainers

Let X be a complete, connected, strictly intrinsic space with curvature $\geq k$

6.9 Definition. Let $m \in \mathbb{N}$ and fix $\epsilon_0 = \frac{1}{100m}$. Let $\epsilon \in (0, \epsilon_0)$. A point $p \in X$ is an (m, ϵ)-strained point if $\exists m$ pairs of points (a_i, b_i) in X such that $\forall i, j \in \{1, \dots, m\}$

$$\widetilde{\angle} a_i p b_i > \pi - \epsilon,$$

$$\widetilde{\angle} a_i p a_j > \pi/2 - 10\epsilon,$$

$$\widetilde{\angle} a_i p b_j > \pi/2 - 10\epsilon,$$

$$\widetilde{\angle} b_i p b_j > \pi/2 - 10\epsilon.$$

The collection $(a_i, b_i)_{i=1,\dots m}$ is called an (m, ϵ)-strainer. $\widetilde{\angle}$ denotes the comparison angle in the k-plane. If $\epsilon > 0$ is given, we call an (m, ϵ)-strained point an m-strained point.

Remark. Given an (m, ϵ)-strainer (a_i, b_i) the quadruple condition

$$\widetilde{\angle} a_i p a_j + \widetilde{\angle} a_j p b_i + \widetilde{\angle} b_i p a_i \leq 2\pi$$

for $(p; a_i, a_j, b_i)$ yields $\widetilde{\angle} a_i p a_j \leq \pi + 11\epsilon$ and $\widetilde{\angle} a_i p b_j \leq \pi + 11\epsilon$ $\forall i, j \in \{1, \dots, m\}$. The set of points that are (m, ϵ)-strained by $(a_i, b_i), i = 1, \dots, m$, is open.

6.10 Definition. The strainer number of X is the supremum of $m \in \mathbb{N}$ such that there exists an (m, ϵ)-strainer for some $\epsilon \in (0, \epsilon_0)$. A strainer number at a point $p \in X$ is the supremum of numbers m such that every neighborhood of x contains an m-strained point.

Remark. X admits a $(1, \epsilon)$-strainer for all $\epsilon > 0$ unless $X = \{pt\}$. Pick $a, b \in X$ and let p be a δ-midpoint for $\delta > 0$ small enough (a, b) is a $(1, \epsilon)$ -strainer of p.

6.11 Proposition. *1. If (a_i, b_i) is an (m, ϵ)-strainer for $p \in X$ and $a'_i \in [p, a_i]$ and $b'_i \in [p, b_i]$, $i = 1, \dots, m$, where $[p, a_i], [p, b_i]$ are shortest paths, then (a'_i, b'_i) is an (m, ϵ)-straider for p as well. In particular, there exists an (m, ϵ)-straider arbitrarily close to p.*

2. If (a_i, b_i) is an (m, ϵ)-strainer for p then $\forall i, j \in \{1, \dots, m\}$

$$\angle a_i p b_i > \pi - \epsilon,$$
$$\angle a_i p a_j > \pi/2 - 10\epsilon,$$
$$\angle a_i p b_j > \pi/2 - 10\epsilon,$$
$$\angle b_i p b_j > \pi/2 - 10\epsilon.$$

Conversely, if these inequalities hold for p, (a_i, b_i) and $\epsilon \in (0, \epsilon_0)$, then p is (m, ϵ)-strained.

Proof. The statements are a direct consequence of the comparison angle monotonicty. □

Coordinates at strainer points. Let (a_i, b_i) be an (m,ϵ)-strainer at p. Define $f : U \to \mathbb{R}^m$ via

$$f(x) = (|xa_1|, \ldots, |xa_m|)$$

where U is a small neighborhood of p. f is a Lipschitz map because $x \mapsto |xa_i|$ are Lipschitz functions. Since the set S of points that are (m,ϵ)-strained by (a_i, b_i), we assume that $U \subset S$.

6.12 Example. Consider $X = \mathbb{R}^m$. Then a family of pairs (a_i, b_i), $i = 1, \ldots, m$, such that a_i and b_i are on the same straight line through 0, and such that $\langle a_i, b_j\rangle = \langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0$, is an (m,ϵ)-strainer and $0 \in \mathbb{R}^m$ is (m,ϵ)-strained. The level set functions $\hat{f}(x) = |x - a_i|$ in intersect almost orthogonally. Hence, for $(r_1, \ldots, r_m)$ near $(|x - a_1|, \ldots, |x - a_m|)$ the level set spheres have a unique intersection point near p.

We omit the proof of the following proposition.

6.13 Proposition. *Let $p \in X$ be (m,ϵ)-strained with $\epsilon \in (0, \epsilon_0)$. Then there exists a neighborhood U of p such that $f : U \to \mathbb{R}^m$ from above is an open map.*

6.14 Corollary. *The strainer number of X is not greater than the Hausdorff dimension.*

Proof. Let p be an (m,ϵ)-strained point as in the previous proposition. It follows that there exists a map $f : U \ni p \to \mathbb{R}^m$ that is open and Lipschitz. Assume U is open. Then $f(U)$ is open. Hence Proposition 1.25 implies

$$m = \dim_H f(U) \leq \dim_H U \leq \dim_H X.$$

□

We also omit the proof of the following theorem.

6.15 Theorem. *If $p \in X$ is an (m,ϵ)-strained point such that m equals the strainer number at p. Then p has a neighborhood which is bi-Lipschitz homeomorphic to an open region in $\mathbb{R}^m$. The bi-Lipschitz map is provided by the map f from above.*

6.16 Corollary. *All finite dimensional Alexandrov spaces with curvature $\geq k$ (complete, connected, (strictly) intrinsic) are locally compact.*

Remark. Strictly intrinsic is not need for the statement. It follows from the claim and since X is intrinsic (a length space) by metric Hopf-Rinow theorem.

We need the following lemma

6.17 Lemma. *Let X be a complete locally compact Alexandrov space with curvature $\geq k$, $p \in X$, and $0 < \lambda < 1$. We define a map $f : X \to X$ as follows. Let $f(x)$ be the point in a shortest path $[px]$ such that $|pf(x)| = \lambda|px|$. Then*

1. *If $k \geq 0$, then $|f(x)f(y)| \geq \lambda \cdot |xy|$ for all $x, y \in X$.*
2. *If $k < 0$, then for every $R > 0$ there is a positive number $c(k, \lambda, R)$ such that $|f(x)f(y)| \geq c(k, \lambda, R)|xy|$ for all $x, y \in B_R(p)$.*

Proof of the lemma. Consider comparison triangles $\Delta\bar{x}\bar{p}\bar{y}$ and $\Delta\overline{f(x)}\bar{p}\overline{f(y)}$. By the angle monotonicity condition for lower curvature it follows that

$$\widetilde{\angle}_k f(x)pf(y) \geq \widetilde{\angle}_k xpy.$$

In particular, it follows that $|f(x)f(y)| = |\overline{f(x)}\ \overline{f(y)}| \geq |\widetilde{x}\widetilde{y}|$ where $\widetilde{x}, \widetilde{y}$ are the points on the segments $[\bar{p}\bar{x}]$ and $[\bar{p}\bar{y}]$ such that $|\bar{p}\widetilde{x}| = \lambda|\bar{p}\bar{x}|$ and $|\bar{p}\widetilde{y}| = \lambda|\bar{p}\bar{y}|$.

For the case $k = 0$, we have that $|\widetilde{x}\widetilde{y}| = \lambda|\bar{x}\bar{y}| = \lambda|xy|$. This is the first claim.

For $k = -1 < 0$ a straightforward computation on 2-dimensional k-plane (using Jacobi fields or Fermi coordinates) yields the optimal estimate

$$|\widetilde{x}\widetilde{y}| \geq \frac{\sinh(\lambda R)}{\sin R}|\bar{x}\bar{y}| = \frac{\sinh(\lambda R)}{\sin R}|xy|.$$

This proves the second claim. □

Proof of the corollary. Let m be the strainer number of X: $m \leq \dim_H X < \infty$. Let $p \in X$ be m-strained. Hence, $\exists\ U$ that is homeomorph to an open set in $\mathbb{R}^m$. Hence U is locally compact. In particular, $\exists r > 0$ such that $B_r(p) \subset U$ is pre-compact.

We show that $B_R(p)$ is pre-compact $\forall R > 0$. This implies X is locally compact.

Assume there exists $R > 0$ such that $B_R(p)$ is not pre-compact. Consequently $\forall \epsilon > 0$ $\exists$ infinite ϵ-separated set $S \subset B_R(p)$.

The previous lemma yields a homothety map $f : B_R(p) \to B_r(p)$ for $\lambda = \frac{r}{R}$, i.e.

$$|f(x)f(y)| \geq c(\lambda, k, R)|xy|$$

and for all $x, y \in S$ we have

$$|f(x)f(y)| \geq \frac{c(\lambda, k, R)}{2}|xy| \geq C(\lambda, k, R)/2\epsilon.$$

Hence $B_r(p)$ contains an infinite set that is ϵ'-separated for $\epsilon' \in (0, C(\lambda, k, R)/2\epsilon)$, hence $B_r(p)$ is not pre-compact. □

6.18 Corollary. *Let X be a Alexandrov space with curvature $\geq k$ (complete, connected, strictly intrinsic). Then the Hausdorff dimension of X equals the strainer number.*

In particular, every Alexandrov space X has integer or infinite Hausdorff dimension.

Proof. Let $m \in \mathbb{N} \cup \{\infty\}$ be the strainer number. We know that $m \leq \dim_H X$. If $m = \infty$ we are done.

Assume first that $\dim_H X < \infty$. Then X is locally compact and $\exists U \subset X$ open bi-Lipschitz homeomorphic to $V \subset \mathbb{R}^m$. It follows that $m = \dim_H U$, and $\exists B_r(x) \subset U$ and $\dim_H B_r(x) = \dim_H U$.

Hence, it suffices to show that $\dim_H(B_r(x)) = \dim_H(X)$. We will show that $\dim_H(B_r(x)) = \dim_H(B_R(x))$ for all $R > r$. The theorem then follows because X can be obtained as union of balls $B_R(x)$ with $R = r+1, r+2, \ldots$. If the Hausdorff dimension of all these balls is equal to $\dim_H(B_r(p))$, it follows by Proposition 1.25 that $\dim_H(X) = \dim_H(B_r(x))$.

For $\forall R > 0\ \exists$ a homothety map $f : B_R(x) \to B_r(x)$ such that $f^{-1} : f(B_R(x)) \to B_R(x)$ is Lipschitz.

The Proposition 1.25 yields

$$\dim_H(B_R(p)) \leq \dim_H(f(B_R(p)) \leq \dim_H(B_r(p)) \leq \dim_H B_R(p).$$

This finishes the proof of the Theorem. □

Recall that a space form is a simply connected complete space of constant curvature, i.e. a sphere, Euclidean space, or a hyperbolic space. For an integer $n \geq 2$ we denote the n-dimensional space form of curvature k by $\mathbb{M}^n_k$. We set $\mathbb{M}^1_k = \mathbb{R}$ if $k \leq 0$ and $\mathbb{M}^1_k = \mathbb{S}^1_{1/\sqrt{k}}$. For $n \geq 1$ fixed let V^k_r and S^k_r be the volume of the r-ball and the $(n-1)$-dimensional area of the r-sphere in $\mathbb{M}^n_k$.

6.19 Theorem. *Let X be a locally compact Alexandrov space of curvature $\geq k$ and $n \in \mathbb{N}$. Then for every $p \in X$ the ratio*

$$\frac{\mathcal{H}^n_X(B_r(p))}{V^k_r}$$

is nonincreasing in r where $\mathcal{H}^n_X$ is the n-dimensional Hausdorff measure. In other words, if $R \geq r > 0$ then

$$\frac{\mathcal{H}^n_X(B_R(p))}{V^k_R} \leq \frac{\mathcal{H}^n_X(B_r(p))}{V^k_r}.$$

Proof for $k = 0$. Let $f : X \to X$ be the (r/R)-homothety map at p. f maps $B_R(p)$ to $B_r(p)$. f is injective and its inverse f^{-1} is Lipschitz with Lipschitz constant R/r. Hence

$$\mathcal{H}^n_X(B_R(p)) \leq \left(\frac{R}{r}\right)^n \mathcal{H}^n_X(B_r(p)).$$

This is the claim. □

Remark. Note that for $k \neq 0$ the same argument proves an inequality of the form

$$\mathcal{X}^n_X(B_R(p)) \leq C(r, R, k, n)\mathcal{H}^n_X(B_r(p))$$

for some nonoptimal constant $C(r, R, k, n)$.

6.20 Corollary. *The Hausdorff measure of a finite dimensional, bounded Alexandrov space is positive and finite.*

Proof. Let $n \in \mathbb{N}$ be the dimension of X. Then there exists an n-strained point p and consequently a bi-Lipschitz map $f : U \ni p \to V \subset \mathbb{R}^n$ for an open set U. Hence, the Hausdorff measure of a ball $B_r(p) \subset U$ is finite.

Consider $R > 0$ such that $B_R(p) = X$. Then the Bishop-Gromov volume monotonicity implies $\mathcal{H}^n_X(X) < \infty$. Moreover $0 < \mathcal{H}^n_X(B_r(p)) \leq \mathcal{H}^n_X(X)$. □

For $k \in \mathbb{R}$, $D > 0$ and $n \in \mathbb{N}$ we define

$$\mathcal{M}(n, k, D)) = \{X : X \text{ is } n\text{-dimensional Alexandrov space with curv} \geq k,\ \text{diam}_X \leq D\}.$$

In particular, for $k > 0$ we set $\mathcal{M}(n, k, \pi/\sqrt{k}) =: \mathcal{M}(n, k)$.

6.21 Theorem (Gromov's compactness theorem)**.** *The class $\mathcal{M}(n, k, D)$ is pre-compact w.r.t. the GH topology.*

Proof. We fix $\epsilon > 0$. By the Bishop-Gromov inequality there exists $c(n, k, D, \epsilon) > 0$ such that

$$\mathcal{H}^n_X(B_r(p)) \geq c(n, k, D, \epsilon)\mathcal{H}^n_X(B_D(p)) = c(n, k, D, \epsilon)\mathcal{H}^n_X(X).$$

for every $p \in X$. It follows, if $B_\epsilon(p_i)$ is a finite collection of N disjoint balls, then

$$\mathcal{H}^n_X(X) \geq \sum_{i=1}^{N} \mathcal{H}^n_X(B_\epsilon(p_i)) \geq Nc(n, k, D, \epsilon)\mathcal{H}^n_X(X).$$

Hence, $N \in \mathbb{N}$ cannot be larger than $c(n, k, D, \epsilon)^{-1} = N_0$, or, in other words, a 2ϵ-separated set cannot contain more than N_0 points. Since a maximal 2ϵ-separated set is an 2ϵ-net, we have that for every $\epsilon > 0$ we find a finite ϵ-net. Thus $\mathcal{M}(n, k, D)$ is uniformly totally bounded, and hence GH precompact. □

6.22 Proposition. *Let $\{X_n\}$ be a sequence of Alexandrov spaces with curvature $\geq k$ and assume $X_n \overset{GH}{\to} X$ for a complete metric space. Assume X is locally compact. Then X is an Alexandrov space with curvature $\geq k$. The same is true for GH limits of pointed spaces.*

Proof. By Theorem 5.29 we have that X is a length space. Since X is locally compact it is strictly intrinsic. Hence it is enough to show the quadruple condition.

Consider a quadruple $(a; b, c, d)$ in X. Since $X_n \overset{GH}{\to}$ there exists a sequence of quadruples $(a_n; b_n, c_n, d_n)$ in X such that $|a_n b_n| \to |ab|$, $|ac_n| \to |ac|$ etc. Then the quadruple condition in the limit follows since the comparison angles $\tilde{\angle}_k(\cdots)$ depend only and continuously on the distance between the involved points. □

6.23 Corollary. *A GH limit of compact Alexandrov spaces with curvature $\geq k$ and dimension not greater than $n \in \mathbb{N}$ is an Alexandrov space with curvature $\geq k$ and dimension not greater than n.*

The same is true for GH limits of pointed spaces.

Remark. In particular $\mathcal{M}(n, k, D)$ is compact w.r.t. the GH distance.

Proof. Let $\{X_i\}$ be a sequence of Alexandrov spaces with curvature $\geq k$, $\dim_H(X_i) \leq n$ $\forall i \in \mathbb{N}$ such that $X_i \overset{GH}{\to} X$ as $i \to \infty$. We have that X is compact and has curvature $\geq k$. Assume that $\dim_H(X) > n$. Since $\dim_H X$ is equal to the strainer number of X. There exists an $(n + 1)$-strained point $p \in X$. We fix an $(n + 1, \epsilon)$-strainer $\{(a_j, b_j)\}_{j=1,\ldots,n+1}$ for p and $\epsilon \in (0, \frac{1}{100(m+1)})$. If the GH distance between X_i and X is small enough, we can find points p', a_j, b_j in X_i whose distances from one another are almost equal to the respective distances between p, a_j, b_j in X. In particular the comparison angles involving these points in X_i are almost equal to the corresponding comparison angles in X (because $\tilde{\angle}$ is a continuous function of the distances). Hence, provided i is large enough, X_i contains an $(n + 1, \epsilon)$-strained point for the same $\epsilon > 0$ as above and hence $\dim_H X_i \geq n + 1$ for i large enough. This is a contradiction. □

6.3 Space of directions

Let X be an Alexandrov space with curvature $\geq k$, $k \in \mathbb{R}$, and $\dim_H X = n \in \mathbb{N}$, and let $p \in X$.

Given two shortest paths γ, σ, parametrized by arc length and starting in p, we can define

$$\angle\gamma\sigma = \lim_{s,t\to 0} \tilde{\angle}_0\gamma(t)p\sigma(s) = \lim_{s,t\to 0} \tilde{\angle}_k\gamma(t)p\gamma(s) \leq \pi.$$

We set $\gamma \sim \sigma$ if and only if $\angle\gamma\sigma = 0$ and define

$$\Sigma_p'\{[\gamma] : \gamma : [0,\epsilon] \to X \text{ shortest path, } \epsilon > 0, \gamma(0) = p\}.$$

Then Σ_p' equipped with $\angle$ is a metric space.

The space of directions Σ_p at p is the completion of Σ_p' w.r.t. $\angle$.

Remark. There can be $x \in \Sigma_p$ that are not represented by a shortest path. For instance, let $X = \overline{B}_1(0) \subset \mathbb{R}^2$. The space of directions at $p = (1,0) \in X$ is $\Sigma_p \simeq [0,\pi] \simeq \mathbb{S}^1 \cap (-\infty, 0] \times \mathbb{R}$ but there are no geodesics in derection of $(0,1)$ or in direction of $-(0,1)$.

We can define $\exp_p : [\gamma] \in \Sigma_p' \mapsto \gamma(1)$ whenever $[\gamma]$ is represented by a geodesic $\gamma : [0,\epsilon] \to X$ for $\epsilon \geq 1$.

More generally, one can consider the Euclidean cone $C(\Sigma_p')$ and define $(r, [\gamma]) \mapsto \gamma(r)$ whenever $[\gamma]$ is represented by geodesic $\gamma : [0,\epsilon] \to X$ such that $\epsilon \geq r$.

The logarithm map $\log_p : X \to C(\Sigma_p)$ is define via $x \mapsto (|px|, [\gamma_x])$ where γ_x is a shortest path between p and x, parametrized by arc length. The logarithm map is in general multivalued.

Remark.

- $\log_p : X \to C(\Sigma_p)$ is noncontracting if $k \geq 0$. Indeed, let $x, y \in X$, then

$$\begin{aligned} |xy|^2 &= |px|^2 + |px|^2 - 2|px||py| \cos \tilde{\angle} xpy \\ &\leq |px|^2 + |px|^2 - 2|px||py| \cos \angle\gamma_x\gamma_y = |\log_p x \log_p y|^2. \end{aligned}$$

- If $k < 0$, one defines $\log_p^k : X \to C^k(\Sigma_p)$ where $C^k(\Sigma_p)$ is the k-cone over Σ_p. Then $\log_p^k$ is noncontracting.

6.24 Proposition. *Let X be a finite dimensional Alexandrov space with curvature $\geq k$. Then Σ_p is compact $\forall p \in X$.*

We first prove the following lemma.

6.25 Lemma. *Let $R > 0$ and $r \in (0,R)$. $\exists C = C(n,k,R) > 0$ such that the following holds if X is an n-dimensional Alexandrov space of curvature $\geq k$, $p \in X$, $\epsilon, r \in (0,1)$, then the ball $B_r(p)$ cannot contain an ϵr-separated set of more than C/ϵ^n points.*

Proof. **1.** We first consider $\mathbb{R}^n$ and a ball $B_r(0) \subset \mathbb{R}^n$. Let $\{p_i\}_{i=1,\dots,N}$ be an ϵr-separated set in $B_r(0)$. It follows that $B_{\epsilon r/2}(p_i)$ is a family of disjoint balls in $B_{2r}(0)$. Moreover $\mathcal{H}^n(B_{\epsilon r/2}(p_i)) = c(n)\left(\frac{\epsilon r}{2}\right)^n = \frac{1}{2}\epsilon^n \mathcal{H}^n(B_{2r}(0))$. Then

$$\mathcal{H}^n(B_2r(0)) \geq \sum_{i=1}^N \mathcal{H}(B_{\epsilon r/2}(p_i)) = N\frac{1}{2}\epsilon^n \mathcal{H}^n(B_{2r}(p)).$$

Hence $N \leq \frac{2}{\epsilon^n}$.

2. Let $p \in X$ be (n, ϵ)-strained and let $f : U = B_r(p) \to f(U) \subset \mathbb{R}^n$ be bi-Lipschitz with $\operatorname{dil} f, \operatorname{dil} f^{-1} \leq c_1$.

If $S \subset B_r(p)$ is ϵr-separated, then $f(S)$ is $\frac{\epsilon r}{c_1}$-separated in $\mathbb{R}^n$ in a ball of radius $c_1 r$. Hence, by the first step $f(S)$ cannot have more than $2c_1^{2n}\frac{1}{\epsilon^n}$ points.

3. Finally let $p \in X$ be arbitrary. The set of (n, ϵ)-strained points is dense in X. Hence, we can choose an (n, ϵ)-strained point q arbitrarily close to p.

We apply the previous step for $\delta = r$ and $U = B_\delta(q)$ with $\delta \in (0, r)$ small. Let $f : B_{2r}(q) \to B_\delta(q)$ be the $\frac{\delta}{2r}$-homothety map, i.e.

$$|f(x)f(y)| \geq \frac{\sinh(\sqrt{-k}\frac{\delta}{2r}r)}{\sinh(\sqrt{-k}r)}|xy| \geq \underbrace{\frac{\sinh(\sqrt{-k}\frac{\delta}{2})}{\sinh(\sqrt{-k}R)}}_{=:C(\delta,k,R)}|xy|.$$

If S is ϵr-separated in $B_r(p) \subset B_{2r}(q)$, then we have that $f(S)$ is $c(\delta, k, R)\epsilon r$-separated in $B_\delta(q)$. Hence the cardinality of S is not larger than $\frac{2c_1^{2n}}{c(\delta,k,R)}\frac{1}{\epsilon^n}$. □

Proof of proposition. Let $S = \{x_i\}_{i=1,\dots,N}$ be an ϵ-separated set in Σ_p and assume $x_i = [\gamma_i]$ (recall Σ'_p is dense in Σ_p).

For $t \downarrow 0$ we have $\tilde{\angle}\gamma_i(t)p\gamma_j(t) \to \angle\gamma_i\gamma_j = |x_i x_j| \geq \epsilon$.

There exists $r > 0$ such that for $t \in (r, 2r)$ we have $\tilde{\angle}\gamma_i(t)p\gamma_j(t) \geq \frac{\epsilon}{2}$ and

$$|\gamma_i(t)\gamma_j(t)|^2 = 2t^2(1 - \cos\tilde{\angle}\gamma_i(t)p\gamma_j(p)) \geq 2r^2(1 - \cos\frac{\epsilon}{2}) = 2r^2 2\sin^2\frac{\epsilon}{4}.$$

Hence

$$|\gamma_i(t)\gamma_j(t)| \geq r\sin\frac{\epsilon}{4} \geq r\frac{\epsilon}{2}.$$

Consequently $\{\gamma_i(t)\}_{i=1,\dots,N}$ is $2r\frac{\epsilon}{4}$-separated in $B_{2r}(p)$. By the previous lemma we have that $N \leq C(k, n, R)/(\frac{\epsilon}{4})^n$ (for some $R > r$).

It follows that a maximal ϵ-separated set has finitely many points. Hence we found finite ϵ-net in Σ_p. □

6.26 Definition. A GH-tangent cone (K, o) in $p \in X$ is the pointed GH limit of $(\frac{1}{r}X, p)$ for $r \downarrow 0$ (if the limite exists).

6.27 Theorem. *Let X be as before. Then the GH tangent cone exists at every $p \in X$ and is isometric to $C(\Sigma_p)$.*

Proof. Let B be the unit ball in $C(\Sigma_p)$. We show that $(B_r(p), \frac{1}{r}d_X, p) = \frac{1}{r}B_r(p)$ converges to B in GH sense for $r \downarrow 0$.

It suffices to show that $\forall\epsilon > 0\ \exists\ \epsilon$-nets $\{x_i\}$ in B and $\{y_i\}$ in $\frac{1}{r}B_r(p)$ and $\{y_i\} \overset{GH}{\to} \{x_i\}$ as $r \downarrow 0$.

We pick an ϵ-net $\{x_i\}$ such that $x_i = (r_i, [\gamma_i]) \in \Sigma'_p$ such that γ_i has speed one.

Let $r > 0$ be small enough such that $\gamma_i(rr_i) = p_i$ is defined $\forall i$. We then have

$$|x_i x_j| = r_i^2 + r_j^2 - 2r_i r_j \cos\angle\gamma_i\gamma_j$$

as well as

$$|p_i p_j|^2 = r^2 \left(r_i^2 + r_j^2 - 2 r_i r_j \cos \widetilde{\angle} p_i p p_j \right).$$

Since $\widetilde{\angle} p_i p p_j \to \angle \gamma_i \gamma_j$ for $r \downarrow 0$, it follows that

$$\left| \frac{|p_i p_j|}{r} - |x_i x_j| \right| < \delta \quad \forall r > 0 \text{ sufficiently small.}$$

Let y_i be the points in $\frac{1}{r} B_r(p)$ that correspond to x_i. Then

$$||y_i y_j| - |x_i x_j|| < \delta \quad \text{for all } r > 0 \text{ sufficiently small.}$$

Hence $\{y_i\} \subset \frac{1}{r} B_r(p) \overset{GH}{\to} \{x_i\} \subset B$ what was to prove. □

6.28 Corollary. *The space of directions Σ_p is an $(n-1)$-dimensional Alexandrove space with curvature ≥ 1.*

Proof. The rescaled space $\frac{1}{r} X$ is an n-dimensional Alexandrov space with curvature $\geq rk$. Hence the pointed limit of $(\frac{1}{r} X, p)$, i.e. $(C(\Sigma_p), o)$ has curvature bouned from below by rk for every $r > 0$, and hence has curvature bounded from below by 0. It follows that Σ_p has curvature ≥ 1.

Moreover $\dim_H C(\Sigma_p) \leq n$. On the other hand $\log_p : B_1(p) \to B$ satisfies

$$|\log_p x \log_p y| \geq C(k)|xy|.$$

Hence $n \dim_H B_1(p) \leq \dim_H B$. Hence $C(\Sigma_p)$ has Hausdorff-dimension n and by another contradiction argument involving strainers we get $\dim_H \Sigma_p = n - 1$. □

Remark. The corollary allows to prove statements about finite dimensional Alexandrov space via induction over the dimension.