Riemannsche Geometrie

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Der vorliegende Text ist ein Skript zur Vorlesung "Riemannsche Geometrie", die der Autor im Sommersemester 2023 am Mathematischen Institut der Universität Freiburg gelesen hat. Auf Wunsch der Zuhörer wurde die Vorlesung in englischer Sprache gehalten und das Skript ab Seit 7 auf Englisch fortgesetzt.

The given text are lecture notes for the lecture "Riemannian Geometry" which the author taught during the summer semester of 2023 at the Mathematical Institute of the University of Freiburg. In response to the audience's request, the lecture was conducted in the English, and the notes continue in English from page 7 onwards.

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Historische Anmerkungen zur Entwicklung der Riemannschen Geometrie

- Die Riemannsche Geometrie ist eine natürliche Erweiterung der Differentialgometrie für Flächen in $\mathbb{R}^3$.

  J. Gauß(1777-1855): Theorema Egregium (Die Gaußkrümmung ist eine Größe der inneren Geometrie einer Fläche)

- B. Riemann (1826-1866) skizziert in seiner Antrittvorlesung (1854) (*Über die Hypothesen welche der Geometrie zu Grunde liegen*) abstrakte Räume, in denen man Längen und Winkel messen kann. Heute sprechen wir von einer Riemannschen Metrik.

- A. Einstein (1879-1955) wendet das Konzept der Riemannschen Metrik in einer veränderten Form an, um seine Allgemeine Relativitätstheorie zu entwickeln (1915).

- Der Begriff der Mannigfaltigkeit in der heutigen form wurde 1913 durch H. Weyl (1885-1955) eingeführt, eine formale Präzisierung der Arbeit von Riemann.

Literatur

- Mandred P. do Carmo *Riemannian Geometrie*
- John Lee *Riemannian Manifolds*
- Jeff Cheeger, David G. Ebin *Comparison Theorems in Riemannian Geometrie*

Themen der Vorlesung

- Definition Riemannsche Metrik, Beispiele, Eigenschaften
- Riemannscher Zusammenhang
- Krümmungsbegriffe
- Einführung des Begriffs der Geodätischen; Existenz kürzester Kurven
- Jacobi Gleichungen
- Räume mit konstanter Krümmung
- Varation der Bogenlänge, der Satz von Myers
- Krümmungsvergleichssätze: Rauchscher Vergleichsatz
- Isometrische Immersionen, Fundamentalgleichungen

1 Differenzierbare Mannigfaltigkeiten

Wir wiederholen zunächst kurz einige grundlegenden Definitionen.

Seien U und V offene Teilmengen der topologischen Räume X und Y. Eine Abbildung $\varphi : U \to V$ heißt Homeomorphismus, falls φ bijektiv ist und φ sowie φ^{-1} stetig sind.

Sei M ein topologischer Raum.

1. Eine n-dimensionale Karte von M ist eine Homeomorphismus $\varphi : U \to \varphi(U) = V$, wobei $U \subset M$ und $V \subset \mathbb{R}^n$ offen Teilmengen sind.

2. Ein n-dimensionaler C^0-Atlas $\mathcal{A}$ von M ist eine Familie von n-dimensionalen Karten $\varphi_i : U_i \to V_i$ für i aus einer Indexmenge I, so dass $\bigcup_{i\in I} U_i = M$.

3. Zwei Karten φ_1, φ_2 heißen C^k-kompatibel für $k \in \mathbb{N} \cup \{\infty\}$, falls der *Koordinatenwechsel*
$$\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1\cap U_2)} : \varphi_1(U_1 \cap U_2) \subset \mathbb{R}^n \to \varphi_2(U_1 \cap U_2) \subset \mathbb{R}^n$$
ein C^k-Diffeomorphismus ist. Ein n-dimensionaler C^0-Atlas $\mathcal{A}$ heißt C^k-Atlas, falls alle Karten C^k-kompatibel sind.

4. Ein n-dimensionaler C^k-Atlas $\mathcal{A}$ heißt maximal, wenn jede n-dimensionale Karte, die mit den Karten in $\mathcal{A}$ C^k-kompatibel ist, bereits zu $\mathcal{A}$ gehört.

1.1 Remark. Jeder maximale C^k-Atlas $\mathcal{A}$ enhält einen maximalen C^∞-Atlas $\mathcal{A}'$. Ein maximaler C^∞-Atlas $\mathcal{A}$ heißt differenzierbare Struktur.

1.2 Definition. Eine n-dimensionale differenzierbare Mannigfaltigkeit ist ein topologischer Raum M mit abzählbarer Umgebungsbasis so dass die Hausdorff-Eigenschaft erfüllt ist und es gibt eine differenzierbare Struktur.

Bemerkung.

- Eine topologischer Raum X erfüllt die Hausdorff-Eigenschaft bzw. heißt Hausdorffsch falls für je 2 Punkte $x, y \in X$ mit $x \neq y$ zugehörige offene Umgebungen U und V existieren, so dass $U \cap V = \emptyset$.

- Ein System offener Mengen $\mathcal{B}$ in einem topologischen Raum X heißt Basis der Topologie, falls jede offene Menge Vereinigung von Mengen aus $\mathcal{B}$ ist.

1.3 Definition. Seien M, N differenzierbare Mannigfaltigkeiten der Dimension m bzw. n. Eine Abbildung $F : M \to N$ heißt C^k-differenzierbar in $p \in M$, falls es Karten $\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^m$ und $\psi : V \subset N \to \psi(V) \subset \mathbb{R}^n$ gibt mit

1. $p \in U$ und $F(U) \subset V$

2. $\psi \circ F \circ \varphi^{-1}$ is C^k-differenzierbar in $\varphi(p) \in \mathbb{R}^m$.

Die Abbildung F ist in der Klasse $C^k(M, N)$ für $k \in \mathbb{N} \cup \{\infty\}$, falls F in jedem Punkt $p \in M$ C^k-differenzierbar ist.

Im Fall $N = \mathbb{R}$ bezeichnen wir $C^k(M, N)$ mit $C^k(M)$. Ist $F \in C^\infty(M, N)$ bezeichnen wir F einfach als differenzierbare Abbildung.

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Sei M eine n-dimensionale differenzierbare Mannigfaltigkeit (Mgft).

1.4 Definition. Eine differenzierbare Abbildung $\alpha : (-\epsilon, \epsilon) \to M$ heißt Kurve in M. Sei $\alpha(0) = p$ und $\mathcal{D}_p$ die Menge der Funktionen $f : M \to \mathbb{R}$ welche differenzierbar in p sind.

Der Tangentenvektor der Kurve α in $t = 0$ ist eine lineare Abbildung $\alpha'(0) : \mathcal{D}_p \to \mathbb{R}$ gegeben durch

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}.$$

Ein Tangentenvektor in $p \in M$ ist der Tangentenvektor einer Kurve $\alpha : (-\epsilon, \epsilon) \to M$ in $t = 0$ mit $\alpha(0) = p$. Die Menge aller Tangentenvektoren in p ist T_pM.

Bezeichnung. Sei $\varphi : U \subset M \to \varphi(U)$ eine Karte mit $p \in U$ und $\varphi(p) = x_0 \in \mathbb{R}^n$. Wir können die Kurve $\alpha(t) = \varphi^{-1}(x_0^1, \dots, x_0^{i-1}, x_0^i + t, x_0^{i+1}, \dots, x_0^n)$ betrachten. Den dazugehoerigen Tangentialvektor bezeichnen wir mit

$$\alpha'(0) =: \frac{\partial^\varphi}{\partial x^i}\Big|_p =: \frac{\partial}{\partial x^i}\Big|_p.$$

Für $f \in \mathcal{D}_p$ berechnen wir

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0} = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(x_0).$$

T_pM ist ein n-dimensionaler Vektorraum und nach Wahl einer Karte φ is $\left(\frac{\partial}{\partial x^i}\Big|_p\right)_{i=1,\dots,n}$ eine Basis von T_pM.

1.5 Proposition. *Seien M und N differenzierbare Mannigfaltigkeiten und $F : M \to N$ eine differenzierbare Abbildung. Zu $p \in M$ und $v \in T_pM$ wählen wir eine differenzierbare Kurve $\alpha : (-\epsilon, \epsilon) \to M$ mit $\alpha(0) = p$ und $\alpha'(0) = v$. Sei $\beta = F \circ \alpha$. Die Abbildung*

$$DF_p : T_pM \to T_{F(p)}N, \quad DF_pv = \beta'(0)$$

ist eine lineare Abbildung und hängt nicht von der Wahl von α für v ab.

Die lineare Abbildung DF_p heißt differential von F in p. Für $f \in C^1(M)$ schreiben wir $df_p = DF_p$.

1.6 Definition. Sei M eine differenzierbare Mannigfaltigkeit. Die Menge

$$TM = \bigcup_{p \in M} T_pM$$

heißt Tangentialbündel über M. TM ist eine differenzierbare Mannigfaltigkeit der Dimension $2n$.

TM ist ein Vektorbündel über M mit Projektionsabbildung

$$\pi : TM \to M, \quad \pi(v) = p \iff v \in T_pM.$$

π is eine differenzierbare Abbildung.

Sei T_p^*M der Dualraum von T_pM. Die Menge $T^*M = \bigcup_{p\in M} T_p^*M$ heißt Kotangentialbündel. Ist eine Karte $\varphi : U \to \varphi(U)$ gegeben, dann ist $D\varphi_p^i = d\varphi_p^i$, $i = 1, \dots, n$, $p \in U$, eine Basis von T_p^*M dual zu $\frac{\partial}{\partial x^i}\Big|_p$.

Für eine Karte schreibt man auch $\varphi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$, also $\varphi^i = x^i$. Dann ist $d\varphi^i = dx^i$.

Wir definieren auch

$$T_p^{(r,s)}M = \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{r\text{-mal}} \otimes \underbrace{T_pM \otimes \cdots \otimes T_pM}_{s\text{-mal}}$$

und

$$T^{(r,s)}M = \bigcup_{p\in M} T_p^{(r,s)}M.$$

Eine Abbildung $T : M \to T^{(r,s)}M$ heißt Tensorfeld, falls $T(p) \in T_p^{(r,s)}M$ $\forall p \in M$. Wir schreibene $T(p) =: T_p =: T\big|_p$.

2 Riemannsche Metriken

Betrachten wir ein $(2,0)$-Tensorfeld g und eine Karte $\varphi : U \to V \subset \mathbb{R}^n$. Es gilt

$$g|_U = \sum_{i,j=1}^{n} g_{ij} d\varphi^i \otimes d\varphi^j$$

wobei

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) : U \to \mathbb{R}.$$

g heißt differenzierbar (C^∞), falls die Funktionen $g_{ij} : U \to \mathbb{R}$ differenzierbar (C^∞) sind für jede Karte φ. Wir schreiben $g \in \Gamma(T^{(2,0)}M)$.

2.1 Definition. Eine Riemann'sche Metrik g auf M ist ein C^∞ $(2,0)$-Tensorfeld, so dass

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

nicht ausgeartet, symmetrisch und positiv definit ist. Das Paar (M, g) heißt Riemannsche Mannigfaltigkeit.

Bemerkung. Sei V ein n-dimensionaler $\mathbb{R}$-Vektorraum und $b : V \times V \to \mathbb{R}$ eine symmetrische Bilinearform. Ist $e_1, \dots, e_n$ eine Basis von V, dann sei $b_{ij} = b(e_i, e_j)$.

1. b heißt nicht ausgeartet, falls $\det{(b_{ij})}_{i,j=1,\dots,n} \neq 0$ für eine (jede) Basis.

2. b heißt positiv (negativ) definit, falls $b(v,v) > 0 (< 0)$ $\forall v \in V \backslash \{0\}$.

3. $\mathrm{Ind}(b) = \max\{\dim U | U \subset V$ Untervektorraum, so dass $b|_{U\times U}$ negativ definit$\}$
   Also $\mathrm{Ind}(b) = 0 \Rightarrow$ positiv definit.

2.2 Definition. Sei (M,g) eine Riemannsche Mannigfaltigkeit, und $c : [a,b] \to M$ eine stückweise C^1 Kurve, d.h. $\exists\ a = t_0 \leq \cdots \leq t_N = b$, so dass $c|_{[t_{i-1},t_i]}$ stetig differenzierbar ist $\forall i = 1,\ldots,N$. Es sei

$$g_{c(t)}(c'(t),c'(t))^{\frac{1}{2}} = |c'(t)|^g = |c'(t)|.$$

Die (Bogen)länge von c bzgl. g ist definiert durch

$$L^g(c) := \int_a^b |c'(t)|dt.$$

Bemerkung. Sei $\varphi : [\widetilde{a},\widetilde{b}] \to [a,b]$ bijektive, stückweise C^1 und $\widetilde{c} = c\circ\varphi$. Es gilt $L(\widetilde{c}) = L(c)$.

Betrachte g_p und eine ONB $(e_1,\ldots,e_n)$ in (T_pM, g_p). Seien $v_1,\ldots,v_n \in T_pM$. Dann gibt es $a_{ij} \in \mathbb{R}$, so dass $v_i = \sum_{j=1} a_{ij}e_j$ und $D(v_1,\ldots,v_n) := \det(a_{ij})_{i,j=1,\ldots,n}$ definiert eine n-form auf T_pM (eine n-multilineare Abbildung $T_pM^n \to \mathbb{R}$). Die Zahl $D(v_1,\ldots,v_n)$ ist das orientierte Volumen des Parallelotops, das von $v_1,\ldots,v_n$ aufgespannt wird).

Sei nun $v_i = \frac{\partial}{\partial x^i}\Big|_p$, $i = 1,\ldots,n$. Wir berechnen

$$g_{ij} = g_p(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p) = \sum_{k,l=1}^n a_{ik}a_{jl}g_p(e_l,e_k) = \sum_{k,l=1}^n a_{ik}a_{jl}.$$

Also $(g_{ij}) = (a_{ij})\cdot(a_{ij})$. Somit folgt aus dem Determinanten-Produktsatz

$$|\det(a_{ij})| = \sqrt{|\det(g_{ij})|}.$$

2.3 Definition. Sei $\varphi : U \to V$ eine Karte und $A \subset U$ meßbar, d.h. $\varphi(A) \subset \mathbb{R}^n$ ist messbar. Dann definieren wir

$$\mathrm{vol}^g(A) =: \mathrm{vol}(A) = \int_{\varphi(A)} |\det(g_{ij})|^{\frac{1}{2}} \circ \varphi^{-1}(x)\, \underbrace{d\mathcal{L}^n(x)}_{dx}.$$

Allgemeiner können wir eine meßbare Zerlegung $(A_\alpha)_{\alpha\in\Lambda}$ von M wählen (mit Λ abzählbar), d.h. $A_\alpha \cap A_\beta$ hat Maß 0, $M\setminus\bigcup_{\alpha\in\Lambda} A_\alpha$ hat Maß 0, und für jedes α existiert eine Karte $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ mit $A_\alpha \subset U_\alpha$. Für eine meßbare Menge $A \subset M$ definieren wir dann

$$\mathrm{vol}^g(A) = \sum_{\alpha\in\Lambda} \mathrm{vol}^g(A \cap A_\alpha).$$

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Let M be a manifold and let (N,h) be a Riemannian manifolds. We consider a map $F : M \to N$ that is smooth (C^∞, differentiable). The *pull-back metric* of h under F is a $(2,0)$-tensor field on M defined by

$$(F^*h)_p(v,w) = h_{F(p)}(DF_p v, DF_p w) \ \forall p \in M \text{ and } v,w \in T_pM.$$

2.4 Lemma. *$g = F^*h$ is a Riemannian metric on M if and only if F is an immersion. g is called induced metric on M.*

2.5 Remark. A smooth map $F : M \to N$ is an immersion if $DF_p : T_pM \to T_{F(p)}N$ is injective as a linear map $\forall p \in M$. In particular it follows $\dim_M \leq \dim_N$. If, in addition, F is a homeomorphism onto $\varphi(M) \subset N$, where $\varphi(M)$ has the standard subspace topology induced from N, we call F an embedding.

Proof. For $p \in M$ fixed F^*h_p is positive definit since $DF_p v \neq 0 \ \forall v \in T_pM\backslash\{0\}$, and F^*h_p is also non-degenerated. Let $\varphi : U^\varphi \to V^\varphi$ be a chart and $p \in U$. Moreover let $\psi : U^\psi \to V^\psi$ be a chart in a neighborhood of $F(p)$. We set $U = U^\varphi \cap F^{-1}(U^\psi)$ that is still an open neighborhood of p. We compute the local represenation of $F^*h = g$ on U. Let $i,j \in \{1,\dots,\dim_M\}$ and $q \in U$. Then

$$g_{ij}(q) = g_q(\frac{\partial^\varphi}{\partial x^i}|_q, \frac{\partial^\varphi}{\partial x^j}|_q) = h_{F(q)}(DF_q \frac{\partial^\varphi}{\partial x^i}|_q, DF_q \frac{\partial^\varphi}{\partial x^j}|_q).$$

Recall that $DF_q \frac{\partial^\varphi}{\partial x^i}|_q = \frac{\partial(\psi\circ F\circ\varphi^{-1})^k}{\partial x^i} \circ \varphi(q) \frac{\partial^\psi}{\partial x^k}|_q$. Plugging this back into the previous formula shows that g^φ_{ij} is a smooth functions on $U \ni p$. Since $p \in U^\varphi$ was arbitrary, g^φ_{ij} is a smooth function on U^φ. $\square$

2.6 Example. Assume $M \subset N$ is an immersed or an embedded submanifold of (N,h) and let $\iota : M \to N$ be the inclusion map. The induced metric on M is the pull-back metric $\iota^*h = g$. With this metric M is called a Riemannian (immersed or embedded) submanifold.

2.7 Example (Metrics in graph coordinates). Let $U \subset \mathbb{R}^n$ be open and $f \in C^\infty(M)$. The graph of f is the set $\mathrm{graph}(f) = \{(x,f(x)) : x \in U\} \subset \mathbb{R}^{n+1}$ which is an embedded submanifold of dimension n of $\mathbb{R}^n$. The map $X : U \to \mathbb{R}^{n+1}$, $X(x) = (x,f(x))$, is an embedding and the induced Riemannian metric on U is

$$g = X^*g^{eucl} = \sum_{i=1}^n dx^i \otimes dx^i + df \otimes df.$$

2.8 Remark. Recall the symmetric product between 1-forms $\alpha,\beta \in \Gamma(T^*M)$: $\alpha \vee \beta = \frac{1}{2}(\alpha\otimes\beta + \beta\otimes\alpha)$. Given a chart $\varphi = (x^1,\dots,x^n)$ on a Riemannian manifold (M,g), we can write for the local representation of g:

$$g = \sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j = \frac{1}{2}\sum_{i,j=1}^n \left(g_{ij}dx^i \otimes dx^j + g_{ij}dx^j \otimes dx^i\right) = \sum_{i,j=1}^n g_{ij} dx^i \vee dx^j.$$

2.9 Definition. A (C^∞)-diffeomorphism $F : M \to N$ (i.e. F is a differentiable bijection with a differentiable inverse) is called an isometry if

$$F^*h = g \tag{1}$$

A map $F : M \to N$ is called a local isometry at $p \in M$ if $\exists$ a neighborhood $U \subset M$ of p such that $F : U \to F(U)$ is a diffeomorphism satisfying (1).

2.10 Remark. Let (M, g) be a Riemannian manifold. g induces a vector bundle isomorphism $\flat : TM \to T^*M$ given by

$$\flat|_{T_pM} : T_pM \to T_p^*M \quad \text{via} \quad \flat(v)(w) = g_p(v, w) \;\forall v, w \in T_pM.$$

The inverse map of $\flat$ is $\sharp : T^*M \to TM$. For instance, given a chart $\varphi : U \to V$ on M it follows

$$T_pM \ni v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}|_p \quad \longrightarrow \quad b(v) = \sum_{i,j=1}^n g_{ij}(p) v^i dx_p^j.$$

This follows since

$$\flat(v)(\frac{\partial}{\partial x^j}) = g(v, \frac{\partial}{\partial x^j}) = \sum_{i=1}^n v^i g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \sum_{i=1}^n v^i g_{ij}.$$

Hence, the coefficients of the cotangent vector $\flat(v)$ w.r.t. the ONB dx^j are $\sum_{i=1}^n g_{ij} v^i =: v_j$. Similarly, we can compute the coefficients of $\sharp(\alpha)$ for $\alpha \in T_p^*M$. We consider the represenation of $\sharp(\alpha)$ w.r.t. the basis $\frac{\partial}{\partial x^i}$: $\sharp(\alpha) = \sum_{i=1}^n w^i \frac{\partial}{\partial x^i}$. Then

$$v_i = \alpha(\frac{\partial}{\partial x^i}) = g(\sharp(\alpha), \frac{\partial}{\partial x^i}) = \sum_{i=1}^n w^i g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

where v_i is the coefficient of α w.r.t. dx^i. Hence $\sum_{k=1}^n g^{ij} v_i = w^j$ where (g^{ij}) is the inverse matrix of (g_{ij}).

The maps $\flat$ and $\sharp$ are called the musical isomorphisms between TM and T^*M because they lower and raise the indices of the coefficient functions.

Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Recall that the differential $df_p : T_pM \to \mathbb{R}$, $p \in M$, of f is defined as $df_p(v) = v(f)$ $\forall v \in T_pM$. Given a chart φ one has

$$df_p = \sum_{j=1}^n \frac{\partial(f \circ \varphi^{-1})}{\partial x^j} \circ \varphi(p) dx_p^j.$$

df is a smooth $(1, 0)$-tensor field, a smoooth 1-form.

2.11 Definition. The smooth vector field

$$\sharp(df) =: \nabla f =: \text{grad} f \in \Gamma(TM)$$

is called gradient of f (w.r.t. g). In local coordinates $\varphi : U \to V$ the gradient ∇f writes as

$$\nabla f|_U = \sum_{i,j=1}^n g^{ij} \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \circ \varphi \frac{\partial}{\partial x^j}.$$

2.12 Remark. Let $f \in C^\infty(M)$. If $r \in f(M)$ is a regular value of f ($r \in f(M)$ is a regular value $\Leftrightarrow df_p \neq 0\ \forall p \in f^{-1}(\{r\})$), then $f^{-1}(\{r\}) = N$ is a $n-1$-dimensional submanifold of M.

Claim. $g_p(\nabla f(p), v) = df_p(v) = v(f) = 0\ \forall p \in N$ and $\forall v \in T_pN$.

Given $v \in T_pM$ we write $v^\perp$ for all $w \in T_pM$ with $g_p(v, w) = 0$. Hence $\nabla f(p)^\perp = T_pN$.

Proof. Let $c : (-\epsilon, \epsilon) \to N$ with $c'(0) = v$. Then $f \circ c \equiv r$ by definiiton of N and hence

$$g_p(\nabla f(p), v) = v(f) = \frac{d(f \circ c)}{dt}\Big|_{t=0} = 0.$$

$\square$

2.13 Example. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x^1, \ldots, x^n) = \sum_{i=1}^n (x^i)^2 - 1$. Then 0 is a regular value of f and $f^{-1}(0) = \{x \in \mathbb{R}^n : \sum_{i=1}^n (x^i)^2 = 1\} =: \mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$. The metric on $\mathbb{S}^{n-1}$ that is induced by the Euclidean metric is called canonical metric of $\mathbb{S}^{n-1}$ or standard metric.

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Riemannian distance We consider a Riemannian manifold (M, g).

A C^1 curve $c : [a, b] \to M$ is called regular if $c'(t) \neq 0$ for all $t \in [a, b]$.

We call a curve $c : [a, b] \to M$ piecewise regular if $\exists \{a = t_0 \leq \dots t_N = b\} = P$ such that $\gamma|_{[t_{i-1}, t_i]}$ is a regular C^1 curve. We call such a partition P as before admissible.

A reparametrization of c is a homeomorphism $\varphi : [\widetilde{a}, \widetilde{b}] \to [a, b]$ such that there is a partition $\widetilde{a} = t_0 \leq \dots t_N = \widetilde{b}$ of $[\widetilde{a}, \widetilde{b}]$ such that $\varphi|_{(t_{i-1}, t_i)}$ is a diffeomorphism onto its image. In particular we have that $L(c \circ \varphi) = L(c)$.

Since φ is a homeomorphism between intervals it is either increasing or decreasing. In the first case we call φ a forward reparametrization and otherwise backward reparametrization. If φ is a C^1 diffeomorphism, one has $\varphi(t) > 0$ in the first case and otherwise $\varphi(t) < 0$, for all $t \in (\widetilde{a}, \widetilde{b})$.

If c is differentiable in $t \in [a, b]$, we define the speed of c in t as $|c'(t)| = |c'(t)|_g = g(c'(t), c'(t))^{\frac{1}{2}}$. We say a C^1 curve c is a unit speed curve if $|c'(t)| = 1\ \forall t$ and constant speed if $|c'(t)| = const$. If c is piecewise C^1 we say c has unit speed if $|c'(t)| = 1$ whenever c is differentiable in t.

The arc-length function of a piecewise C^1 curve $c : [a, b] \to M$ is defined as

$$s(t) = L(c|_{[a,t]}) = \int_a^t |c'(s)| ds.$$

2.14 Lemma.

1. *Every regular curve $c : [a, b] \to (M, g)$ has unit speed forward reparametrization.*

2. *Every piecewise regular curve has a unique forward reparametrization by arc length.*

Proof. (1) We choose $t_0 \in [a, b]$ and define $s : [a, b] \to \mathbb{R}$ by

$$s(t) = \int_{t_0}^t |c'(s)|_g ds.$$

Since $s'(t) = |c'(t)| > 0$, it follows that s is an increasing local C^1 diffeomorphimus and thus a C^1 diffeomorphism from $[a, b]$ to an interval $[\widetilde{a}, \widetilde{b}] \subset \mathbb{R}$. We define $\varphi = s^{-1} : [\widetilde{a}, \widetilde{b}] \to [a, b]$. Hence φ is forward reparametrization and for $\widetilde{c} = c \circ \varphi$ we compute

$$|\widetilde{c}'(t)| = |\varphi'(t) c'(\varphi(t))| = |\varphi'(t)||c'(\varphi(t))| = \frac{1}{|s'(\varphi(t))|} |c'(\varphi(t))| = 1.$$

Hence $\widetilde{c}$ is a unit speed reparametrization of c.

If c is piecewise regular, we prove the existence statement for the reparametrization by induction on the number of smooth segments N for an admissible partition. If there is only one segment, then the statement follows by (1). Assume the statement is true for partitions with N segments. If $c : [a, b] \to M$ is a piecewise regular curve such that $\exists a = t_0 \leq \dots \leq t_{N+1} = b$ with $c|_{[t_{i-1}, t_i]}$ is regular. Then there exists the desired reparametrizations $\varphi : [0, c] \to [a, t_N]$ and $\psi : [0, d] \to [t_N, b]$ for $c|_{[a, t_N]}$ and $c|_{[t_N, b]}$ respectively. Then we define

$$\widetilde{\psi}(s) = \begin{cases} \varphi(s) & s \in [0, c], \\ \psi(s - c) & s \in [c, c + d]. \end{cases}$$

Then $\widetilde{\varphi} : [0, d+c] \to [a,b]$ is a desired reparametrization for c.

We prove uniqueness. If $\widetilde{c} = c \circ \widetilde{\varphi}$ and $\hat{c} = c \circ \widetilde{\varphi}$ are both forward reparametrizations of c by arc length. Since $\widetilde{c}$ and $\hat{c}$ have the same arc length and have both speed 1, they are defined on intervals of the same length $L(c)$. Up to translation $\widetilde{\varphi}$ and $\hat{\varphi}$ are therefore both homeomorphisms from $[0, L(c)]$ to $[a,b]$. If we define $\eta = \widetilde{\varphi}^{-1} \circ \psi$, then η is a piecewise regular increasing homeomorphism that satisfies $\hat{c} = c \circ \varphi \circ \eta = \widetilde{c} \circ \eta$. For all $s \in [0, L(c)]$ except for finitely many, where $\widetilde{\gamma}$, $\hat{\gamma}$ and η are not smooth, we can compute

$$1 = |\hat{c}'(s)| = |\widetilde{c}'(\eta(s))\eta'(s)| = |\widetilde{c}'(\eta(s))|\eta'(s) = \eta'(s).$$

Since η is continuous and $\eta(0) = 0$, it follows $\eta(s) = s$ for all $s \in [0, L(c)]$. Hence $\widetilde{c} = \hat{c}$. $\square$

One of the most important concepts in Riemannian geometry is the distance between points that we can define as follows.

2.15 Definition (Riemannian distance function)**.** Let $p, q \in (M, g)$ and define

$$d_g(p,q) = \inf L(c)$$

where we take the infimum w.r.t. all piecewise regular curves $c : [a,b] \to M$ with $c(a) = p$ and $c(b) = q$.

The following lemma guarantees that d_g is well-defined as long as M is connected.

2.16 Lemma. *If M is a connected smooth manifold then for any two points in M there exists a piecewise regular curve that connects them.*

Proof. Let $p, q \in M$. Since a connected manifold is path-connected, p and q can be joined by a continuous path $c : [a,b] \to M$. By compactness of $[a,b]$ and its image $c([a,b])$ there exists a partition $\{a = t_0 \leq \cdots \leq t_N = b\}$ such that $c([t_{i-1}, t_i])$ is contained in the domain of a single smooth coordinate chart $\varphi : U \to V$. We can also assume that V is a ball. Therefore we can replace each such segment by the image under φ^{-1} of straight line in U. This yields a piecwise regular curve between q and q. $\square$

2.17 Theorem. *Let (M, g) be a connected Riemannian manifold. (M, d_g) is a metric space whose topology is the same as the given topology of the manifold.*

Proof. By definition $d_g(p,q) \geq 0$ $\forall p, q \in M$ and $d_g(p,p) = 0$ as well as symmetry in p and q.

The triangle inequality follows because given piecewise regular curves $c : [a,b] \to M$ and $\widetilde{c} : [\widetilde{a}, \widetilde{b}] \to M$ such that $c(b) = \widetilde{c}(\widetilde{a})$, then

$$\hat{c}(s) = \begin{cases} c(s) & s \in [a,b] \\ \widetilde{c}(s - b + \widetilde{a}) & s \in [b, b + \widetilde{b} - \widetilde{a}] \end{cases}$$

is a piecewise regular curve as well. Since $L(\hat{c}) = L(c) + L(\widetilde{c})$, it follows

$$d_g(c(a), \widetilde{c}(\widetilde{b})) \leq L(\hat{c}) = L(c) + L(\widetilde{c}).$$

Hence, taking the infimum w.r.t. c and $\widetilde{c}$ yields the triangle inequality.

We need to show that $d_g(p,q) > 0$ if $q \neq p$. We first prove the following Lemma.

2.18 Lemma. *Let $V \subset \mathbb{R}^n$ be open, let g be a Riemannian metric on V and let g_{euc} be the Euclidean Riemannian metric on V. Let $K \subset V$ be compact. Then, $\exists c, C$ such that*

$$c|v|_{g_{euc}} \leq |v|_g \leq C|v|_{g_{euc}} \quad \forall x \in K \text{ and } \forall v \in T_xU \simeq \mathbb{R}^n \text{ with } v \neq 0.$$

2.19 Remark. If $\varphi : U \subset M \to V$ is a chart of (M, g), then we can consider the Riemannian metric $(\varphi^{-1})^*g = h$ on V. In particular $\varphi : (U, g) \to (V, h)$ is an isometry.

Proof of the Lemma. Define the continuous function $(x, v) \mapsto |v|_g = g_x(v, v)^{\frac{1}{2}}$ on the compact set

$$L = \{(x, v) \in T_xU : x \in K, |v|_{g_{euc}}\} = K \times \mathbb{S}^{n-1}.$$

Hence, there exist $c, C > 0$ such that

$$c \leq g_x(v, v)^{\frac{1}{2}} \leq C \text{ on } L.$$

If $v \in T_xU$ with $x \in K$ is arbitrary, consider $w = \frac{v}{|v|_{g_{euc}}}$. Then $|w|_{g_{euc}} = 1$, it follows $w \in L$ and by homogeneity of $g_x(v, v)^{\frac{1}{2}}$ in v it follows

$$c \leq \frac{|w|_g}{|w|_{g_{euc}}} \leq C.$$

This is the claim. □

We can now finish the proof of the Theorem. Let $p \neq q \in M$.

We pick a chart $\varphi : U \to V \subset \mathbb{R}^n$ with $p \in U$, $\varphi(p) = p$ and such that $V = B_{2R}(0)$ for $R > 0$. We consider $\overline{B_R(0)} =: K$ and assume the claim of Lemma 2.18 for the pull-back metric $(\varphi^{-1})^*g = \widetilde{g}$ on V.

We treat two cases. Assume first $q \notin \varphi^{-1}(K) = \widetilde{U}$. Let $\gamma : [0, 1] \to M$ be a piecewise C^1 curve with $\gamma(0) = p$ and $\gamma(1) = q$, and set $\tau = \sup\{t > 0 : \gamma(s) \in \widetilde{U} \forall s \in [0, t]\}$. Consider $\widetilde{\gamma} = \varphi \circ \gamma$. It follows

$$\begin{aligned} L^g(\gamma) = \int_0^1 |\gamma'(t)|_g dt \geq \int_0^\tau |\gamma'(t)|_g dt = \int_0^\tau |\widetilde{\gamma}'(t)|_{\widetilde{g}} dt \\ \geq c \int_0^\tau |\widetilde{\gamma}'(t)|_{eucl} dt \geq c|\widetilde{\gamma}(\tau)| \geq cR. \end{aligned}$$

Since γ was arbitrary, it follows $d_g(p, q) \geq cR > 0$.

Now let $q \in \widetilde{U}$ and let γ be as before. If there exists $\tau \in (0, 1)$ such that $\gamma(\tau) \notin \widetilde{U}$, then we get as in the first case, that $L^g(\gamma) \geq cR$.

If $\gamma(t) \in \widetilde{U}$ for all $t \in (0, 1)$, it follows

$$L^g(\gamma) = L^{\widetilde{g}}(\widetilde{\gamma}) \geq c \int_0^1 |\widetilde{\gamma}'(t)|_{eucl} dt \geq c|\widetilde{\gamma}(1)|_{eucl} = c|\varphi(q)|_{eucl} > 0.$$

So we always have $L^g(\gamma) \geq c|\varphi(q)|_{eucl}$. Taking the infimum again yield $d_g(p, q) > 0$. □

03.05.2023

Consider $\mathbb{R}^{n+1}$ together with a symmetric, non-degenerated bilinear form $\langle\cdot,\cdot\rangle$, not necessarily positive definit. The index of $\langle\cdot,\cdot\rangle$ is defined as

$$\operatorname{ind}_{\langle\cdot,\cdot\rangle} = \max\{\dim_U \in \mathbb{N} : \langle\cdot,\cdot\rangle|_{U\times U} \text{ is negative definit}\}.$$

2.20 Example. Consider $\langle v,w\rangle_1 := -v^0w^0 + \sum_{i=1}^n v^1w^1$. We write $\mathbb{R}_1^{n+1} = (\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle_1)$. $\operatorname{ind}_{\langle\cdot,\cdot\rangle_1} = 1$ and $\mathbb{R}_1^{n+1}$ is called $n+1$-dimensional Minkowski space.

Given a non-degenerated, symmetric bilinear form $\langle\cdot,\cdot\rangle$ we set $U^\perp = \{v \in \mathbb{R}^{n+1} : \langle v,u\rangle = 0\ \forall u \in U\}$.

2.21 Fact.

1. *If $n+1>0$, then $\exists v \in \mathbb{R}^{n+1}$ with $\langle v,v\rangle \neq 0$.*
2. *Let $U \subset \mathbb{R}^{n+1}$ be a linear subspace and $\langle\cdot,\cdot\rangle|_{U\times U}$ negative definite. Then* $\dim_U = \operatorname{ind}_{\langle\cdot,\cdot\rangle} \Leftrightarrow \langle\cdot,\cdot\rangle|_{U^\perp\times U^\perp}$ *positive definite.*
3. *Let $U \subset \mathbb{R}^{n+1}$ be a linear subspace with $U\cap U^\perp = \{0\}$. Then $\exists$ basis $e_0,\dots,e_n$ of $\mathbb{R}^{n+1}$ and $\epsilon_0,\dots,\epsilon_n \in \{\pm1\}$ such that* $\operatorname{span}\{e_0,\dots,e_k\} = U$ *and* $\operatorname{span}\{e_{k+1},\dots,e_n\} = U^\perp$ *and $\langle e_i,e_j\rangle = \epsilon_i\delta_{ij}\ \forall i,j \in \{0,\dots,n\}$.*

Proof. (1) If $\langle v,v\rangle = 0\ \forall v \in V$, then

$$\langle u,v\rangle = \frac{1}{2}\left(\langle u+v,u+v\rangle - \langle v,v\rangle - \langle u,u\rangle\right)\ \forall u,v \in V.$$

Hence $\langle\cdot,\cdot\rangle \equiv 0$.
(2) $\Rightarrow$: We show that $U\cap U^\perp = 0$. If $u \in U\cap U^\perp$, then $\langle u,u\rangle = 0$. Since $\langle\cdot,\cdot\rangle|_{U\times U}$ negative definit, it follows $u=0$.
Hence $\dim_U = n+1-\dim_{U^\perp}$.
We show that $\langle v,v\rangle \geq 0\ \forall v \in U^\perp$ (positive semi-definit). Otherwise $\exists v \in U^\perp$ with $\langle v,v\rangle < 0$, and therefore $\angle\cdot,\cdot\rangle|_{W\times W}$ is negative definit with $U + \operatorname{span}(v) = W$ which contradicts the definition of $\operatorname{ind}_{\langle\cdot,\cdot\rangle}$.
We show $\langle\cdot,\cdot\rangle$ is non-degenerated on $U^\perp$. Let v in $U^\perp$ with $\langle v,u\rangle = 0\ \forall u \in U^\perp$. Since $U+U^\perp = \mathbb{R}^{n+1}$, $\langle v,u\rangle = 0\ \forall u \in \mathbb{R}^{n+1}$. From $\langle\cdot,\cdot\rangle$ non-degenerated it follows that $v=0$.
It follows that $\langle\cdot,\cdot\rangle$ is positive definit on $\mathbb{R}^{n+1}$.
$\Leftarrow$: Exercise.
(3) Exercise. One can use (2). □

2.22 Corollary. *Let $v \in \mathbb{R}_1^{n+1}$ with $\langle v,v\rangle_1 < 0$ ("v is timelike"). Then $\langle\cdot,\cdot\rangle_1|_{v^\perp\times v^\perp}$ is positive definit.*

2.23 Example (Lorentz model of hyperbolic space). The subset

$$H^n(r) = \left\{x \in \mathbb{R}^{n+1} : \langle x,x\rangle_1 = -r^2, x_0 > 0\right\}$$

of $\mathbb{R}^{n+1}$ is an n-dimensional Riemannian manifold.
To see this we consider $f(x) = \langle x,x\rangle_1$ that is smooth. Then $df_x(v) = 2\langle x,v\rangle_1$. In particular, if $x \neq 0$, then $df_x \neq 0$. Hence $-r^2 \neq 0$ is a regular value of the function f, and therefore $f^{-1}(\{-r^2\}) = H^n(r)$ is a smooth manifold (a hypersurface). We note that

$$\forall x \in H^n(r)\colon\ T_xH^n(r) = \ker df_x = \{w \in \mathbb{R}^{n+1}_1 : \langle x, w\rangle_1 = 0\} = \{x\} \times \{x\}^\perp.$$

With the previous corollary one has that $\langle \cdot, \cdot\rangle|_{\{x\}^\perp \times \{x\}^\perp}$ is positive definit.

Consider also the inclusion map $i : H^n(r) \to \mathbb{R}^{n+1}$, $i(x) = x$. Then

$$i^*\langle v, w\rangle_1|_x = \langle Di|_x v, Di|_x w\rangle_1|_x = \langle v, w\rangle_1 \text{ for } v, w \in T_xH^n(r).$$

We set $\mathbb{H}^n := H^n(1)$.

2.24 Definition. The Riemannian manifold $\mathbb{H}^n$ with $g = i^*\langle \cdot, \cdot\rangle_1$ is called Lorentz model of the hyperbolic Riemannian space.

2.25 Remark. $\mathbb{H}^n$ is diffeomorphic to $\mathbb{R}^n$: the map $x \in \mathbb{R}^n \mapsto (\sqrt{1 + |\widetilde{x}|^2}, \widetilde{x}) \in H^n$ is a diffeomorphism.

2.26 Definition. A Riemannian manifold (M, g) is called frame homogeneous ("Raum freier Beweglichkeit") if the following holds: Let $x, y \in M$ and let $(v_1, \dots, v_n)$, $(w_1, \dots, w_n)$ be ONB of T_xM and T_yM respectively w.r.t. g. Then there exists an isometry $F : (M, g) \to (M, g)$ such that $F(x) = y$ and $DF_xv_i = w_i\ \forall i = 1, \dots, n$.

2.27 Proposition. $(H^n, i^*\langle \cdot, \cdot\rangle_1)$ *is frame homogeneous.*

Proof. Let $x, y \in H^n$ and $(v_1, \dots, v_n)$, $(w_1, \dots, w_n)$ ONB of T_xH^n and T_yH^n respectively. $\Rightarrow$ $(x = v_0, v_1, \dots, v_n)$ and $(y = w_0, w_1, \dots, w_n)$ are ONBs of $\mathbb{R}^{n+1}_1$. Then

$$\exists A \in O(n + 1, 1) = \{A \in GL(n, \mathbb{R}) : \langle Av, Aw\rangle_1 = \langle v, w\rangle\}$$

s.t. $Av_i = w_i\ \forall i = 0, \dots, n$.

In particular $A(H^n(1)) = H^n(1)$ (by definition of $H^n(1)$).

We define $F = A|_{H^n(1)}$. Then $F : H^n(1) \to H^n(1)$ is a diffeomorphism with $F^{-1} = A^{-1}|_{H^n(1)}$ and $F(x) = y$ as well as

$$DF_xv_i = w_i\ \ \forall i = 1, \dots, n$$

since $DF_xv_i(g) = v(g \circ F) = v(g \circ A) = (g \circ A \circ c)'(0) = dg(Ac'(0)) = dgw_i = w_i(g)$ for $g \in C^\infty(H^n(1))$.

Moreover F is an isometry of H^n, since

$$F^*i^*\langle v, w\rangle_1 = \langle Av, Aw\rangle_1 = \langle v, w\rangle_1.$$

□

2.28 Remark. $(\mathbb{R}^n, \langle \cdot, \cdot\rangle_{eucl})$ is a frame homogeneous, und ebenso $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle_{eucl} = 1\}$.

Let $x, y \in \mathbb{R}^n$, und $(v_1, \dots, v_n), (w_1, \dots, w_n)$ be ONBs.

To see this let $x, y \in \mathbb{R}^n$ and let $(v_i)_{i=1,\dots,n}, (w_i)_{i=1,\dots,n}$ ONBs at $\mathbb{R}^n$. Choose $A \in O(n)$ such that $Av_i = wi\ \forall i = 1, \dots, n$, and define $F(z) = A(z - x) + y$. Moreover $DF_x = A$.

In the case of $\mathbb{S}^n$ note that $T_x\mathbb{S}^n = x^\perp$ and $A\mathbb{S}^n = \mathbb{S}^n\ \forall A \in O(n + 1)$. We choose $A \in O(n + 1)$ now such that $Ax = y$ and $Av_i = w_i\ \forall i = 1, \dots, n$ for ONBs (v_i) and (w_i) of $T_x\mathbb{S}^n$ and $T_y\mathbb{S}^n$ respectively.

2.29 Definition. The Riemannian manifold $\mathbb{B} = (M, g)$ where $M = B_1(0) = \{x \in \mathbb{R}^n : \langle x, x\rangle_{eucl} < 1\}$ und $g^P_x = \frac{4}{(1-|x|)^2_{eucl}} \langle \cdot, \cdot \rangle_{eucl}$ is called Poincaré model of the hyperbolic space.

2.30 Lemma. *The map $F : \mathbb{H}^n \to \mathbb{B}^n$, $F(x_0, \widetilde{x}) = \frac{1}{1+x_0}\widetilde{x}$, is an isometry between $\mathbb{H}^n$ and $\mathbb{B}^n$.*

Proof. The inverse map is $F^{-1}(y) = \frac{1}{1-|y|^2}(1 + |y|^2, 2y_1, \ldots, 2y_n)$. Hence F is a diffeomorphism.
We show that $F^*g^P = i^*\langle\cdot,\cdot\rangle_1$. More precisely

$$g^P_{F(x)}(DF_x v, DF_x w) = \langle v, w\rangle_1$$

$\forall (x, v), (x, w) \in T_x\mathbb{H}^n$ ($\Leftrightarrow$ $\langle x, v\rangle_1 = 0 = -x_0v_0 + \sum_{i=1}^n x_iv_i\rangle$).

$$DF_x v = \frac{d}{dt}\Big|_{t=0} F(x + tv) = \frac{1}{1 + x_0}\widetilde{v} - \frac{v_0}{(1 + x_0)^2}\widetilde{x}$$

and

$$DF_x w = \frac{1}{1 + x_0}\widetilde{w} - \frac{w_0}{(1 + x_0)^2}\widetilde{x}.$$

It follows

$$\begin{aligned}
&g^P_{F(x)}(DF_x v, DF_x w)\\
&= \underbrace{\frac{4}{(1 - |F(x)|)^2}}_{(1+x_0)^2}\left(\frac{1}{(1 + x_0)^2}\sum_{i=1}^n v_iw_i + \frac{v_0w_0}{(1 + x_0)^4}|\widetilde{x}|^2 - \frac{1}{(1 + x_0)^3}\Big(\underbrace{w_0\sum_{i=1}^n x_iv_i}_{v_0x_0w_0} + \underbrace{v_0\sum_{i=1}^n x_iw_i}_{v_0x_0w_0}\Big)\right)\\
&= \sum_{i=1}^n v_iw_i + v_0w_0\left(\frac{|\widetilde{x}|^2}{(1 + x_0)^2} - \frac{2x_0}{1 + x_0}\right) = \langle(v_0, \widetilde{v}), (w_0, \widetilde{w})\rangle_1.
\end{aligned}$$

□

2.31 Definition. The Riemannian mfd $H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$ with the metric $g^H_x = \frac{1}{x_n^2}\langle\cdot,\cdot\rangle_{eucl}|_x$ is called half space model of the hyperbolic space.

08.05.23

Let (M, g) be a Riemannian manifold and d_g the induced distance function, i.e.

$$d_g(p, q) = \inf_\gamma L^g(\gamma)$$

where $\gamma : [a, b] \to M$ is piecewise regular with $\gamma(a) = p$ and $\gamma(b) = q$.

End of the proof of 2.17 Theorem. It remains to show that the metric topology of d_g is the same as the manifold topology. We will show the following first:
For $p \in M$ and $W \subset M$ open with $p \in W$ there exists a chart $\varphi : U \to V = B_{2R}(0)$ and $C, D > 0$ such that $\varphi(p) = 0$, $U \subset W$ and the following is satisfied:

- If $q \in U' = \varphi^{-1}\left(\overline{B_R(0)}\right)$, then $d_g(p, q) \leq C d_{\bar{g}}(p, q)$ where $\bar{g} = \varphi^*\langle\cdot,\cdot\rangle_{eucl}$ on U'.

- If $q \notin U'$, then $d_g(p,q) \geq D$.

Indeed we can find a chart $\varphi : U' \to V$ such that $p \in U$, $U \subset W$ such that $\varphi(p) = 0 \in V = B_{2R}(0)$.

- Consider $q \in U' = \varphi^{-1}\left(\overline{B_R(0)}\right)$ and $c(t) = t\varphi(q)$. Then $L^{eucl}(c) = |\varphi(q)|_{eucl}$. Consider the pull-back metric $(\varphi^{-1})^* g$ on V. Wie Lemma 2.18 we compute

$$L^{(\varphi^{-1})^* g}(c) = \int_0^1 |c'(t)|_{(\varphi^{-1})^* g} dt \leq C \int_0^1 |c'(t)|_{eucl} dt = L^{eucl}(c) = |\varphi(q)|_{eucl}.$$

  Moreover, the map $\varphi : U \to V$ is an isometry between (U, g) and $(V, (\varphi^{-1})^* g)$ as well between $(U, \bar{g} = (\varphi)^* g_{eucl})$ and (V, g_{eucl}). Hence $|\varphi(q)|_{eucl} = d_{\bar{g}}(p,q)$ and

$$L^{(\varphi^{-1})^* g}(c) = L^g(\varphi^{-1} \circ c) \geq d_g(p,q)$$

  since $\varphi^{-1} \circ c$ is a C^1 curve between p and q. Then the first claim follows.

- We showed this already before.

Consider $U \subset M$ open. For every $p \in U$ we can choose a coordinate chart $\varphi : U \to V \subset \mathbb{R}^n$ as before. The second statement in the previous lemma implies that for $q \in M$ with $d_g(p,q) < D$ it follows $q \in U' \subset U$. Hence $B_D(p) \subset U$.

If $A \subset M$ is open w.r.t. the metric topology, then $\exists \epsilon > 0$ such that $B_\epsilon^{d_g}(p) \subset A$. Let W and $\varphi : U \to V$ be as in the previous lemma. Choose $\delta > 0$ small enough such that $C\delta < \epsilon$. By the first statement in Lemma 2.18 it follows that $\varphi^{-1}\left(B_\epsilon^{eucl}(0)\right) \subset B_\delta(p) \subset A$. Since $\varphi^{-1}\left(B_\epsilon^{eucl}(0)\right)$ is open w.r.t. the manifold topology, the claim follows. $\square$

Some more examples

1. *Products.* Consider Riemannian manifolds (M_0, g_0) and (M_1, g_1). The product manifold $M_0 \times M_1$ inherits the natural Riemannian metric $g := g_0 \oplus g_1$ defined via

$$g_{(p_0,p_1)}(v,w) = g_0|_{p_0}(D\pi_0|_{(p_0,p_1)} v, D\pi_0|_{(p_0,p_1)} w) + g_1|_{p_1}(D\pi_1|_{(p_0,p_1)} v, D\pi_1|_{(p_0,p_1)} w)$$

   where $\pi_i : M_i \to M_0 \times M_1$ is $\pi_i((p_0,p_1)) = p_i$, $i = 0,1$ and $D\pi_i|_{(p_0,p_1)} : T_{(p_0,p_1)}(M_0 \times M_1) \to T_{p_i} M_i$. The map

$$D(\pi_0, \pi_1)_{(p_0,p_1)} = (D\pi_0|_{p_0}, D\pi_1|_{p_1}) : T_{(p_0,p_1)}(M_0 \times M_1) \to T_{p_0} M_0 \times T_{p_1} M_1$$

   is a vector space isomorphism and hence we can identify each $v \in T_{(p_0,p_1)}(M_0 \times M_1)$ with $(v_0, v_1) \in T_{p_0} M_0 \times T_{p_1} M_1$. We then write for $(v_0,v_1), (w_0,w_1) \in T_{(p_0,p_1)} M_0 \times M_1$:

$$g_{(p_1,p_2)}((v_0,v_1),(w_0,w_1)) = g_0|_{p_0}(v_0,w_0) + g_1|_{p_1}(v_1,w_1).$$

   Given charts $\varphi_i : U_i \to V_i \subset \mathbb{R}^{n_i}$, $i = 0,1$, then each metric g_i has the local expression $g_i|_{U_i} = \sum_{k,l=1}^{n_i} (g_i)_{kl} d\varphi_i^k \otimes d\varphi_i^l$. Recall $\varphi_i = (\varphi_i^1, \ldots, \varphi_i^{n_i})$.

   A chart on $M_0 \times M_1$ is given by $(\varphi_0, \varphi_1) : U_0 \times U_1 \to V_0 \times V_1 \subset \mathbb{R}^{n_0+n_1}$ and local expression of g in this chart has the following coefficient matrix

$$(g_{kl})_{k,l=1,\ldots,n_0+n_1} = \begin{pmatrix} (g_0)_{kl}, k,l = 1,\ldots,n_0 & 0 \\ 0 & (g_1)_{kl}, k,l = 1,\ldots,n_1 \end{pmatrix}.$$

Example: n-Torus.
Consider $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with the restricted metric $g = i^*\langle \cdot, \cdot \rangle_{eucl}$. Then the Riemannian manifold

$$\mathbb{T}^n = \Big(\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n-\text{times}}, \underbrace{g \oplus \cdots \oplus g}_{n-\text{times}} \Big)$$

is called n-torus.

2. *Warped Products.* Again let $(M_i, g_i), i = 0, 1$, be Riemannian manifolds and let $f : M_0 \to (0, \infty)$. The warpred product $M_0 \times_f M_1$ is the Riemannian manifold given by the product $M_0 \times M_1$ together with the Riemannian metric $g := g_0 \oplus f^2 g_1$ defined by
$$g_{(p_0,p_1)}((v_0, v_1), (w_0, w_1)) := g_0|_{p_0}(v_0, v_0) + f(p_0)^2 g_1|_{p_1}(v_1, w_1)$$
where $(v_0, v_1), (w_0, w_1) \in T_{p_0}M_0 \times T_{p_1}M_1 \simeq T_{(p_0,p_1)}(M_0 \times M_1)$.
Important examples of warped products are
Surfaces of Revolution.
Let C be an embedded smooth 1-dimensional submanifold in $H := \{(x,z) \in \mathbb{R}^2 : x > 0\}$. C equipped with the restricted Euclidean metric $i^*\langle \cdot, \cdot \rangle_{eucl}$ is a Riemannian manifold. The surface of revolution determined by C is the subset
$$S_C = \{(x,y,z) : \left(\sqrt{x^2 + y^2}, z\right) \in C\} \subset \mathbb{R}^3.$$
Let $\psi^{-1} = c : (\alpha, \omega) \to H$ be a chart of C, i.e. $c = (a, b)$ is regular curve, and if $|c'| \equiv 1$, then c is an isometry. Consider the map
$$X(t, \theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t)),\ \ X : (\alpha, \omega) \times \mathbb{S}^1 \to C$$
where we identify $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with $\mathbb{R}/2\pi\mathbb{Z}$ via $\theta \mapsto (\cos\theta, \sin\theta)$.
The pull-back Riemannian metric is
$$X^*\langle \cdot, \cdot \rangle_{eucl}(v_0, v_1, w_0, w_1) = \langle DXv, DXw \rangle_{eucl} = \sum_{i=1}^{3} (dx^i)^2(DXv, DXw) = (*)$$
where $(v_0, v_1), (w_0, w_1) \in T_{(t,\theta)}((\alpha, \omega) \times \mathbb{S}^1)$. Note that
$$DX_{t,\theta} = \begin{pmatrix} a'(t)\cos\theta & -a(t)\sin\theta \\ a'(t)\cos\theta & a(t)\sin\theta \\ b'(t) & 0 \end{pmatrix}.$$
Inserting this back in the previous equation and a computation yields
$$(*) = (\underbrace{a'(t)^2 + b'(t)^2}_{=1})v_0 w_0 + a(t)^2 v_1 w_1 = (dt)^2(v_0, w_0) + a(t)^2 (d\theta)^2(v_1, w_1).$$
Hence, the restricted Euclidean metric on S_C is isometric to the warped product $(\alpha, \beta) \times_a \mathbb{S}^1$ where $\mathbb{S}^1$ is equipped with the metric $d\theta^2$.
$\mathbb{R}^n \backslash \{0\}$ *as warped product.*
Consider $(0, \infty) \times_f \mathbb{S}^{n-1}$ with $f(t) = t$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x|_{eucl} = 1\}$ equipped with the restricted Euclidean metric. Then $\Phi(t, \theta) = t \cdot \theta \in \mathbb{R}^n \backslash \{0\}$ is an isometry between $(0, \infty) \times_f \mathbb{S}^{n-1}$ and $\mathbb{R}^n \backslash \{0\}$ (Exercise).

15.05.2023

3 Connections

Consider a k-dimensional vector bundle $\pi : E \to M$ over M (e.g. the tangent bundle TM). In particular E is a smooth manifold, π is a smooth map, and $\pi^{-1}(\{x\}) = E_x$ is a linear space of dimension k for all $x \in M$.

$$\Gamma(E) := \{s \in C^\infty(M,E) : \pi \circ s = \mathrm{id}_M\}$$

is the family of smooth sections (e.g. $\Gamma(TM)$ vector field).

Question: How can we differentiate a section $s \in \Gamma(E)$ in direction of a vector field $V \in \Gamma(TM)$?

Note that we can already differentiate functions $f \in C^\infty(M)$ in direction of a vector $V \in \Gamma(TM)$ via

$$V(f)(p) = \frac{d(f \circ c)}{dt}|_{t=0} =: \nabla_V f|_p$$

where $c : (-\epsilon, \epsilon) \to M$ such that $c'(0) = V_p$.

The operator $(V,f) \in \Gamma(TM) \times C^\infty(M) \mapsto \nabla_V f \in C^\infty(M)$ has the following properties:

(1) ∇ is bilinear, i.e. $\forall V, W \in \Gamma(TM)$, $\forall f, f_0, f_1 \in C^\infty(M)$ and $\forall \alpha, \beta \in \mathbb{R}$

$$\nabla_{\alpha V} f = \nabla_V(\alpha f) = \alpha \nabla_V f,\ \nabla_{V+W} f = \nabla_V f + \nabla_W f \ \&\ \nabla_V(f_0 + f_1) = \nabla_V f_0 + \nabla_V f_1.$$

(2) ∇ is C^∞-homogeneous in V, i.e. $\nabla_{gV} f = g \nabla_V(f)$.

(3) ∇ satisfies the product rule

$$\nabla_V(g \cdot f) = V(g)f + g\nabla_V f, \quad \forall V \in \Gamma(TM),\ \forall f, g \in C^\infty(M).$$

3.1 Definition. A linear connection (or covarariant derivative, or gauge potential) on a vectorbundle $\pi : E \to M$ is an operator

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (V,s) \mapsto \nabla_V s$$

that satisfies the previous three points (1), (2) and (3) where $f, f_0, f_1 \in C^\infty$ are replaced with $s, s_0, s_1 \in \Gamma(E)$.

Most relevant for us will be connections on TM.

3.2 Example. Consider the trivial bundle $E = M \times \mathbb{R}^k \to M$ with $\pi(x,v) = x$. The sections $E_1, \dots, E_k$ with $E_i|_x = (x, e_i)$, $\forall x \in M$, satisfy that $(E_1|_x, \dots, E_k|_x)$ is a basis of $\{x\} \times \mathbb{R}^k = \pi^{-1}(\{x\}) \simeq \mathbb{R}^k$ for every $x \in M$. Hence, every $s \in \Gamma(E)$ has a unique representation as $s = \sum_{i=1}^k s^i E_i$ with $s^i \in C^\infty(M)$.

($s^i \in C^\infty(M)$ since s is a smooth map M from to E.)

The standard connection on E is defined as

$$\bar{\nabla}_V s := \sum_{i=1}^k V(s^i) E_i.$$

The 3 properties (1), (2) and (3) follows directly from the corresponding properties for $V(f) = \nabla_V f$, $f \in C^\infty(M)$ and $V \in \Gamma(TM)$.

This example for instance applies to the tangent bundle $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

3.3 Example. Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. We know that $TM \subset T\mathbb{R}^n$, i.e. $T_xM \subset T_x\mathbb{R}^n \simeq \mathbb{R}^n$ is m-dimensional linear subspace. There is the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. Then

$$(x, v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \text{ with } x \in M \quad \Rightarrow \quad (x, v) = (x, v^\top) + (x, v^\perp)$$

where $(x, v^\top) \in T_xM$ and $(x, v^\perp) \in (T_xM)^\perp$ in $\{x\} \times \mathbb{R}^n$ where for $A \subset T_xM$ the orthogonal complement is $A^\perp = \{v \in T_x\mathbb{R}^n : \langle v, w \rangle = 0,\ \forall w \in A\}$.

Consider the standard connection $\bar{\nabla}$ on $T\mathbb{R}^n$ and $v \in T_pM$ and $X \in \Gamma(TM)$.

First we find an extenstion $\bar{X} \in \Gamma(T\mathbb{R}^n)$ such that $\bar{X}|_M = X$.

For that we pick $\varphi = (x^1, \dots, x^n) : U \to V$ such that $\varphi(U \cap M) = V \cap \mathbb{R}^m \times \{0\} =: W$. A vector field on W is defined by

$$D\varphi X|_{\varphi^{-1}(x)} = (Y^1, \dots, Y^m, 0, \dots, 0)$$

where $Y^i = X(x^i)(\varphi^{-1}(x))$ defined for $x \in W$. Let $P(x^1, \dots, x^n) = (x^1, \dots, x^m, 0, \dots, 0)$ be the projection map, define $\bar{Y}^i(y) = Y^i(P(y))$ and $\bar{X}(p) = D\varphi^{-1}\bar{Y}|_{\varphi(p)}$ for $p \in U$. By definition it follows $\bar{X}|_p = X_p$ for $p \in M \cap U$. By partition of unity we find an extension of X in a neighborhood of M.

The connection $\bar{\nabla}$ induces a connection on TM as follows

$$V, X \in \Gamma(TM) \mapsto \nabla_V X := (\bar{\nabla}_{\bar{V}} \bar{X})^\top.$$

This does not depend on the extension $\bar{X}$, since we can write $V(\bar{X}^i)(p) = (\bar{X}^i \circ c)'(0) = (X^i \circ c)'(0)$ for a curve $c : (-\epsilon, \epsilon) \to M$ for $c'(0) = V_p$.

Let M be an m-dimensional smooth manifold. We can consider

$$\bigcup_{p \in M} \operatorname{End}(T_pM) =: \operatorname{End}(TM) \simeq T^*M \otimes TM$$

where $\operatorname{End}(T_xM) = \{A : T_xM \to T_xM \text{ linear}\}$. This is a smooth manifold.

Note that given a chart $\varphi : U \to V$, $\varphi(p) = (x^1, \dots, x^m)$ the local representation of $T \in T_p^*M \otimes T_pM$ is $\sum_{i,j=1}^m a_i^j dx^i|_p \otimes \frac{\partial}{\partial x^j}|_p$. The coefficient matrix $(a_i^j)_{i,j=1,\dots,m}$ determines $A \in \operatorname{End}(T_pM)$.

3.4 Lemma. *Let ∇ be a connection on TM. Let $X \in \Gamma(TM)$. Then $\exists L \in \Gamma(\operatorname{End}(TM))$ such that $(\nabla_V X)|_p = L_p V_p$. In particular $\nabla_V X|_p$ depends only on the value of V in p.*

Notation. We will also write $\nabla_V X|_p =: \nabla_v X$ where $v := V_p$.

Proof. Let $V, \bar{V} \in \Gamma(TM)$ with $V_p = \bar{V}_p = v$. We have to show that $\nabla_V X|_p = \nabla_{\bar{X}} X|_p$ for every $X \in \Gamma(TM)$. Then we can set $L_p v := \nabla_V X$.

Since $V \to \nabla_V$ is linear, it is enough to show that $0 = V_p$ implies $\nabla_V X|_p = 0$.

We pick a chart $\varphi = (x^1, \dots, x^m) : U \to V$ on M around p such that we can write

$V|_U = \sum_{i=1}^m V(x^i)\frac{\partial}{\partial x^i}$.
Now we choose a function $\lambda \in C^\infty(M)$ such that $\operatorname{supp}\lambda \subset U$ and $\lambda(p) = 1$. We write

$$\lambda^2 V|_U = \sum_{i=1}^m (\lambda V(x^i))(\lambda \frac{\partial}{\partial x^i}).$$

Note here that $\lambda \cdot V(x^i) \in C^\infty(M)$ is well defined since λ is smooth and has compact support in U, and also $\lambda\frac{\partial}{\partial x^i} \in \Gamma(TM)$ is a well-defined vector field on M.
Then it follows with the $C^\infty(M)$ homogeneity of ∇ that

$$\begin{aligned}\nabla_{\lambda^2 V} X = \lambda^2 \nabla_V X \ \Rightarrow\ \nabla_V X|_p &= \lambda(p)\nabla_V X|_p = \nabla_{\lambda^2 V} X|_p \\ &= \sum_{i=1}^m \lambda(p) \underbrace{V(x^i)(p)}_{=0 \text{ da } V_p = 0} \nabla_{\lambda\frac{\partial}{\partial x^i}} X|_p = 0.\end{aligned}$$

Hence $L_p v = \nabla_V X|_p$ with $V_p = v$ is well-defined. The matrix representation in local coordinates of $p \mapsto L_p$ is $(a_i^j(p))_{i,j=1,\dots,m}$ where these coefficients are defined via

$$L_p \frac{\partial}{\partial x^i} = \nabla_{\lambda\frac{\partial}{\partial x^i}} X|_p = \sum_{i,j=1}^m v^i \underbrace{\left(\nabla_{\frac{\partial}{\partial x^i}} X\right)|_p(x^j)}_{a_i^j(p)} \frac{\partial}{\partial x^j}.$$

Hence $p \mapsto L_p$ is a smooth section of $\operatorname{End}(TM)$, that is $L \in \Gamma(\operatorname{End}(TM))$. □

3.5 Lemma. *Let $X_0, X_1 \in \Gamma(TM)$ such that $X_0|_U = X_1|_U$ for $U \subset M$ open. Then $\nabla_X X_0|_U = \nabla_X X_1|_U$ for all $X \in \Gamma(TM)$ (i.e. ∇_X is a local operator).*

Proof. Since $\nabla_V X$ is linear in X, it suffices to show that $X|_U = 0$ implies $\nabla_V X|_U = 0$. We choose $\lambda \in C^\infty(M)$ such that $\operatorname{supp}\lambda \subset U$ and $\lambda(p) = 1$ for $p \in U$. Then it follows

$$(1-\lambda)X = X \quad\Rightarrow\quad \nabla_V X|_p = \nabla_V (1-\lambda)X|_p = V_p(1-\lambda)X_p + (1-\lambda)(p)\nabla_V X|_p = 0.$$

□

3.6 Definition (Christoffel Symbols)**.** Let ∇ be a linear connection on TM and let $E_1, \dots, E_m : U \subset M \to TM|_U$ vector field defined on U such that $(E_1(p), \dots, E_m(p))$ is a basis for every $p \in M$ (for instance $E_i = \frac{\partial}{\partial x^i}$ for a chart $\varphi = (x^1, \dots, x^n) : U \to V$). One calls $E_1, \dots, E_m$ a local frame. Then $\nabla_{E_i} E_j|_U = \sum_{k=1}^m \Gamma_{ij}^k E_k$ where Γ_{ij}^k are called Christoffel symbols of ∇ w.r.t. $E_1, \dots, E_m$.

3.7 Remark. For $X, Y \in \Gamma(TM)$ and let $E_1, \dots, E_m$ be a local frame. Then we can write $X|_U = \sum_{i=1}^m X^i E_i$, $Y|_U = \sum_{i=1}^m Y^i E_i$ and

$$\begin{aligned}\nabla_X Y|_U = \sum_{i,j=1}^m X^i \nabla_{E_i}(Y^j E_j)|_U &= \sum_{i,j=1}^m X^i E_i(Y^j) E_j + X^i Y^j \nabla_{E_i} E_j \\ &= \sum_{k=1}^m X(Y^k) E_k + \sum_{i,j,k=1}^m X^i Y^j \Gamma_{ij}^k E_k.\end{aligned}$$

In particular, we see that $\nabla_X Y$ only depends on $Y \circ c : (-\epsilon, \epsilon) \to TM$ for some curve c such that $c'(0) = X$.

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Let Γ_{ij}^k be the Christoffel Symbols of a connection ∇ on TM w.r.t. to a local frame $E_1, \dots, E_m$ on $U \subset M$. If E_i, $i = 1, \dots, m$, are the coordinate vectorfields $\frac{\partial}{\partial x^i}$ of a chart $\varphi = (x^1, \dots, x^m) : U \to V$, then we also write ${}^\varphi\Gamma_{ij}^k$.

Let $e^1, \dots, e^m$ be the local frame for T^*M dual to $E_1, \dots, E_m$. The Christoffel symbols $\Gamma_{ij}^k, i = 1, \dots, m$, define 1-forms via

$$\omega_j^k = \sum_{i=1}^m \Gamma_{ij}^k e^i$$

on U that are called connection 1-forms of ∇ w.r.t. $E_1, \dots, E_m$. Then we can write

$$\nabla_V X|_U = \sum_{k=1}^m \left(V(X^k) + \sum_{j=1}^m X^j \omega_j^k(V) \right) E_k.$$

Notations. If $U \subset M$ is open, $V \in \Gamma(TM)$ and $X \in \Gamma(TU)$, then we define $p \in U \mapsto \nabla_V^U X|_p = \nabla_V(\lambda X)|_p$ where $\lambda \in C^\infty(M)$ with $\operatorname{supp} \lambda \subset U$ and $\lambda \equiv 1$ in a neighborhood of p in U.

3.8 Theorem (Levi-Civita connection). *Let (M, g) be a Riemannian manifold. Then there exists exactly one linear connections on TM such that $\forall V, X, Y \in \Gamma(TM)$ the following properties hold:*

(1) $\nabla_X Y - \nabla_Y X = [X, Y]$ *(∇ is torsionfree, or symmetric)*
$[X, Y] \in \Gamma(TM)$ *denotes the Lie bracket, i.e.* $[X, Y](f) = X(Y(f)) - Y(X(f))$.

(2) $V(g(X, Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y)$ *(∇ is a Riemannian connection)*
We note that $p \in M \mapsto g(X, Y)(p) = g_p(X_p, Y_p) \in C^\infty(M)$ *since g is smooth.*

This connection ∇ is defined through

$$\begin{aligned} 2g(\nabla_X Y, Z) = Xg(Y, Z) &+ Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &+ g(Y, [Z, X]) + g(Z, [X, Y]) = \Theta_{X,Y}(Z). \end{aligned} \tag{2}$$

That is $\nabla_X Y = \frac{1}{2} \sharp \Theta_{X,Y}$.

If $\varphi = (x^1, \dots, x^m)$ is a chart, it follows for the Christoffel symbols ${}^\varphi\Gamma_{ij}^k$ of ∇ w.r.t. $\frac{\partial}{\partial x^i}$, $i = 1, \dots, m$:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^m g^{kl} \left(\frac{\partial}{\partial x^j} g_{il} + \frac{\partial}{\partial x^i} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right)$$

where $(g^{kl})_{k,l=1,\dots,m}$ is the inverse matrix of $(g_{ij})_{i,j=1,\dots,m}$.

From this formula we can see that ${}^\varphi\Gamma_{ij}^k$ is symmetric in i, j.

Proof. Uniqueness. Assume we have a connection with the two properties (1) and (2). Then for $X, Y, Z \in \Gamma(TM)$ we compute the following:

$$\begin{aligned}
Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) = g(\nabla_Y Z, X) + g(Z, \nabla_X Y) - g(Z, [X,Y]) \\
Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = g(\nabla_X Z, Y) - g([X,Z], Y) + g(X, \nabla_Z Y).
\end{aligned}$$

This yields the formula (2).

If we show that the right hand side in (2) defines a 1-form. Then we can define $\nabla_X Y := \frac{1}{2}\sharp\Theta_{X,Y}$. For this it is enough to show that the right hand side in (2) is $C^\infty(M)$-homogeneous in Z.

We compute

$$\begin{aligned}
Xg(Y,fZ) &= X(fg(Y,Z) = X(f)g(Y,Z) + fXg(Y,Z), \\
Yg(X,fZ) &= Y(f)g(X,Z) + fYg(X,Z).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
-g(X,[Y,fZ]) &= -g(X, Y(f)Z + f[Y,Z]) = -Y(f)g(X,Z) - fg(X,[Y,Z]), \\
g(Y,[fZ,X]) &= X(f)g(Y,Z) + fg(Y,[Z,X]).
\end{aligned}$$

Plugging this into (2) yields $C^\infty(M)$-homogenity.

Hence, the right hand side defines a 1-form and therefore via $\sharp$ a vector field on M, and we showed that for every connection ∇ with the properties (1) and (2) $\nabla_X Y$ coincides with this vector field. This gives uniqueness of ∇.

Existence. We need to show that the right hand side in (2), or more precisely $\sharp\Theta_{X,Y} =: \nabla_X Y$ really defines a connection.

- Check that $(X,Y) \in \Gamma(TM) \times \Gamma(TM) \mapsto \nabla_X Y$ is $\mathbb{R}$-bilinear.
- Check that $\nabla_X Y$ is $C^\infty(M)$ homogeneous in X.
- Check that $\nabla_X Y$ satisfies a product rule in Y.

We will only check the last point. For this we compute:

$$\begin{aligned}
2g(\nabla_X(fY), Z) &= Xg(fY,Z) + fYg(Z,X) - Zg(X,fY) \\
&\qquad - g(X,[fY,Z]) + g(fY,[Z,X]) + g(Z,[X,fY]) \\
&= X(f)g(Y,Z) + fXg(Y,Z) + fYg(Z,X) - Z(f)g(X,Y) - fZg(X,Y) \\
&\qquad + Z(f)g(X,Y) - fg(X,[Y,Z]) + fg(Y,[Z,X]) + X(f)g(Z,Y) + fg(Z,[X,Y]) \\
&= g(2X(f)Y, Z) + g(2\nabla_X(fY), Z) \quad \forall X,Y,Z \in \Gamma(TM) \text{and } \forall f \in C^\infty(M).
\end{aligned}$$

Finally, we show the formula for the Christoffel symbols. Let $\varphi = (x^1, \ldots, x^m)$ be a chart, and set $\frac{\partial}{\partial x^i} = X$, $\frac{\partial}{\partial x^j} = Y$ and $\frac{\partial}{\partial x^k} = Z$. Then

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = \sum_{k=1}^m {}^\varphi\Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

It follows with (2):

$$2\sum_{k=1}^{m} {}^{\varphi}\Gamma_{ij}^{k} g_{kl} = 2g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = \frac{\partial}{\partial x^i} g_{jl} + \frac{\partial}{\partial x^j} g_{li} - \frac{\partial}{\partial x^l} g_{ij}.$$

We note here, that the Lie bracket between the coordinate vector fields $\frac{\partial}{\partial x^i}$ vanishes. This is not true in general for a local frame E_i.

Keeping i, j fixed, this is a vector w.r.t. $l = 1, \ldots, m$ that we get by applying $(g_{kl})_{kl}$ to $({}^{\varphi}\Gamma_{ij}^{k})_{k=1,\ldots,m}$. Hence applying the inverse $(g^{sl})_{s,l=1,\ldots,m}$ from the left, yields the desired formula for ${}^{\varphi}\Gamma_{ij}^{k}$. $\square$

3.9 Examples. (1) The standard connection $\bar{\nabla}$ on $\mathbb{R}^m$ is the Levi-Civita connection of $\langle\cdot,\cdot\rangle_{eucl}$.

(2) Let $M \subset \mathbb{R}^m$ be a submanifold with $i : M \to \mathbb{R}^n$, $i(x) = x$ and let $g = i^*\langle\cdot,\cdot\rangle_{eucl}$. We defined a connection via $(X, Y) \in \Gamma(TM) \times \Gamma(TM) \mapsto \nabla_X Y := (\nabla_{\widetilde{X}} \widetilde{Y})^{\top}$ where $\widetilde{X}, \widetilde{Y}$ are extensions of X, Y to $\mathbb{R}^m$ and $(p, v)^{\top}$ is the component of $(p, v) \in T_p\mathbb{R}^n$ tangential to T_pM.

Claim: ∇ is the Levi-Civita connection of g.

Proof of the claim. We compute for $X, Y, Z \in \Gamma(TM)$.

$$\begin{aligned} Xg(Y,Z)(p) = X_p\langle \widetilde{Y}, \widetilde{Z}\rangle_{eucl} &= \langle \nabla_{X_p}\widetilde{Y}|_p, \widetilde{Z}(p)\rangle_{eucl}(p) + \langle \widetilde{Y}(p), \nabla_{X_p}\widetilde{Z}|_p\rangle_{eucl} \\ &= \langle (\nabla_{X_p}\widetilde{Y}|_p)^{\top}, Z_p\rangle_{eucl} + \langle Y_p, (\nabla_{X_p}\widetilde{Z}|_p)^{\top}\rangle_{eucl} \\ &= g(\nabla_{X_p}Y|_p, Z_p) + g(Y_p, \nabla_{X_p}Z|_p). \end{aligned}$$

Moreover

$$\nabla_X Y|_p - \nabla_Y X|_p = (\bar{\nabla}_{\widetilde{X}}\widetilde{Y}|_p - \bar{\nabla}_{\widetilde{Y}}\widetilde{X}|_p)^{\top} = ([\widetilde{X}, \widetilde{Y}](p))^{\top} = [X, Y](p).$$

For the last identity we note that for $f \in c^{\infty}(\mathbb{R}^n)$ we have $[\widetilde{X}, \widetilde{Y}]_p(f) = \widetilde{X}_p(\widetilde{Y}(f)) - \widetilde{Y}_p(\widetilde{X}(f)) = X_p(Y(f|_M)) - Y_p(X(f|_M)) = [X, Y]_p(f|_M)$. $\square$

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In the following let (M, g) be a Riemannian manifold and let ∇ be its LC-connection.

3.10 Definition. A smooth vector field along a $\gamma \in C^\infty(I, M)$, for an interval $I \subset \mathbb{R}$, is a map $v : I \to TM$ such that $v(t) \in T_{\gamma(t)}M \ \forall t \in I$. The set of all smooth vector fields along γ is denoted with $\Gamma(\gamma^*TM)$.

More generally, one can consider a smooth section s of a vector bundle $\pi : E \to M$ along γ: $s \in C^\infty(I, E)$ such that $\pi \circ s(t) = \gamma(t)$.

The definition of vector field along a curve γ also includes the case when $\gamma \equiv \mathit{const} = p \in M$. Then $v : I \to T_pM$.

3.11 Theorem. *There is a unique operator $\nabla_t : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ such that*

(1) ∇_t is linear,

(2) $v \in \Gamma(\gamma^*TM), f \in C^\infty(I, \mathbb{R}) \Rightarrow \nabla_t(f \cdot v) = f'v + f\nabla_t v$,

(3) If $V \in \Gamma(TM)$ and $v = V \circ \gamma$, then $\nabla_t v|_t = \nabla_{\gamma'(t)} V|_{\gamma(t)}$.

∇_t is also called covariant derivative along γ.

Proof. Let $\varphi = (x^1, \dots, x^m) : U \to V$ be a chart and assume $\gamma(I) \subset U$. Given $v \in \Gamma(\gamma^*TM)$ we can then write

$$v(t) = \sum_{i=1}^m v^i(t) \frac{\partial}{\partial x^i}|_{\gamma(t)}$$

where $v^i(t) \in C^\infty(I, \mathbb{R})$ for $i = 1, \dots, m$.
Uniqueness of ∇_t if $\gamma(I) \subset U$. Assume there is an operator ∇_t with the properties (1), (2) and (3) on vector fields along γ with $\gamma(I) \subset U$. Then

$$\begin{aligned}
\nabla_t v|_t &\overset{(1)}{=} \sum_{i=1}^m \nabla_t \left(v^i \frac{\partial}{\partial x^i} \circ \gamma(t) \right) \\
&\overset{(2)}{=} \sum_{i=1}^m (v^i)'(t) \frac{\partial}{\partial x^i} \circ \gamma(t) + v^i(t) \nabla_t \left(\frac{\partial}{\partial x^i} \circ \gamma(t) \right) \\
&\overset{(3)}{=} \sum_{i=1}^m (v^i)'(t) \frac{\partial}{\partial x^i} \circ \gamma(t) + v^i(t) \left(\nabla_{\gamma'(t)} \frac{\partial}{\partial x^i} \right) \circ \gamma(t) \\
&= \sum_{i=1}^m (v^i)'(t) \frac{\partial}{\partial x^i} \circ \gamma(t) + \sum_{j=1}^m v^i(t) (\gamma'(t))^j \underbrace{\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \right) \circ \gamma(t)}_{\sum_{k=1}^m {}^{\varphi}\Gamma_{ij}^k \circ \gamma(t) \frac{\partial}{\partial x^k} \circ \gamma(t)} =: \nabla_t^{\varphi} v|_t.
\end{aligned}$$

Since the Christoffel symbols Γ_{ij}^k for the chart φ are unique, also the operator ∇_t is uniquely determined on U throught this formula.
Claim. Let $\gamma \in C^\infty(I, M)$ and $v \in \Gamma(\gamma^*TM)$. Then $\nabla_t(v|_J)|_{t_0} = \nabla_t v|_{t_0}$ for any $J \subset I$ with $\gamma(J) \subset U$ for a chart $\varphi : U \to V$.

Proof of the claim. We define an operator on $\Gamma((\gamma|_J)^*TM)$ via $\widetilde{\nabla}_t\widetilde{v}|_{t_0} := \nabla_t(f\widetilde{v})|_{t_0}$ for $t_0 \in J$ where $f \in C^\infty(I,[0,1])$ with $f \equiv 1$ on $(t_0-\epsilon, t_0+\epsilon)$ and $\operatorname{supp} f \subset J$. The definition of $\widetilde{\nabla}_t$ is independent of the choice of f by the product rule for ∇_t. This follows exactly like the statement of Lemma 3.5. Then

(1) $\widetilde{\nabla}_t$ is linear,

(2) $\widetilde{\nabla}_t$ satisfies the product rule:

$$\widetilde{\nabla}_t(\widetilde{g}\widetilde{v})|_{t_0} = \nabla_t(f^2\widetilde{g}\widetilde{v}) = (\widetilde{g}f)'(t_0)(f\widetilde{v})|_{t_0} + \widetilde{g}f(t_0)\nabla_t(f\widetilde{v})|_{t_0} = \widetilde{g}'(t_0)\widetilde{v}(t_0) + \widetilde{g}(t_0)\widetilde{\nabla}_t\widetilde{v}|_{t_0}.$$

(3) If $X \in \Gamma(TM)$, then

$$\widetilde{\nabla}_t(X\circ\gamma|_J)|_{t_0} = \nabla_t(f(X\circ\gamma|_J))_{t_0} = \nabla_t(f(X\circ\gamma))_{t_0} = \nabla_{\gamma'(t_0)}X\circ\gamma(t_0) = \nabla_{(\gamma|_J)'(t_0)}X\circ(\gamma|_J)(t_0).$$

Hence, $\widetilde{\nabla}_t$ is an operator that satisfies the properties (1), (2), (3) and hence coincides with ∇_t for $v \in \Gamma((\gamma|_J)^*TM)$ as long as $\gamma(J) \subset U$ for a charte $\varphi : U \to V$ (both have to be ∇_t^φ).

Hence $\nabla_t(v|_J)_{t_0} = \nabla_t v(t_0)$ for $t_0 \in J$ and with $\gamma(J) \subset U$. □

Existence. We define $\nabla_t : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ through $\nabla_t = \nabla_t^\varphi$ via the previous formula on the domain of a given chart φ.

For definition of ∇_t for general γ and $v \in \Gamma(\gamma^*TM)$ we choose a covering of M with charts $\varphi : U \to V$ and define $\nabla_t v|_t$ as $\nabla_t^\varphi v|_t$ for $\gamma(t) \in U$.

If $J \subset I$ with $\gamma(J) \subset U$ for such a chart $\varphi : U \to V$, then ∇_t satisfies (1), (2), (3) for $t \in J$.

Claim. ∇_t does not depend on the covering with charts.

If $\psi = (y^1, \dots, y^m) : \widetilde{U} \to \widetilde{V}$ such that $U \cap \widetilde{U} \neq \emptyset$, then the computation we did for uniquness shows that for $\gamma(t) \in U \cap \widetilde{U}$ we have

$$\nabla_t^\psi v|_t = \nabla_t^\varphi v|_t.$$

Hence, the definition of ∇_t does not depend on covering of M with charts. □

3.12 Examples.

- Consider $M = \mathbb{R}^n$, $TM = \mathbb{R}^n \times \mathbb{R}^n$, $\gamma : I \to \mathbb{R}^n$ and $V \in \Gamma(\gamma^*TM)$. A global chart of M is $\varphi(p^1, \dots, p^n) = (p^1, \dots, p^n)$ and ${}^\varphi\Gamma_{ij}^k \equiv 0$.
  Hence $\bar{\nabla}_t V|_t = ((V^1)'(t), \dots, (V^n)'(t))$.
- Let $M \subset \mathbb{R}^n$ be a m-dimensional submanifold, $\gamma : I \to M$ a smooth curve. The operator $\nabla_t : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ is given by

  $$(\bar{\nabla}_t V|_t)^\top =: \nabla_t V|_t.$$

  Indeed, it satifies (1), (2) and (3).

3.13 Definition. We say $v \in \Gamma(\gamma^*TM)$ is parallel along γ if $\nabla_t v \equiv 0$.

3.14 Remark.

- $v, w \in \Gamma(\gamma^*TM)$ parallel and $\alpha, \beta \in \mathbb{R}$, then $\alpha v + \beta w$ parallel as well.

- $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Then $\nabla_t v = \sum_{i=1}^n (v^i)'(t) e_i$ where e^i is a basis of $\mathbb{R}^n$. Hence v is parallel if and only if $v(t) = v(0) = v_0 \in \mathbb{R}^n$.

3.15 Theorem. *Let $\gamma \in C^\infty(I, M)$, $a \in I$ and $v \in T_{\gamma(a)}M$. Then there exists exactly one parallel $v \in \Gamma(\gamma^* TM)$ with $v(a) = v_a$.*

Proof. Let $\varphi = (x^1, \dots, x^m) : U \to V$ be a chart such that $\gamma(a) \in U$. Then $v(t) = \sum_{i=1}^m v^i(t) \frac{\partial}{\partial x^i}|_{\gamma(t)}$ if $\gamma(t) \in U$. It follows that v is parallel on $J \subset I$ with $\gamma(J) \subset U$ if and only if

$$(v^i)'(t) + \sum_{j=1} v^j(t) \omega_j^i(\gamma'(t)) = 0 \quad \forall t \in J, \ \forall i = 1, \dots, m. \tag{3}$$

Recall that $\omega_j^i(\gamma'(t)) = \sum_{k=1}^m \Gamma_{jk}^i (\gamma'(t))^k$ are the connection 1-forms.

The equation (3) is an ordinary, linear, $\mathbb{R}^m$-valued, differential equation with smooth coefficients on J. Hence it has a unique solution everywhere on J (because it is linear with smooth coefficients) for the initial value $(v^1(a), \dots, v^m(a))$.

If $\widetilde{\varphi}$ another chart with $U \cap \widetilde{U} \neq \emptyset$, the value of v at some $t_0 \in I$ with $\gamma(t_0) \in U \cap \widetilde{U}$ then determines v for any $t \in I$ with $\gamma(t) \in \widetilde{U}$.

Hence, the existence and uniqueness of such a parallel v on I follows from successively solving the equation (3) on coordinate charts. $\square$

3.16 Definition. Let $\gamma : I \to M$ as before and $s, t \in I$. The parallel transport along γ from $\gamma(s)$ to $\gamma(t)$ is the map $P_{s,t}^\gamma : T_{\gamma(s)}M \to T_{\gamma(t)}M$ defined as follows: If $v \in T_{\gamma(s)}M$, we consider the $v \in \Gamma(\gamma^* TM)$ parallel with $v(s) = v$, and we set $P_{s,t}^\gamma(v) = v(t)$.

3.17 Remark. The map $P_{s,t}^\gamma$ is a vector space isomorphism between $T_{\gamma(s)}M$ and $T_{\gamma(t)}M$ where the inverse is given by $P_{t,s}^\gamma$.

This follows from the fact that solutions of (3) are a vector space and unique for a given initial value $v(a) = v_a$ (the parallel v with $v(a) = 0$ is the vector field $v \equiv 0$). This implies that $P_{s,t}^\gamma$ is a linear map that is injective and therefore an isomorphism.

3.18 Lemma. *Consider $\gamma \in C^\infty(I, M)$ and $V, W \in \Gamma(\gamma^* TM)$. Then*

$$\frac{d}{dt} g(V(t), W(t)) = g(\nabla_t V|_t, W(t)) + g(V(t), \nabla_t W|_t).$$

Proof. Exercise. $\square$

3.19 Corollary. *Consider a smooth curve $\gamma : I \to M$ and $s, t \in I$. Then $P_{s,t}^\gamma$ is an orthogonal map.*

Proof. $V, W \in \Gamma(\gamma^* TM)$ parallel. Then

$$\frac{d}{dt} g(V(t), W(t)) = g(\nabla_t V|_t, W(t)) + g(V(t), \nabla_t W|_t) = 0.$$

Hence $t \in I \mapsto g(V(t), W(t))$ is constant. $\square$

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3.20 Lemma. *Let $\gamma \in C^\infty(I, M)$, $0 \in I$ and $W \in \Gamma(\gamma^* TM)$. Then*

$$\nabla_t W|_{t=0} = \frac{d}{dt}\Big|_{t=0} P^\gamma_{t,0} W(t).$$

Proof. Let $e_1, \ldots, e_m \in T_{\gamma(0)}M$ be an ONB and let $E_1(t), \ldots, E_m(t) \in \Gamma(\gamma^* TM)$ be parallel such that $E_i(0) = e_i \ \forall i = 1, \ldots, m$, that is $P^\gamma_{0,t} e_i = E_i(t)$. Then $E_1(t), \ldots, E_m(t)$ is an ONB of $\Gamma(\gamma^* TM)$ since $P^\gamma_{0,t}$ is orthogonal. Hence

$$W(t) = \sum_{i=1}^m W^i(t) E_i(t)$$

and

$$P^\gamma_{t,0} W(t) = \sum_{i=1}^m W^i(t) P^\gamma_{t,0} E_i(t) = \sum_{i=1}^m W^i(t) e_i.$$

Therefore it follows

$$\frac{d}{dt}|_{t=0} P^\gamma_{t,0} W(t) = \sum_{i=1}^m (W^i)'(0) e_i = \sum_{i=1}^m \Big((W^i)'(0) E_i(0) + \underbrace{W^i \nabla_t E_i|_{t=0}}_{=0} \Big) = \nabla_t W|_{t=0}.$$

□

4 Curvature tensor

Let (M, g) be a Riemannian manifold and ∇ the LC connection.

4.1 Definition. The map

$$X, Y, Z \in \Gamma(TM) \mapsto R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$$

definiert ein tensor feld in $\Gamma(T^*M \otimes T^*M \otimes T^*M \otimes TM)$. R is called the Riemannian curvature tensor of (M, g).

4.2 Remark. It is clear that $R(X,Y)Z = -R(Y,X)Z$. Hence $R \equiv 0$, if $\dim_M = 1$.
We need to show that R is indeed a tensor field. For this we show $C^\infty(M)$ homogenenity in $X, Y, Z \in \Gamma(TM)$:

$$fR(X,Y)Z = R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ).$$

For instance, we compute

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X (Y(f)Z + f\nabla_Y Z) - \nabla_Y (X(f)Z + f\nabla_X Z) - [X,Y](f)Z - f\nabla_{[X,Y]} Z \\ &= X(Y(f))Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla_X \nabla_Y Z \\ &\qquad - Y(X(f)) - X(f)\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y \nabla_X Z - [X,Y](f) - f\nabla_{[X,Y]} Z = \ldots \end{aligned}$$

A geometric interpretation of R via parallel transport. Let $\alpha : W \subset \mathbb{R}^2 \to M$ be smooth such that $(0,0) \in W$, $\alpha(0,0) = p \in M$ (for instance, if $\varphi : U \to V$ is charte, α can be $\varphi^{-1} : V \cap (\mathbb{R}^2 \times \{(0,\dots,0)\} \subset \mathbb{R}^m) \simeq W \to M$).
We set $\alpha(s,t) = \alpha^t(s) = \alpha_s(t)$ and define $v : W \to TM$ via

$$v(s,t) = P^{\alpha_s}_{0,t} \circ P^{\alpha^0}_{0,s} v_0$$

for some $v_0 \in TM$. Then $\pi \circ v(s,t) = \alpha(s,t)$. We also define

$$w(s,t) = P^{\alpha_0}_{t,0} \circ P^{\alpha^t}_{s,0}(v(s,t)).$$

It follows $\pi \circ w(s,t) = \alpha(0,0) = p$, that is $w(s,t) \in T_pM$. In particular $w(0,0) = v(0,0) = v_0$.

4.3 Theorem. *Let α, v and w as before. Then it follows*

$$R\left(\frac{\partial\alpha}{\partial t}\Big|_{(0,0)}, \frac{\partial\alpha}{\partial s}\Big|_{(0,0)}\right) v_0 = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{(0,0)} w(s,t)$$

or equivalently

$$w(s,t) = v_0 + stR\left(\frac{\partial\alpha}{\partial t}\Big|_{(0,0)}, \frac{\partial\alpha}{\partial s}\Big|_{(0,0)}\right) v_0 + o(|(s,t)|).$$

4.4 Lemma. *Let $\alpha : W \to M$ as before and let $v : W \to TM$ be such that $\pi \circ v = \alpha$. Then*

$$R(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t})v = \nabla_s\nabla_t v - \nabla_t\nabla_s v.$$

Proof of the lemma. We fix $(s_0,t_0) \in W$. Assume first that $\frac{\partial\alpha}{\partial s}$, $\frac{\partial\alpha}{\partial t}$ are linear independent at (s_0,t_0).
$\Rightarrow \exists$ a chart φ such that $\varphi \circ \alpha(s,t) = (s,t,0,\dots,0)$ in a small open neighborhood of (s_0,t_0). Computations take place in this neighborhood.
$\Rightarrow \exists$ local vector fields $X, Y, V \in \Gamma(TM)$ such that $[X,Y] = 0$ and such that

$$X \circ \alpha = \frac{\partial\alpha}{\partial s}, \quad Y \circ \alpha = \frac{\partial\alpha}{\partial t} \text{ and } V \circ \alpha = v.$$

Since R is a tensor (that means $(R(X,Y)V)|_p = R(X_p,Y_p)V_p$), it follows that

$$\begin{aligned} R\left(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\right) v = (R(X,Y)V) \circ \alpha &= \nabla_X(\nabla_Y V) \circ \alpha - \nabla_Y(\nabla_X V) \circ \alpha - 0 \\ &= \nabla_{\frac{\partial\alpha}{\partial s}}(\nabla_Y V) \circ \alpha - \nabla_{\frac{\partial\alpha}{\partial t}}(\nabla_X V) \circ \alpha \\ &= \nabla_s(\nabla_Y V \circ \alpha(s,t)) - \nabla_t(\nabla_X V \circ \alpha(s,t)) \\ &= \nabla_s(\nabla_t(V \circ \alpha(s,t))) - \nabla_t(\nabla_s(V \circ \alpha(s,t))) \end{aligned}$$

This is the claim, since $V \circ \alpha(s,t) = v(s,t)$.

If $\frac{\partial \alpha}{\partial s}_{(s_0,t_0)}$ and $\frac{\partial \alpha}{\partial t}_{(s_0,t_0)}$ are not linear independent where $\alpha(s_0,t_0)=p$, then we consider again a chart φ with $\varphi(p)=0$ and we consider

$$\underbrace{D\varphi|_p \frac{\partial \alpha}{\partial s}|_{(s,t)}}_{=\widetilde{v}_1(s,t)} + \tau z_1, \quad \underbrace{D\varphi|_p \frac{\partial \alpha}{\partial t}|_{(s,t)}}_{=\widetilde{v}_2(s,t)} + \tau z_2$$

such that these vectors are linear independent for all (s,t) in a neighborhood $\widetilde{W}$ of (s_0,t_0). Then we replace α with

$$\alpha_\tau(s,t) := \widetilde{\alpha}(s,t,\tau) := \varphi^{-1}(\varphi\circ\alpha(s,t) + s\tau z_1 + t\tau z_2).$$

It follows that

$$\frac{\partial \alpha_\tau}{\partial s}\Big|_{(s,t)} = D\varphi^{-1}(\widetilde{v}_1 + \tau z_1)\Big|_{(s,t)}, \quad \frac{\partial \alpha_\tau}{\partial t}\Big|_{(s,t)} = D\varphi^{-1}(\widetilde{v}_2 + \tau z_2)\Big|_{(s,t)}$$

are linear independent in $T_{\alpha(s,t)}M$. Moreover, we define a smooth local vectorfield $V: U \to TM$ such that $V\circ\alpha = v$, and also $v_\tau(s,t) = V\circ\alpha_\tau(s,t)$ for $(s,t)\in\widetilde{W}$.

In local coordinates w.r.t. the chart φ we have $v_\tau(s,t) = \sum_{i=1}^m v_\tau^i(s,t)\frac{\partial}{\partial x^i}\circ\alpha_\tau(s,t)$ with smooth coefficients. We see that $v_\tau \to v$ on $\widetilde{W}$ as $\tau \downarrow 0$. We compute then

$$\nabla_t v_\tau(s,t) = \sum_{i=1}^m \left(\frac{\partial}{\partial t}(v_\tau^i)(s,t) + \sum_{k,l=1}^m v_\tau^k(s,t)\left(\frac{\partial \alpha_\tau}{\partial t}\right)^l (s,t)\Gamma_{k,l}^i\circ\alpha_\tau(s,t)\right)\frac{\partial}{\partial x^i}\circ\alpha_\tau(s,t)$$

where $\left(\frac{\partial \alpha_\tau}{\partial t}\right)^l(s,t) = (\widetilde{v}_2(s,t)+\tau z_1)^l$. Finally we compute

$$\begin{aligned}\nabla_s\nabla_t v_\tau(s,t) = \sum_{i=1}^m \Bigg[&\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}(v_\tau^i)(s,t) + \sum_{k,l=1}^m v_\tau^k(s,t)\left(\frac{\partial \alpha_\tau}{\partial t}\right)^l (s,t)\Gamma_{kl}^i\circ\alpha_\tau(s,t)\right)\frac{\partial}{\partial x^i}\circ\alpha_\tau(s,t) \\ &+ \left(\frac{\partial}{\partial t}(v_\tau^i)(s,t) + \sum_{k,l=1}^m v_\tau^k(s,t)\left(\frac{\partial \alpha_\tau}{\partial t}\right)^l (s,t)\Gamma_{kl}^i\circ\alpha_\tau(s,t)\right)\left(\frac{\partial \alpha_\tau}{\partial s}\right)^j \Gamma_{ij}^s\circ\alpha_\tau(s,t)\frac{\partial}{\partial x^s}\circ\alpha_\tau(s,t)\Bigg].\end{aligned}$$

Hence also $\nabla_s\nabla_t v_\tau(s,t) \to \nabla_s\nabla_t v(s,t)$ on $\widetilde{W}$ as $\tau\downarrow 0$.

We then follow the program of the first part of the proof and obtain

$$R(\frac{\partial \alpha_\tau}{\partial s}\Big|_{(s,t)}, \frac{\partial \alpha_\tau}{\partial t}\Big|_{(s,t)})v_\tau(s,t) = \nabla_{\frac{\partial \alpha_\tau}{\partial s}|_{(s,t)}}\left(\nabla_{\frac{\partial \alpha_\tau}{\partial t}} v_\tau\right)|_{(s,t)} - \nabla_{\frac{\partial \alpha_\tau}{\partial t}|_{(s,t)}}\left(\nabla_{\frac{\partial \alpha_\tau}{\partial s}} v_\tau\right)\Big|_{(s,t)}.$$

We have expressed the right hand side in local coordinates and saw it converges to the correspoinding term with $\tau = 0$.

Since R is a tensor field, if $\tau\to 0$, also left hand side converges to $R(\frac{\partial\alpha}{\partial s}\Big|_{(s,t)}, \frac{\partial\alpha}{\partial t}\Big|_{(s,t)})v(s,t)$.

Hence we get the desired identity and this finishes the proof of the lemma. □

Proof of the theorem. First note that $\nabla_t v|_{(s,t)} = 0$. By the previous lemma we get

$$R(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s})v(s,t) = \nabla_t \nabla_s v|_{(s,t)}.$$

Then, we also compute

$$\frac{\partial}{\partial s} w|_{(0,t)} = \frac{d}{ds}\Big|_{s=0} \left(P^{\alpha_0}_{t,0}(P^{\alpha^t}_{s,0}(v(s,t))) \right) = P^{\alpha_0}_{t,0} \left(\frac{d}{ds}\Big|_{s=0} P^{\alpha^t}_{s,0}(v(s,t)) \right) = P^{\alpha_0}_{t,0} \left(\nabla_s v|_{(0,t)} \right).$$

Hence

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} w|_{(0,0)} = \nabla_t \nabla_s v|_{(0,0)} = R(\frac{\partial \alpha}{\partial t}\Big|_{(0,0)}, \frac{\partial \alpha}{\partial s}\Big|_{(0,0)}) \underbrace{v(0,0)}_{=v_0}.$$

This finishes the proof. $\square$

05.06.2023

Let (M,g) be a Riemannian manifold, ∇ the LC connection and R the curvature tensor.

4.5 Theorem. *The following statements are equivalent:*

(1) $R \equiv 0$,

(2) $\forall p \in M$ $\exists$ *neighborhood* U *of* p *such that: if* $\gamma : [0,1] \to U$ *is smooth with* $\gamma(0) = \gamma(1) = p$, *then* $P^{\gamma}_{0,1} = id_{T_pM}$.
(The parallel transport is locally path independent.)

(3) If $\gamma_0, \gamma_1 : [0,1] \to M$ *are homotopy equivalent via a smooth homotopy* $\alpha : [0,1]^2 \to M$ *with* $\alpha(s,0) = \gamma_0(0) = \gamma_1(0)$ *and* $\alpha(s,1) = \gamma_0(1) = \gamma_1(1)$, *then* $P^{\gamma_0}_{0,1} = P^{\gamma_1}_{0,1}$.

Proof. (1) $\Rightarrow$ (3): Consider a smooth homotopy $\alpha : [0,1]^2 \to M$ between γ_0 and γ_1, that ist $\alpha(0,t) = \gamma_0(t)$ and $\alpha(1,t) = \gamma_1(t)$ for all $t \in [0,1]$ and $\alpha(s,0) = \gamma_0(0) = \gamma_1(0) = p$ and $\alpha(s,1) = \gamma_0(1) = \gamma_1(1) = q$ $\forall s \in [0,1]$. We set $\gamma_s(t) := \alpha(s,t)$.

Let $v_0 \in T_pM$ and define $V(s,t) = P^{\gamma_s}_{0,t} v_0$. Then $\nabla_t V(s,t) = 0$.

By assumption we have $R \equiv 0$. Hence 4.4 Lemma implies that $0 = \nabla_t \nabla_s V(s,t)$. It follows that $t \in [0,1] \mapsto \nabla_s V(s,t)$ is parallel vector field along γ_s.

On the other hand, we have $V(s,0) = v_0$ $\forall s \in [0,1]$. Hence $\nabla_s V(s,0) = 0$ which is the initial value of the parallel vectorfield $t \in [0,1] \mapsto \nabla_s V(s,t)$. Hence $\nabla_s V(s,t) = 0$ $\forall t \in [0,1]$ for all $s \in [0,1]$. Hence $s \in [0,1] \mapsto V(s,t)$ parallel.

Especially for $t = 1$, it follows $\alpha(s,1) = q$ and $P^{\gamma_s}_{0,1} v_0 \in T_qM$ constant in $s \in [0,1]$.

(3) $\Rightarrow$ (2): We can choose U with $p \in U$ that is simply connected, for instance let $U = \varphi^{-1}(B_\epsilon(0))$. Then any closed curve γ is homotopy equivalent to the constant curve p via a smooth homotopy.

(2) $\Rightarrow$ (1): This follows from 4.3 Theorem. Let $w_0, w_1 \in T_pM$ and let $\alpha(s,t)$ be a smooth map into M such that $\frac{\partial \alpha}{\partial s}(0,0) = w_0$ and $\frac{\partial \alpha}{\partial t}(0,0) = w_1$. Consider $w(s,t)$ as in 4.3 Theorem. For s,t small enough this is the parallel transport along a closed curve inside of the neighborhood U given by (ii), it follows $w(s,t) = w(0,0) \in T_pM$. Dann folgt

$$R\Big(\frac{\partial \alpha}{\partial t}\Big|_{(0,0)}, \frac{\partial \alpha}{\partial s}\Big|_{(0,0)}\Big) v_0 = 0.$$

$\square$

4.6 Corollary. *If* $R \equiv 0$ *and if* M *is simply connected, then there exists a global frame on* M *for* TM. *That is* $TM = M \times \mathbb{R}^m$.

Proof. Let $p \in M$ be fixed and let $v_1, \ldots, v_m$ be a Basis of T_pM. We then define $V_i \in \Gamma(TM)$ via $V_i(q) = P^{\gamma}_{0,1} v_i$ where $\gamma : [0,1] \to M$ is a curve that connects p and q. The previous Theorem implies that this definition of V_i does not depend on γ.

Moreover the smoothness of V_i follows from the smooth dependency of solutions of ODEs on a smooth parameter. In this cases the smooth parameter ist $q \in M$. $\square$

4.7 Remark. The frame $(V_1, \dots, V_m)$ is also parallel, that is $\nabla_X V_i = 0$ $\forall i = 1, \dots, m$ and every vector field $X \in \Gamma(TM)$.

4.8 Examples. (1) $M = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. The covering map $\pi : \mathbb{R}^m \to \mathbb{T}^m$ induces locally a Riemannian metric that is locally isometric to the Euclidean metric on $\mathbb{R}^m$. Hence the Christoffel symbols vanish and $R \equiv 0$. But $\mathbb{T}^m$ is not simply connected, despite $T\mathbb{T}^m = \mathbb{T}^m \times \mathbb{R}^m$.

(2) The Moebisu strip $M = \mathbb{R}^2/\Gamma$ admits a metric that is locally isometric to $\mathbb{R}^n$. Hence $R \equiv 0$. But TM is not trivial (since M is not orientable) and M is not simply connected.

In the following we also write $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$ for the Riemannian metric.

07.06.2023

4.9 Lemma (Symmetries of the curvature tensor)**.** *Let $X, Y, Z, W \in \Gamma(TM)$.*

(a) $R(X,Y)Z = -R(X,Y)Z$,

(b) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$,

(c) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,

(d) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

Proof. (a) We observed this already from the definition.

(b) Since R is a tensor, it is enough to consider vectorfields X, Y, Z with $[X,Y]_p = [Y,Z]_p = [Z,X]_p = 0$. Then the claim follows from symmetry of ∇ and the definition.

(c) follows form the fact that ∇ is a metric (Riemannian) connections. We assume again that $[X,Y]_p = 0$. It is enough to show that $\langle R(X_p, Y_p)Z_p, Z_p\rangle = 0$. For this we compute

$$X_p(Y|Z|^2) = 2X_p(\langle \nabla_Y Z, Z\rangle) = 2\langle \nabla_X(\nabla_Y Z)_p, Z_p\rangle + 2\langle \nabla_Y Z_p, \nabla_X Z_p\rangle$$

as well as

$$Y_p(X|Z|^2) = 2\langle \nabla_Y(\nabla_X Z)_p, Z_p\rangle + 2\langle \nabla_X Z_p, \nabla_Y Z_p\rangle.$$

We substract the second from the first line and obtain

$$0 = [X,Y]_p|Z|^2 = \langle R(X_p, Y_p)Z_p, Z_p\rangle.$$

(d) follows form (a), (b) and (c). We skip details. □

4.10 Lemma (R in local coordinates)**.** *Given a chart $\varphi : U \to V$ we compute coefficient functions R^l_{ijk} of R w.r.t. the basis $dx^i \otimes dx^j \otimes dx^k \frac{\partial}{\partial x^l}$, that is*

$$R|_U = \sum_{i,j,k,l=1}^{m} R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$$

where $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = \sum_{l=1}^m R^l_{ijk}\frac{\partial}{\partial x^l}$. It holds

$$R^l_{ijk} = \frac{\partial}{\partial x^i}\Gamma^l_{jk} - \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \sum_{s=1}^{m}\left(\Gamma^s_{ik}\Gamma^l_{js} - \Gamma^s_{jk}\Gamma^l_{is}\right).$$

(We can compute R_p from g_p and the first and second derivatives of g_{ij} at p.)

Proof. Recall that $\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} = \sum_{l=1}^m \Gamma^l_{jk}\frac{\partial}{\partial x^l}$. Moreover

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}.$$

Then we compute

$$\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_{l=1}^{m} \frac{\partial}{\partial x^i} \Gamma^l_{jk} \frac{\partial}{\partial x^l} + \underbrace{\sum_{l,s=1}^{m} \Gamma^l_{ik} \Gamma^s_{jl} \frac{\partial}{\partial x^s}}_{\sum_{l,s=1}^{m} \Gamma^s_{jk} \Gamma^l_{is} \frac{\partial}{\partial x^l}}.$$

That is the claim. □

4.11 Remark.

$$R_{ijkl} := \langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \rangle = \sum_{s=1}^{m} R^s_{ijk} g_{sl} = \flat \circ R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k}.$$

The identities of 4.9 Lemma then write as

(a) $\Leftrightarrow$ $R^l_{ijk} = -R^l_{jik}$,

(b) $\Leftrightarrow$ $R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0$,

(c) $\Leftrightarrow$ $R_{ijkl} = -R_{ijlk}$,

(d) $\Leftrightarrow$ $R_{ijkl} = R_{klij}$.

Sectional curvature. We fix $p \in M$ and T_pM and consider the Grassmannian space

$$G_2(T_pM) = \{E \subset T_pM : E \text{ linear subspace}, \ \dim_E = 2\},$$

as well as $G_2(M) = \bigcup_{p \in M} G_2(T_pM)$.

For $E \in G_2(M)$ let $u, v \in E$ and define

$$Q(u,v) = \langle u,u \rangle \langle v,v \rangle - \langle u,v \rangle^2 = \det \begin{pmatrix} \langle u,u \rangle & \langle u,v \rangle \\ \langle u,v \rangle & \langle v,v \rangle \end{pmatrix}.$$

$Q(u,v) > 0$ if and only if u, v is a Basis of E. In particular, if u, v is orthonormal w.r.t. g, then $Q(u,v) = 1$.

($\sqrt{Q(u,v)}$ is the area of the parallelogram spanned by u and v w.r.t. the inner product $\langle \cdot, \cdot \rangle = g_p$.)

Note that for linear map $A : E \to E$ we have $Q(Au, Av) = (\det A)^2 Q(u,v)$.

4.12 Definition (Sectional curvature)**.** The function $K : G_2(M) \to \mathbb{R}$ given by

$$K(E) := K(u,v) := \frac{\langle R(u,v)v, u \rangle}{Q(u,v)}, \text{ where } u, v \text{ is a basis of } E,$$

is called sectional curvature of (M, g).

4.13 Remark. The definition of the sectional curvature $K(E) = K(u,v)$ does not depend on the choice of u, v. Indeed, if u', v' is another basis, then there exists $A : E \to E$ such that $Au = u'$ and $Av = v'$, and $\langle R(u',v')v', u \rangle = (\det A)^2 \langle R(u,v)v, u \rangle$.

4.14 Examples. 1. $(M,g) = (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{eucl})$, then $R \equiv 0$. Hence $K \equiv 0$.

2. $M \subset \mathbb{R}^3$ 2-dimensional submanifold, then $K(T_pM) = K(p)$ is the Gauß curvature. For this recall that the Gauß curvature, by the Theorema Egregium, is

$$K(p) = \frac{\partial}{\partial x^2}\Gamma^2_{11} - \frac{\partial}{\partial x^1}\Gamma^2_{21} + \sum_{n=1}^{2}\left(\Gamma^n_{11}\Gamma^2_{n2} - \Gamma^n_{21}\Gamma^2_{n1}\right)$$

where we choose a charte such that $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p$ is ONB of T_pM. But the right hand side is also

$$R^2_{211}(p) = R(\frac{\partial}{\partial x^2}\Big|_p, \frac{\partial}{\partial x^1}\Big|_p)\frac{\partial}{\partial x^1}\Big|_p(dx^2_p) = R_{2112}.$$

3. (M,g) frame homogeneous Riemannian manifold (Raum freier Beweglichkeit). Then $K = const = K_0 \in \mathbb{R}$.

   Indeed: Since (M,g) is frame homogeneous, it follows $\forall E, E' \in G_2(M)$ there exists an isometry $F : (M,g) \to (M,g)$ such that $DF(E) = E'$.

   Exercise sheet 7: $K(E) = K(DF(E)) = K(E')$.

Question: Is information lost when going from R to K?

Answer: No. The knowledge of K determines R uniquely.

4.15 Lemma. *Let V be a vector space of dimension ≥ 2 and let $\langle\cdot,\cdot\rangle$ be an inner product on V. Assume $R, R' : V \times V \times V \to V$ are trilinear maps such that the conditions (a), (b) and (c) of 4.9 Lemma are satisfied by*

$$\langle R(x,y)u, w\rangle, \quad \langle R'(x,y)u, w\rangle.$$

If $u, v \in V$ are linear independent, then we define $K(u,v)$ and $K'(u,v)$ as above via $K(u,v) = R(u,v,v,u)/Q(u,v)$. If $K(E) = K'(E)$ for all $G_2(V)$, then $R = R'$.

Proof. Set

$$R(v,w,x,y) = \langle R(v,w)x, y\rangle.$$

We show that $R(v,w,w,v) = 0$ for all $v, w \in V$ implies $R(v,w,x,y) = 0$ for all $v,w,x,y \in V$. Since the set of tensorfields with (a), (b) and (c) is an $\mathbb{R}$ vector space and since the map $R \mapsto K_R$ is linear, then it follows that the map $R \mapsto K_R$ is injective.

If $R(v,w,w,v) = 0$ $\forall v,w \in T_pM$, then $0 = R(u, v+x, v+x, u) = 2R(u,v,x,u) + R(u,v,v,u) + R(u,x,x,u) = 2R(u,v,x,u)$ $\forall u,v,x \in T_pM$.

Hence $R(u,v,x,u) = R(x,u,u,v) = R(u,x,v,u)$.

The same argument in u yields first that $R(u,v,x,w) = -R(u,v,x,w)$ $\forall u,x,v,w \in T_pM$. Then we also have

$$\begin{aligned} R(u,v,x,w) &= R(w,x,v,u) = -R(x,v,w,u) - R(v,w,x,u) = R(x,v,u,w) - R(v,w,x,u) \\ &= -R(v,u,x,w) - R(u,x,v,w) - R(v,w,x,u) = R(u,v,x,w). \end{aligned}$$

Hence $R(u,v,x,w) = 0$ $\forall u,v,w,x \in V$. $\square$

4.16 Corollary. *Let* (M, g) *be Riem. mfd. and* $p \in M$. *Assume* $K|_{G_2(T_pM)} = c = const.$ *Then*

$$R(u, v)w = c\,(\langle v, w\rangle u - \langle u, w\rangle v) \ \ \forall u, v, w \in T_pM.$$

Proof. Set $R'(u, v)w = c\,(\langle v, w\rangle u - \langle u, w\rangle v)$. Then $K_{R'}(u, v) = c = K(u, v)$ and R' satisfies (a), (b), (c) and (d). Hence $R'(u, v)w = R(u, v)w$. $\square$

12.06.2023

5 Geodesics

Let (M, g) be a Riemannian manifold and ∇ the LC connection.

5.1 Definition. Let $I \subset \mathbb{R}$ be an interval. A smooth curve $c : I \to M$ is called geodesic if c' is parallel along c, that is $\nabla_t c' \equiv 0$.

5.2 Remark. Let $\varphi : U \to V$ be a chart. Then $\varphi \circ c =: (c^1, \dots, c^m)$ and

$$\nabla_t c' = 0 \text{ on } J \subset I \text{ with } c(J) \subset U \;\Leftrightarrow\; (c^i)''(t) + \sum_{j,k=1}^{m} (c^j)'(t)(c^k)'(t)\Gamma^i_{jk} \circ c(t) = 0,\ i = 1, \dots, m.$$

We call this the *geodesic equation.*

Proof. First

$$c'(t) = \sum_{i=1}^{m} (c^i)'(t) \frac{\partial}{\partial x^i} \circ c(t) \text{ if } t \in J.$$

Then

$$\nabla_t c'|_t = \sum_{i=1}^{m} (c^i)''(t) + \sum_{i=1} (c^i)'(t) \nabla_{c'(t)} \frac{\partial}{\partial x^i} \circ c(t) = \dots$$

□

5.3 Remark. The geodesic equation is a nonlinear ODE of second order with smooth coefficients. Hence, the Theorem of Picard-Lindelöff guarantees that for initial values $p \in M$ and $v \in T_pM$ there exists $\epsilon > 0$ and a unique geodesic $c_v : (-\epsilon, \epsilon) \to M$ such that $c_v(0) = p$ and $c_v'(0) = v$.

5.4 Remark. (1) If $c(t)$ is a geodesic and $a, b \in \mathbb{R}$, then also $\widetilde{c}(t) = c(ta + b)$ is a geodesic.

(2) If $F : (M, g) \to (\widetilde{M}, \widetilde{g})$ is an isometry and c is geodesic on M, then $F \circ c$ is a geodesic on $\widetilde{M}$.

(3) Let c be a geodesic, then $\langle c'(t), c'(t) \rangle = g_{c(t)}(c'(t), c'(t)) = |c'(t)|^2$ is constant. Indeed

$$\frac{d}{dt} \langle c'(t), c'(t) \rangle = 2 \langle \nabla_t c'|_t, c'(t) \rangle = 0.$$

5.5 Examples. (1) $(M, g) = (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{eucl})$. Then $\varphi(x) = x$ is a global chart and ${}^{\varphi}\Gamma^k_{ij} = 0$ for all $i, j, k = 1, \dots, m$. Hence, the geodesic equation becomes

$$(c^i)'' \equiv 0,\ i = 1, \dots, m,$$

where $(c^1, \dots, c^m) = \varphi \circ c$. For given initial values $p \in \mathbb{R}^m$ and $v \in \mathbb{R}^m$ the unique solution of this equation is $c(t) = p + tv$.

(2) Let $i: M \subset \mathbb{R}^n$ be an embedded m-dimensional submanifold equipped with induced intrinsic metric $i^*\langle\cdot,\cdot\rangle_{eucl}$. The geodesic equation for $c(t) = (c^1(t), \dots, c^n(t)) \in M$ is

$$c''(t)^\top = 0.$$

That is $c''(t)$ is a vector normal to $T_{c(t)}M \subset \mathbb{R}^n$.

(3) A special case of the previous example: $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Recall that $T_p\mathbb{S}^{n-1} = \{p\} \times p^\perp$ for $p \in \mathbb{S}^{n-1}$. Hence $c(t) = (c^1(t), \dots, c^n(t)) \in \mathbb{S}^n$ is geodesic if and only if $c''(t) || \pm N \circ c(t)$ where $N(p)$ is the outer unit normal vector in $p \in \mathbb{S}^{n-1}$. For instance, the curves

$$c(t) = \cos t \cdot p + \sin t \cdot v \in \mathbb{S}^{n-1},\ p \in \mathbb{S}^{n-1}, v \in p^\perp$$

are geodesics.

Moreover $c: \mathbb{R} \to \mathbb{S}^{n-1}$ satisfies $c(0) = p$ and $c'(0) = v$. Hence c is the unique solution of the geodesic equation with intial values p and v.

Notation. $\forall v \in TM$ let $c_v : (\alpha_v, \omega_v) \to M$ be unique maximal solution of the geodesic equation on (M, g) with $c_v'(0) = v$: The interval $(\alpha_v, \omega_v) \ni 0$ is maximal with $\alpha_v, \beta_v \in \mathbb{R} \cup \{\pm\infty\}$.

5.6 Remark (Consequences of the Theorem of Picard-Lindelöff).

- $W := \{(v,t) | v \in TM, t \in (\alpha_v, \omega_v)\}$ is open in $TM \times \mathbb{R}$.
- $TM \times \{0\} \subset W$ and $\forall p \in M$ it holds $\{0_p\} \times \mathbb{R} \subset W$ since $c_{0_p}(t) \equiv p$. Here $0_p = 0 \in T_pM$.
- $c_{sv} = c_v(s)$ and $(\alpha_{sv}, \omega_{sv}) = (\frac{1}{s}\alpha_v, \frac{1}{s}\omega_v)$, $s > 0$.
- W offen $\Rightarrow \mathcal{U} := \{v \in TM : 1 \in (\alpha_v, \omega_v)\} = P_{TM}(W \cap TM \times \{1\})$ offen in TM und $0_p \in \mathcal{U}\ \forall p \in M$.

5.7 Definition (Exponential Map)**.** The map $\exp: \mathcal{U} \to M$ with $\exp(v) = c_v(1)$ is called exponential map.

For $p \in M$ we set $\exp_p := \exp|_{\mathcal{U} \cap T_pM} : \mathcal{U} \cap T_pM \to M$.

5.8 Lemma (Consequences of the Theorem of Picard-Lindelöff)**.**

(1) $\exp$ *is smooth,*

(2) $\exp(tv) = c_v(t)\ \forall t \in (\alpha_v, \beta_v)$,

(3) $D(\exp_p)_{0_p} v = v$ *where* $v \in T_{0_p}(T_pM) \simeq T_pM$. *Note that* $T_pM \simeq \mathbb{R}^m$ *and hence* $T_{0_p}(T_pM) \simeq \mathbb{R}^m \simeq T_pM$.

Hence $D(\exp_p)_{0_p} = id_{T_pM}$ *and in particular there exists an open neighborhood* U *of* 0_p *such that* $\exp_p : U \to \exp_p(U)$ *is a diffeomorphismus.*

Proof. (1) This follows because of smooth dependence of ODEs on intial values.

(2) Follows since $s \in (\frac{1}{t}\alpha_v, \frac{1}{t}\beta_v) \mapsto c_v(st)$ is a maximal solution of $\nabla_t c' = 0$ with $c'(0) = tv$. By uniqueness of solutions of the initial value problem it follows $c_v(st) = c_{tv}(s)$.

(3) Let $v \in T_{0_p}T_pM$ and $\gamma(t) = tv \in T_pM$. Hence $\gamma'(0) = v$. Therefore, we can compute as follows

$$D(\exp_p)_{0_p}v = D(\exp_p)_{0_p}\gamma'(0) = (\exp_p \circ\gamma)'(0) = (\exp_p(tv))'|_{t=0} = c_v'(0) = v.$$

□

5.9 Examples.

(1) $(M, g) = \mathbb{R}^m$: geodesics are curves following straight lines with constant speed $t \mapsto p + tv$. Hence $\exp_p(v) = p + v$.

(2) $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ (embedded sphere). Note that $T_p\mathbb{S}^{n-1} = p^\perp$ and $g_p^{\mathbb{S}^{n-1}} = \langle\cdot,\cdot\rangle_{eucl}|_{T_p\mathbb{S}^{n-1}}$. We know that the geodesic c_v with $c_v(0) = p$ and $c_v'(0) = v$ is given by

$$c_v(t) = \cos t \cdot p + \sin t \cdot v.$$

Hence $\exp_p(tv) = \cos t \cdot p + \sin t \cdot v$. In particular $\exp_p$ is defined everywhere on $T_p\mathbb{S}^{n-1}$ and it is injective on $B_\pi(0_p) \subset T_p\mathbb{S}^{n-1}$.

Similary, one argues for $\mathbb{H}^m \subset \mathbb{R}_1^{m+1}$ (Lorentz-Modell of Hyperbolic space) (Exercise).

(3) Consider a Lie-group G. For $v \in T_eG$ there exists a unique left-invariant vector field $X^v \in \Gamma(TG)$ such that $X^v(e) = v$, that is $X_g^v = Dl_g|_e v$ where $l_g(h) = gh$. The exponential map $\exp : T_eG \to G$ of a Lie group is defined as $\exp(v) = \gamma(1)$ where γ solves $X^v \circ \gamma = \gamma'$ with $\gamma(0) = e$.

Let g be a bi-invariant Riemannian metric on G, that is left- and right-translations l and r are isometries (Exercise sheet 3, Problem 3).

Exercise sheet 7: the flow curve γ of X^v is a geodesic with $\gamma'(0) = v$. Hence

$$\exp(v) = \gamma'(0) = c_v'(0).$$

Hence, the geodesic exponential map $\exp_e$ at $e \in G$ w.r.t. g coincides with exp.

In particular: Consider $G = SO(n)$. $g(A, B) = \langle A, B\rangle = \mathrm{trace}(A^tB)$ induces a bi-invariant metric on G. The exponental map is

$$\exp_E : T_EG \to G, \quad \exp_E(A) = \sum_{k\geq 0} \frac{A^k}{k!} = e^A.$$

5.10 Definition. Let (M, g) be a Riemannian Mfd. $p \in M$ and $U \subset T_pM$ open and star-shaped w.r.t. 0_p such that $\exp_p|_U$ is a diffeomorphsm. Let $L : (T_pM, g_p) \to \mathbb{R}^m, \langle\cdot,\cdot\rangle_{eucl})$ orthogonal. Then we call

$$\varphi : \exp_p(U) \to L(U) \subset \mathbb{R}^m,\ \varphi = L \circ (\exp_p|_U)^{-1}$$

normal chart with center p, or normal coordinates.

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5.11 Theorem. *Let φ be a normal chart on M at p. Then*

$$g_{ij}|_p = \delta_{ij}, \quad {}^{\varphi}\Gamma^k_{ij}(p) = 0.$$

Proof. We compute

$$D\varphi_p v = D\varphi_p c_v'(0) = L(v)$$

and

$${}^{\varphi}g_{ij}|_p = g|_p(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p) = g|_p(D\varphi^{-1}i_p(e_i), D\varphi_p^{-1}(e_j)) = g|_p(L^{-1}e_i, L^{-1}e_j) = \langle e_i, e_j\rangle_{eucl} = \delta_{ij}$$

where we used that L is orthogonal.

Let c be a geodesic with $c(0) = p$ and $c'(0) = v$. Then

$$\varphi \circ c(t) = L \circ (\exp_p|_U)^{-1}(c(t)) = L(tc'(0)) = tLv =: tx.$$

Since L is orthogonal and since the geodesic c was arbitrary, x can be choosen arbitrarily as well.

The geodesic equation for c w.r.t. the normal chart φ is

$$\underbrace{(tx^i)''}_{=0} + \sum_{j,k=1}^m {}^{\varphi}\Gamma^i_{jk} \circ c(t) x^j x^k = 0.$$

At $t = 0$ the equation is

$$\sum_{j,k=1}^m {}^{\varphi}\Gamma^i_{jk}(p) x^j x^k = x^T(\Gamma^i_{jk}(p))_{j,k} x = 0.$$

Since $\Gamma^i_{jk} = \Gamma^i_{kj}$, $(\Gamma^i_{jk}(p))_{jk}$ defines a symmetric bilinear form that maps every $x \in \mathbb{R}^m$ to $0 \in \mathbb{R}^m$. Hence $\Gamma^i_{jk}(p) = 0$ $\forall i, j, k = 1, \ldots, m$. □

5.12 Remark. If $R_p \neq 0$, then there is no chart such that all derivatives of g_{ij} of order 1 and order 2 vanish at p.

5.13 Lemma. *Let $c : [a, b] \times (-\epsilon, \epsilon) \to M$ a smooth map such that $c(\cdot, \tau) =: c_\tau$ is a geodesic for every $\tau \in (-\epsilon, \epsilon)$. Then*

$$\frac{d}{dt}\langle c_\tau'(t), \frac{\partial}{\partial \tau} c|_{(t,\tau)}\rangle = \frac{1}{2}\frac{d}{d\tau}\langle c_\tau'(t), c_\tau'(t)\rangle, \quad \frac{d^2}{dt^2}\langle c_\tau'(t), \frac{\partial c}{\partial \tau}(t, \tau)\rangle = 0 \ \forall t \in [a, b] \quad \forall \tau \in (-\epsilon, \epsilon).$$

Proof.

$$\frac{d}{dt}\langle c_\tau'(t), \frac{\partial}{\partial \tau} c(t, \tau)\rangle = \langle \nabla_t c_\tau'|_t, \frac{\partial}{\partial \tau} c(t, \tau)\rangle + \langle c_\tau'(t), \nabla_t \frac{\partial c}{\partial \tau}(t, \tau)\rangle.$$

This is equal to

$$= \langle c_\tau'(t), \nabla_\tau c_\tau'(t)\rangle = \frac{1}{2}\frac{d}{d\tau}\langle c_\tau'(t), c_\tau'(t)\rangle.$$

where we used $\nabla_t c'_\tau \equiv 0\ \forall \tau$ and $\nabla_\tau \frac{\partial c}{\partial t} = \nabla_t \frac{\partial c}{\partial \tau}$ for the first equality. To see the latter, we compute in local coordinates:

$$\nabla_\tau \left(\sum \frac{\partial c^i}{\partial t} \frac{\partial}{\partial x^i} \circ c \right) = \sum_i \frac{\partial^2 c^i}{\partial \tau \partial t} \frac{\partial}{\partial x^i} \circ c + \sum_{i,j} \frac{\partial c^i}{\partial t} \frac{\partial c^j}{\partial \tau} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \circ c.$$

The last expression on the right hand side is symmetric in t and τ.

Moreover

$$\frac{d^2}{dt^2} \langle c'_\tau, \frac{\partial}{\partial \tau} c \rangle = \frac{1}{2} \frac{d}{d\tau} \frac{d}{dt} \langle c'_\tau, c'_\tau \rangle = \frac{d}{d\tau} \langle \nabla_t c'_\tau, c'_\tau \rangle = 0.$$

□

5.14 Corollary (Gauß Lemma)**.** *Let $U \subset T_pM$ starshaped w.r.t. 0_p and let $\exp_p$ be defined on U. Then for every smooth $v : (-\epsilon, \epsilon) \to U \subset T_pM$ it holds*

$$\langle v(0), v'(0) \rangle_p = \langle c'_{v(0)}(1), (\exp_p \circ v)'(0) \rangle_{\exp_p(v(0))}.$$

Proof. Consider $c(t, \tau) = \exp_p(tv(\tau)) = c_\tau(t)$. Then $c'_\tau(0) = v(\tau)$ and $c'_0(t) = c'_{v(0)}(t)$.

Claim. $\langle c'_0(t), \frac{\partial c}{\partial \tau}\Big|_{(t,0)} \rangle \Big|_{c_0(t)} = t \langle v(0), v'(0) \rangle_p$.

Proof of the claim. For $t = 0$ we have $\langle c'_0(0), \frac{\partial c}{\partial \tau}(0,0) \rangle_{c_0(0)} = 0$ since $c(\tau, 0) = p$ for all τ and $\frac{\partial c}{\partial \tau}(0,0) = 0$.

$$\frac{d}{dt} \langle c'_0(t), \frac{\partial c}{\partial \tau}\Big|_{(t,0)} \rangle \overset{\text{Lemma 5.13}}{=} \frac{1}{2} \frac{d}{d\tau}\Big|_{\tau=0} \langle c'_\tau(t), c'_\tau(t) \rangle = \langle v(0), v'(0) \rangle_p = const.$$

Hence $t \mapsto \langle c'_0(t), \frac{\partial c}{\partial \tau}(t, 0) \rangle = t \langle v(0), v'(0) \rangle_p$ is linear.

For $t = 1$ we have $\langle c'_0(1), \frac{\partial c}{\partial \tau}\Big|_{(1,0)} \rangle = \langle c'_{v(0)}(1), (\exp_p \circ v)'(0) \rangle = \langle v(0), v'(0) \rangle$. That is the statement of the Lemma. □

5.15 Theorem (First variation of arc length)**.** *Set $g = \langle \cdot, \cdot \rangle$ and consider $\alpha : [a, b] \times (-\epsilon, \epsilon) \to M$ smooth, $\alpha_\tau(t) = \alpha(t, \tau)$. We assume $|\alpha'_0(t)| = const = d > 0$. Then*

$$\frac{d}{d\tau} L(\alpha_\tau) = -\frac{1}{d} \int_a^b \langle \nabla_t \alpha'_0|_t, V(t) \rangle dt + \langle \alpha'_0(t), V(t) \rangle \Big|_{t=a}^{t=b}$$

where $V(t) = \frac{\partial \alpha}{\partial \tau}(t, 0) = D\alpha_{(t,0)} \frac{\partial}{\partial \tau}|_t$. The map α is called smooth variation of α_0 and V is called variation vector field along α_0.

Proof of Theorem 5.15.

$$\begin{aligned} \frac{d}{d\tau}\Big|_{\tau=0} L(\alpha_\tau) &= \int_a^b \frac{d}{d\tau}\Big|_{\tau=0} \langle \alpha'_\tau(t), \alpha'_\tau(t) \rangle^{\frac{1}{2}} dt = \frac{1}{d} \int_a^b \langle \nabla_\tau \frac{\partial \alpha}{\partial t}\Big|_{(t,0)}, \alpha'_0(t) \rangle dt \\ &= \frac{1}{d} \int_a^b \langle \nabla_t \frac{\partial \alpha}{\partial \tau}\Big|_{(t,0)}, \alpha'_0(t) \rangle dt = -\frac{1}{d} \int_a^b \langle \frac{\partial \alpha}{\partial \tau}\Big|_{(t,0)}, \nabla_t \alpha'_0(t) \rangle dt + \langle V(t), \alpha'_0(t) \rangle \Big|_a^b. \end{aligned}$$

In the first equality we used that $\alpha : [a, b] \times (-\epsilon, \epsilon) \to M$ is smooth. In the second equality we used the chain rule and that ∇ is a Riemannian connection.

In the third equality we used $\nabla_\tau \frac{\partial \alpha}{\partial t} = \nabla_t \frac{\partial \alpha}{\partial \tau}$. In the fourth inequality we use again that ∇ is a Riemannian connection. □

More generally. A variaton of a piecewise smooth curve $c : [a, b] \to M$ is a continuous map $\alpha : [a, b] \times (-\epsilon, \epsilon) \to M$ such that

1. $\alpha(t, 0) = c(t)\ \forall t \in [a, b]$,
2. $\exists a = t_0 < \dots t_N = b$ such that $\alpha|_{[t_{i-1},t_i]\times(-\epsilon,\epsilon)}$ is smooth $\forall i = 1, \dots, N$.

A variation is called proper if $\alpha(a, \tau) = c(a)$ and $\alpha(b, \tau) = c(b)\ \forall \tau \in (-\epsilon, \epsilon)$. If α is smooth, we call α a smooth variation.

The first variation formula also holds for general (non-smooth) variations with appropriate boundary terms.

5.16 Lemma. *Given a smooth vector field $V(t)$ along a differentiable curve $c : [a, b] \to M$, then there exists a smooth variation $\alpha : [a, b] \times (-\epsilon, \epsilon) \to M$ of c such that $V(t)$ is the corresponding variational vector field of α. If $V(a) = V(b) = 0$, we can choose α as a proper variation.*

Proof. We set $\alpha(t, \tau) = \exp_{c(t)}(\tau V(t))$. Since $c([a, b]) \subset M$ is compact compact, we can find $\epsilon > 0$ such that α is well defined on $\alpha : [a, b] \times (-\epsilon, \epsilon) \to M$. Moreover $\frac{\partial\alpha}{\partial\tau}|_{(t,0)} = V(t)$. If $V(a) = 0$ then $\alpha(a, \tau) = \exp_{c(a)}(0_{c(a)}) = c(a)$ for all $\tau \in (-\epsilon, \epsilon)$ and similarly if $V(b) = 0$. $\square$

5.17 Corollary. *(1) $c : [a, b] \to M$ geodesic, then*

$$\frac{d}{d\tau}L(\alpha_\tau) = 0$$

for every variation α of c.

(2) If $|c'(t)| = const$ and $\frac{d}{d\tau}_{\tau=0}L(\alpha_\tau) = 0$ for every Variation α of $c = \alpha_0$ with $V(a) = V(b) = 0$, then c is a geodesic.

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5.18 Definition. Let (M,g) be a Riemannian manifold, $p \in M$, $U \subset T_pM$ open such that $0_p \in U$ and $\exp_p : U \to V := \exp_p(U)$ is a diffeomorphism.

- $r := r_p : V \to [0,\infty)$, $r(q) = |\exp_p^{-1}(q)| = \sqrt{g(\exp_p^{-1}(q), \exp_p^{-1}(q))}$.

  In particular $r \in C^\infty(V\backslash\{p\})$ and $r(c_v(1)) = r(\exp_p(v)) = |v| = L(c_v|_{[0,1]})$, since $|c_v'(t)| = |c_v'(0)| = |v|$ and therefore $L(c_v|_{[0,1]}) = \int_0^1 |v|dt = |v|$.

- $Q := Q_p : V \to [0,\infty)$, $Q(q) = \langle \exp_p^{-1}(q), \exp_p^{-1}(q)\rangle = (r(q))^2$.

  In particular $Q \in C^\infty(M)$.

- $X := X_p \in \Gamma(TV)$, $X_q = c_v'(1)$ where $v = \exp_p^{-1}(q)$.

5.19 Lemma. $\nabla Q = \sharp \circ dQ = 2X$.

Proof. It is sufficient to show that $dQ_{\gamma(0)}(\gamma'(0)) = 2\langle X(\gamma(0)), \gamma'(0)\rangle$ for every regular curve $\gamma : (-\epsilon,\epsilon) \to M$. We set $v(t) = (\exp_p|_U)^{-1}(\gamma(t))$.

We compute

$$(Q\circ\gamma)'(0) = \frac{d}{dt}\langle v(t), v(t)\rangle|_{t=0} = 2\langle v'(t), v(t)\rangle = 2\langle(\exp_p \circ v)'(0), c_{v(0)}'(1)\rangle = 2\langle \gamma'(0), X(\gamma(0))\rangle.$$

□

5.20 Corollary. *It holds $|\nabla r_p| = const = 1$ on $V\backslash\{p\}$.*

Proof. We compute

$$2X_q = \nabla Q|_q = \nabla(r^2)|_q = 2r(q)\nabla r|_q.$$

For this we can consider $dr_q^2(\gamma'(0)) = \langle \nabla(r^2)_q, \gamma'(0)\rangle$ and apply the chain rule to dr^2.

Moreover, if $\exp_p^{-1}(q) = v$, then $|X_q| = |c_v'(1)| = |v| = |\exp_p^{-1}(q)| = |r(q)|$.

Hence $|\nabla r|_q| = 1$. □

5.21 Remark. Recall $c_v(s) = c_{tv}(s/t)$. Hence

$$tc_v'(s)|_{s=t} = c_{tv}'(1) = X \circ \exp_p(tv) = r \circ \exp_p(tv)\nabla r \circ \exp_p(tv) = t|v|\nabla r \circ c_v(t).$$

If $|v| = 1$, it follows $c_v'(t) = \nabla r \circ c_v(t)$ $\forall t > 0$. It follows that c_v is a flow curve of the vector field ∇r. If $q \in V\backslash\{p\}$, then there exists $v \in U$ such that $\exp_p(v) = q$ and therefore $c_{v/|v|}$ is the unique flow curve of ∇r through q.

Recall $d : M \times M \to [0,\infty)$ defined via

$$d(p,q) = \inf\{L(\gamma) : \gamma : [a,b] \to M \text{ piecewise } C^1 \text{ and } \gamma(a) = p, \gamma(b) = q\}.$$

5.22 Theorem. *Let (M,g) be Riemannian manifold, $p \in M$ and $\rho > 0$ such that $\exp_p|_{B_\delta(0_p)}$ a diffeomorphismus onto its image. Then*

(1) $\forall q \in \exp_p(B_\delta(0_p))$ it holds $r_p(q) = d(p,q)$.

(2) $\forall v \in B_\delta(0_p)$ it holds

$$L(c_v|_{[0,1]}) = d(p, c_v(1)).$$

Hence $c_v|_{[0,1]}$ is a length minimizing curve between p and $c_v(1)$.

Proof. Note that (2) implies (1) since for $q \in \exp_p(B_\delta(0_p))$ we can choose $v \in T_pM$ such that $\exp_p(v) = c_v(1) = q$.

Let $v \in B_\delta(0_p)$ and $c_v(1) = q$, and let $\gamma : [a,b] \to M$ be piecewise regular such that $\gamma(a) = p$ and $\gamma(b) = q$.

Claim. $L(\gamma) \geq |v| = L(c_v|_{[0,1]})$.

Proof of the claim. W.l.o.g. $\gamma(t) \neq p$ $\forall t \neq a$.

1. Assume $\gamma([a,b]) \subset V = \exp_p(B_\delta(0_p))$. Then

$$\begin{aligned}|v| = r_p(q) = r_p(q) - r_p(p) &= \int_a^b (r_p \circ \gamma)'(t)dt = \int_a^b dr_p(\gamma'(t))dt \\ &= \int_a^b \langle \nabla r_p|_{\gamma(t)}, \gamma'(t)\rangle dt \leq \int_a^b |\gamma'(t)|dt = L(\gamma).\end{aligned}$$

Hence $L(c_v|_{[0,1]}) \leq L(\gamma)$.

Note that this estimate is proved for γ regular, but clearly generalizes for γ piecewise regular.

Assume we have γ such that equality holds in the previous estimate. Then, whenever γ is differentiable in t we have equality in

$$\langle \nabla r_p \circ \gamma(t), \gamma'(t)\rangle \leq |\gamma'(t)||\nabla r_p \circ \gamma(t)|,$$

and equality holds if and only if $\gamma'(t) = \lambda(t)\nabla r_p \circ \gamma(t)$ for $\lambda(t) \geq 0$. This is the equality case of the Cauchy-Schwarz inequality. Hence $|\gamma'(t)| = \lambda(t)$ when γ is differentiable in t.

In this case, by Lemma 2.14, we know that $\gamma \circ \varphi^{-1}(t) =: c(t), t \in [0, L(\gamma)]$ with $\varphi(t) = \int_a^t \lambda(s)ds$ is an arclength reparametrization of γ.

Moreover, if c is differentiable in t, then

$$c'(t) = \gamma' \circ \varphi^{-1}(t)\frac{1}{\varphi' \circ \varphi^{-1}(t)} = \nabla r_p \circ c(t).$$

Hence c is a gradient flow curve of ∇r_p and in particular also differentiable in every t since left and right hand derivatives of c in every point t are both equal to $\nabla r_p(c(t))$. It follows there exists $w \in T_pM$ with $|w| = 1$ such that $c = c_w$ with $c_w(L(\gamma)) = q$. More precisely, by uniqueness of the gradient flow curve $c_{v/|v|}$ of r_p through q we have $c_w = c_{v/|v|}$, or $w = \frac{1}{|\exp_p^{-1}(q)|}\exp_p^{-1}(q)$.

2. Assume $\gamma([a,b]) \cap V^c \neq \emptyset$.

Consider $\delta' \in (0,\delta)$. Then $\exp_p|_{B_{\delta'}(0_p)}$ is still a diffeomorphim. Moreover $\exp_p(\overline{B_{\delta'}0_p})) = \overline{V'}$ is compact and $\partial V' = \exp_p(\partial B_{\delta'}(0_p))$.

Let $t_0 = \inf\{t \in (a,b] : \gamma(t) \in (V')^c\}$. By continuity of γ $t_0 \in (a,b]$ and $\gamma(t_0) \in \partial V'$. We apply the first case to $\gamma|_{[a,t_0]}$. Define $w = \exp_p^{-1}(\gamma(t_0))$. Then it follows $L(\gamma) \geq L(\gamma|_{[a,t_0]}) \geq |w| = \delta'$. This holds for every $\delta' \in (0,\delta)$ and hence $L(\gamma) > |v|$ since $v \in B_\delta(0_p)$. Hence, the equality case can only appear when $\gamma([a,b]) \subset V$ and in this case γ is a reparametrization of $c_{v/|v|}|_{[0,|v|]}$. $\square$

5.23 Remark.

- The previous theorem implies $\exp_p(B_\delta(0_p)) \subset B_\delta^{d_g}(p)$ if $\exp_p|_{B_\delta(0_p)}$ is a diffeomorphism.

  On the other hand, let $r > \delta > 0$ be small enough such that $B_\delta^{d_g}(p) \subset \exp_p(B_r(0_p))$ and such that $\exp_p|_{B_r(0_p)}$ is a diffeomorphism. If $q \in B_\delta^{d_g}(p)$, let $v = \exp_p^{-1}(q)$. Then $c_v : [0,1] \to M$ is curve that connects $p = c_v(0)$ and $q = c_v(1)$ with $\delta > d(p, c_v(1)) = L(c_v) = |v|$ according to the previous theorem and in this case $B_\delta^{d_g}(p) = \exp_p(B_\delta(0_p))$.

- Let $p, q \in M$ with $p \neq q$, then $d_g(p,q) > 0$.

  Proof. For $\delta > 0$ small $\exp_p|_{B_\delta(0_p)}$ is a diffeomorphism and $\exp_p(B_\delta(0_p)) = B_\delta^{d_g}(p)$ is an open neighborhood w.r.t. the topology on M. In particular the manifold topology and the metric topology coincide. Since M is Hausdorffsch, there exists $\delta > 0$ such that $B_\delta^{d_g}(p) \cap B_\delta^{d_g}(q) = \emptyset$. Then it follows $d_g(p,q) > \delta$ since otherwise $q \in B_\delta^{d_g}(p)$ by the previous theorem. $\square$

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5.24 Theorem (Hopf-Rinow)**.** *Let (M, g) be Riemannian manifold. The following statements are equivalent:*

1. (M, d_g) *is a complete metric space.*
2. $\exists p \in M$ *such that* $\exp_p$ *is defined on* T_pM.
3. $\forall v \in TM$ c_v *is defined on* $\mathbb{R}$.

Any of the previous statements implies that

4. $\forall p, q \in M$ $\exists$ *a geodesic* $c : [a, b] \to M$ *such that* $c(a) = p$, $c(b) = q$ *and* $d(p, q) = L(c)$.

A Riemannian manifold (M, g) is called complete if one of the previous statements 1. or 2. or 3. is satisfied.

5.25 Examples. (1) M compact (without boundary points), then M is complete, since 1. is satisfied.

(2) $(\mathbb{R}^m, \langle\cdot,\cdot\rangle_{eucl})$, $(\mathbb{S}^{m-1}, i^*\langle\cdot,\cdot\rangle_{eucl})$ and $(\mathbb{H}^{m-1}, i^*\langle\cdot,\cdot\rangle_1)$ are complete, since 2. is satisfied.

(3) Let $M \subset \mathbb{R}^n$ be an embedded submanifold, such that M is a closed subset of $\mathbb{R}^n$. Then $(M, i^*\langle\cdot,\cdot\rangle_{eucl} = g)$ is complete. This can be seen as follows.

$d_g(p, q) \geq |p - q|_{eucl} \Rightarrow \overline{B_r^{d_g}(p)} \subset M \cap \overline{B_r^{\langle\cdot,\cdot\rangle_{eucl}}(p)}$. The set on the RHS is compact in $\mathbb{R}^n$. Moreover $\overline{B_r^{d_g}(p)}$ is closed w.r.t. the d_g topology and hence w.r.t. the manifold topology that comes from $\mathbb{R}^n$. Hence $\overline{B_r^{d_g}(p)}$ is a closed set in $\mathbb{R}^n$. Since it is also subset of a compact set in $\mathbb{R}^n$, it is a compact subset in $\mathbb{R}^n$ itself. Then it follows also that $\overline{B_r^{d_g}(p)}$ is compact w.r.t. d_g. Therefore (M, d_g) is complete as a metric space.

5.26 Lemma. *Assume* $\exp_p|_{B_r(0_p)}$ *is a diffeomorphism, and let* $\rho \in (0, r)$*. Then it holds:* $\forall q \in M \backslash \overline{B_\rho^{d_g}(p)}$ *there exists* $q' \in \partial B_\rho^{d_g}(p)$ *such that* $d(p, q) = d(p, q') + d(q', q)$.

Proof. Note first that $\overline{B_\rho^{d_g}(p)} = \exp_p(\overline{B_\rho(0_p)})$ is compact. Here we use that $\rho \in (0, r)$.

It follows that also $\partial B_\rho^{d_g}(p)$ is compact.

Since $d_g(x, q) = r_q(x)$ is continuous on $\partial B_\rho^{d_g}(p)$, there exists $q' \in \partial B_\rho^{d_g}(p)$ such that $d(q', q) = \min_{\widetilde{x} \in \partial B_\rho^{d_g}(p)} d(x, q)$.

Let $\gamma : [0, 1] \to M$ be piecewise C^1 such that $c_v(0) = p$ and $c_v(1) = q$. There exists $t_0 \in [0, 1]$ such that $\gamma(t_0) \in \partial B_\delta(p)$. Hence $L(\gamma|_{[0,t_0]}) \geq \delta = d(p, q')$.

On the other hand it hold $L(\gamma|_{[t_0,1]}) \geq d(\gamma(t_0), q) \geq d(q', q)$.

Hence $L(\gamma) \geq d(p, q') + d(q', q)$, and so $d(p, q) \geq d(p, q') + d(q', q)$, and therefore we have equality. $\square$

Proof of the Theorem of Hopf-Rinow. **1.** Assume 2. We show 4. : $\exists$ shortest curve between p and any $q \in M$.

Choose $\delta \in (0, d_g(p,q))$ as in the previous Lemma. We set

$$v(x) := \frac{(\exp_p|_{\overline{B_\delta(0_p)}})^{-1}(x)}{|\cdots|} \Rightarrow |v(x)| = 1.$$

Because of 2. $\exp_p(tv(x)) = c_{v(x)}(t)$ is defined on $\mathbb{R}$.

For $q \in M$ let $q' \in \partial B^{d_g}_\delta(p)$ be as in 5.26 Lemma where we choose $\delta > 0$ small enough such that $\exp_p^{-1}$ is defined on $B^{d_g}_{2\delta}(p)$. Consider $v(q') = v$ and $c_v : \mathbb{R} \to M$.

Claim. $c_v(d_g(p,q)) = q$. Then it follows $L(c_v|_{[0,d_g(p,q)]}) = d_g(p,q)$ and c_v is the desired shortes curve.

Let $t \in [\delta, d_g(p,q)]$ be maximal such that $d_g(p,q) = t + d_g(c_v(t), q)$ (this is true for $t = \delta$ by 5.26 Lemma).

Assume $t < d_g(p,q)$. Then we choose $\delta'' > 0$ small enough such that $q \notin B^{d_g}_{\delta''}(c_v(t))$ and as in 5.26 Lemma. Let $q'' \in \partial B_{\delta''}(c_v(t))$ such that

$$d_g(c_v(t), q) = d_g(c_v(t), q'') + d_g(q'', q)$$

and let $w \in T_{c_v(t)}M$ with $|w| = 1$ such that $c_w(\delta'') = q''$.

It follows

$$d_g(p,q) = t + d(c_v(t), q) = t + \delta'' + d_g(q'', q).$$

Note that $t = L(c_v|_{[0,t]}) \geq d_g(p, c_v(t))$ and $\delta'' = d(c_v(t), q'')$. Hence

$$d_g(p,q) \geq d_g(p, c_v(t)) + d_g(c_v(t), q'') + d_g(q'', q) \geq d_g(p,q).$$

In particular, it follows $t = L(c_v|_{[0,t]}) = d_g(p, c_v(t))$.

We define $\gamma = c_w|_{[0,\delta'']} * c_v|_{[0,t]}$. It follows that $L(\gamma) = t + \delta'' = d_g(p, q'')$.

Hence γ is piecewise smooth curve, that is also length minimizer. Hence it ist geodesic. In particular $c_v'(t) = c_w'(0) = w$ and γ is a smooth extension of c_v beyond t

Therefore we have

$$d(p,q) = (t + \delta'') + d_g(q'', q) = t + \delta'' + d_g(c_v(t + \delta''), q).$$

That is in contradiction to the maximality of t.

2. 2. $\Rightarrow$ 1.

Claim. $\exp_p\left(\overline{B_r(0_p)}\right) = \overline{B^{d_g}_r(p)}\ \forall r > 0$.

Proof of the claim. $\subset$: Let $v \in T_pM$ with $|v| \leq r$. Then it follows with $c_v : [0,1] \to M$ satisfies $c_v'(t) = |v|$. Hence $d(p, c_v(1)) \leq L(c_v) = |v| \leq r$, and therefore $\exp_p(v) \in \overline{B^{d_g}_r(p)}$.

$\supset$: Consider $q \in M$ with $d_g(p,q) \leq r$. Because **1.** there exists $v \in T_pM$ such that $c_v(1) = \exp_p(v) = q$ and $|v| = L(c_v|_{[0,1]}) = d(p,q) \leq r$. Hence $v \in B_r(0_p)$.

The claim implies that $\overline{B_r(p)}$ is compact for all $r > 0$. Hence (M, d_g) is complete.

3. 1. $\Rightarrow$ 3.

Assume there exists $v \in TM$ such that $\omega_v < \infty$. Then we choose $t_i \uparrow \omega_v$ and $c_v(t_i)$ are Cauchysequence w.r.t. d_g. For this note that

$$d_g(c_v(t_i), c_v(t_{i+1})) \leq L(c_v|_{[t_i,t_{i+1}]}) \leq (t_{i+1} - t_i)|v|.$$

Since (M, d_g) is a complete metric space, $\exists q \in M$ such that $c_v(t_i) \to q$.

Set $w_i = c_v'(t_i)$. In particular $|w_i| = |v|$. Since $c_v(t_i) \to q$ and since $|w_i| = |v|$, there exists a subsequence of w_i that converges to $w \in T_qM$.

On the other hand, we know that $W = \{(\widetilde{v}, t) \in TM \times \mathbb{R} : t \in (\alpha_{\widetilde{v}}, \omega_{\widetilde{v}})\}$ is open. Hence we can choose $t_0 > 0$ such that $(w, t_0) \in W$. It follows that $i \in \mathbb{N}$ big enough $(w_i, t_0) \in W$. By uniqueness of geodesics we have that $c_{w_i}(t_0) = c_v(t_i + t_0)$. Hence $t_i + t_0 < \omega_v$ for i sufficiently big. But we assumed $t_i \uparrow \omega_v$. This is a contradiction.

3. $\Rightarrow$ 2. is obviously true. $\square$

5.27 Proposition. *$\exists V \subset M \times M$ neighborhood of $\{(p,p) \in M \times M : p \in M\}$ and $V' \subset TM$ a neighborhood of $\{0_p \in T_pM : p \in M\}$ such that $\pi \times (\exp|_{V'}) : V' \to V$ diffeomorphismus.*

Proof. **1.** $\forall p \in M$ we have that $D(\pi \times \exp)_{0_p} : T_{0_p}TM \to T_pM \times T_pM$ is an isomorphism. Indeed: let $v \in T_pM$. Then

$$D(\pi \times \exp)_{0_p} v = \frac{d}{dt}|_{t=1}(\pi \times \exp)(0_p + tv) = (0_p, c_v'(1)) \in \{0_p\} \times T_pM \backslash \{(0_p, 0_p)\}.$$

If $\gamma : (-\epsilon, \epsilon) \to M$ and $\gamma(0) = p$, then $\widetilde{\gamma}(t) = 0_{\gamma(t)}$ is a horizontal curve in TM. It follows $\pi \times \exp(\widetilde{\gamma}) = (\gamma(t), \gamma(t))$ and

$$\frac{d}{dt}|_{t=0}\pi \times \exp(\widetilde{\gamma}(t)) = (\gamma'(0), \gamma'(0)) \subset T_pM \times T_pM \backslash \{(0_p, 0_p)\}.$$

The span of these two images generates all of $T_pM \times T_pM$. Hence $D(\pi \times \exp)_{0_p}$ is surjective and hence an isomorphism.

2. It already follows that in a neighborhood of every 0_p the map $\pi \times \exp$ is a diffeomorphism (*).

$\forall p \in M$ there exists $\epsilon_p > 0$ such that $\forall v \in T_pM$ with $g_p(v, v) < \epsilon_p$ we have that $D(\pi \times \exp)_v$ is an Isomorphism.

Let $U \subset M$ be open such that $\overline{U}$ is compact. The dependence of ϵ_p on p is continuous. Hence there exists $\epsilon > 0$ such that $\forall v \in TU$ with $|v|^2 < \epsilon$ we have that $D(\pi \times \exp)_v$ is a diffeomorphism. We set $TU^\epsilon = \{v \in TU : |v| < \epsilon\}$.

Moreover we can choose $\epsilon > 0$ small enough such that $\pi \times \exp|_{TU^\epsilon}$ is injective. Otherwise $\exists \epsilon_n$ with $\epsilon_n \downarrow 0$ and $v_n, w_n \in TU^{\epsilon_n}$ with $\pi(v_n) = \pi(w_n) = p_n$ and $\exp(v_n) = \exp(w_n) = q_n$. Since $\overline{U}$ compact, there exists a subsequence n_i such that $(p_{n_i}, q_{n_i}) \to (p, q)$ and $v_{n_i}, w_{n_i} \to 0_p$. Hence $q = p$. But this contradicts (*).

3. We can find a countable cover of M by such U_is with corresponding ϵ_is. We set $V = \bigcup_i TU_i^{\epsilon_i}$ that is a neighborhood of $\{0_p : p \in M\}$. Then $\pi \times \exp|_V$ is a local diffeomorphism. But it is also injective, since otherwise $\exists v, w \in V$ with $\pi(v) = \pi(w) = p$. Assume $v \in TU_i^{\epsilon_i}$ and $w \in TU_j^{\epsilon_j}$. If $\epsilon_i \leq \epsilon_j$ then $v, w \in TU_i^{\epsilon_i}$ and by the previous step we would get $v = w$. $\square$

26.06.2023

6 Jacobi Fields

Let (M,g) be a Riemannian manifold. We consider $\alpha : [a,b] \times (-\epsilon,\epsilon) \to M$ smooth such that $\forall \tau \in (-\epsilon,\epsilon)$ the curvces $c_\tau(t) = \alpha(t,\tau)$ are geodesics. Set $c_0 = c$. (For instance we can consider $\alpha(t,\tau) = \exp_p(tv(\tau))$ where $v : (-\epsilon,\epsilon) \to T_pM$ is smooth.)

6.1 Lemma. *The variation vector field $Y(t) = \frac{\partial}{\partial\tau}\alpha(t,0) \in \Gamma(c^*TM)$ of α along c satisfies*

$$\nabla_t\nabla_t Y + R(Y,c')c' = 0 \quad (*).$$

Proof. Since the Levi-Civita connection is symmetric, it follows that

$$\nabla_t \frac{\partial\alpha}{\partial\tau}|_{(t,\tau)} = \nabla_\tau \frac{\partial\alpha}{\partial t}|_{(t,\tau)}.$$

Then, together with 4.4 Theorem it follows

$$\nabla_t\nabla_t \frac{\partial\alpha}{\partial\alpha}|_{(t,0)} = \nabla_t\nabla_\tau \frac{\partial\alpha}{\partial t}|_{(t,0)} = \left(\nabla_\tau\nabla_t \frac{\partial\alpha}{\partial t}\right)|_{(t,0)} + R(\frac{\partial\alpha}{\partial t}|_{(t,0)}, \frac{\partial\alpha}{\partial\tau}|_{(t,0)})\frac{\partial\alpha}{\partial t}|_{(t,0)}.$$

Since $\alpha(\cdot,\tau)$ is geodesic $\forall\tau \in (-\epsilon,\epsilon)$, the first term on the right hand side vanishes. □

6.2 Definition. Let $c : [a,b] \to M$ be a geodesic. Solutions of the equation $(*)$ are called Jacobi fields along c. $(*)$ is called Jacobi equation.

6.3 Remark. If Y is a Jacobi field along c and $\widetilde{c}(t) = c(a+tb)$ a linear reparametrization of c with $a,b \in \mathbb{R}$, then $\widetilde{Y}(t) = Y(a+tb)$ is Jacobi field along $\widetilde{c}$.

6.4 Remark. Let $E_i, i = 1,\dots,m$, be parallel vectorfields along c such that $(E_i(t))_{i=1,\dots,m}$ is an orthonormal basis $\forall t \in [a,b]$ and $E_1(t) = c'(t)$. Then for any $Y \in \Gamma(c^*TM)$ we can write $Y = \sum_{i=1}^m Y^i(t)E_i(t)$ for $Y^i \in C^\infty([a,b],\mathbb{R})$. We compute then

$$\nabla_t Y|_t = \sum_{i=1}^m (Y^i)'(t)E_i(t) \text{ and } \nabla_t\nabla_t Y|_t = \sum_{i=1}^m (Y^i)''(t)E_i(t).$$

Moreover

$$R(Y(t),c'(t))c'(t) = \sum_{i=1}^m Y^i(t)R(E_i(t),E_1(t))E_1(t) = \sum_{i=1}^m\sum_{k=1}^m Y^i(t)R^k_{i11}(t)E_k(t)$$

where $\sum_{k=1}^m R^k_{i11}(t)E_k(t) = R(E_i(t),E_1(t))E_1(t)$ and $R^k_{i11} \in C^\infty([a,b],\mathbb{R})$. Note that $R^k_{111} = R^1_{i11} = 0$ for all $k,i = 1,\dots,m$.

Therefore, the Jacobi equation $(*)$ is equivalent to

$$(Y^k)''(t) + \sum_{i=2}^m R^k_{i11}(t)Y^i(t) = 0,\ k = 2\dots,m, \text{ and } (Y^1)''(t) = 0.$$

This is a linear System of ODEs of second order and has a unique solution for initial values $Y(t_0) = v$ and $Y'(t_0) = w$ with $t_0 \in [a,b]$. In the following let $[a,b] = [0,T]$ and $t_0 = 0$.

It follows that $\{Y \in \Gamma(c^*TM) : Y \text{ Jacobi}\}$ is a $2m$-dimensional $\mathbb{R}$-vector space.

The Jacobi equation is the linearisation of the geodesic equation.

6.5 Lemma. *Let c be a geodesic and $Y \in \Gamma(c^*TM)$. We can write $Y = Y^\top + Y^\perp$ where $Y^\top || c'$ and $Y^\perp \perp c'$, i.e. $\langle Y^\perp, c'\rangle = 0$. (If E_i is as before, then $Y^\top = Y^1 E_1$ and $Y^\perp = Y - Y^\top$.)*

Then Y is a Jacobi field if and only if $Y^\top$ and $Y^\perp$ are Jacobi field.

Proof. One direction is clear.

Assume Y is Jacobi. Then Y^1 satisfies $(Y^1)''(t) = 0$. Hence $\exists a, b \in \mathbb{R}$ such that $Y^1(t) = a + tb$ and therefore $Y^\top = (a+tb)c'$ that satisfies the Jacobi equation. Hence $Y^\top$ is a Jacobi field.

Since the Jacobi equation is linear, also $Y^\perp = Y - Y^\top$ is a Jacoi field. $\square$

6.6 Remark. $\dim\{Y \in \Gamma(c^*TM) : Y \perp c' \text{ and } Jacobi\} = 2m - 2$.

6.7 Example. Let (M, g) be Riem. mfd. with constant sectional curvature $K_0 \iff R(u,v)w = K_0\left(\langle v, w\rangle u - \langle u, w\rangle v\right)$.

Let $E_i, i = 1, \ldots, m$ be orthonormal, parallel frame along c with $c'(t) = E_1(t)$. Then $R(E_j, E_m)E_m = K_0\left(E_j - \delta_{jm}E_m\right)$ and

$$R^i_{j11} = K_0\left(\delta^i_j - \delta_{j1}\delta^i_1\right) = 0 \text{ if } j = 1, i = 1 \text{ or } j \neq i, i, j > 1, \text{ and } = K_0 \text{ if } i = j > 1.$$

It follows that $Y = \sum_{i=2}^m Y^i E_i$ Jacobi field (with $Y \perp c'$)

$$\iff (Y^i)'' + K_0 Y^i = 0 \ \forall i = 2, \ldots, m$$

$\iff \exists A, B \in \Gamma(c^*TM)$ parallel along c d.h. $\nabla_t A = \nabla_t B = 0$ such that

$$Y(t) = \begin{cases} \cos\left(\sqrt{K_0}t\right) A(t) + \sin\left(\sqrt{K_0}t\right) B(t) & K_0 > 0, \\ A(t) + tB(t) & K_0 = 0, \\ \cosh(\sqrt{|K_0|}t)A(t) + \sinh(\sqrt{|K_0|}t)B(t) & K_0 < 0. \end{cases}$$

6.8 Lemma. *Let $c(t) = \exp_p(tv) = c_v(t)$, $t \in [0, T]$, be a geodesic with $v \in T_pM$, let $w, z \in T_pM$ and Y the Jacobi field with $Y(0) = w$ and $Y'(t) = z$. We choose $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = w$, and $X \in \Gamma(\gamma^*TM)$ with $X(0) = v$ and $\nabla_t X = z$. Then*

$$Y(t) = \frac{\partial}{\partial\tau}|_{\tau=0} \exp_{\gamma(\tau)}(tX(\tau)) = \frac{\partial}{\partial\tau}|_{\tau=0} \exp(tX(\tau)).$$

In particular, if $w = 0$, then $\gamma \equiv p$ and $X(t) \in \Gamma(T_pM)$. It follows in this case

$$Y(t) = \frac{\partial}{\partial\tau}\Big|_{\tau=0} \exp_p(tX(\tau)) = D(\exp_p)_{tv}(tz) = tD(\exp_p)_{tv}(z).$$

Every Jacobi field is the variation vector field of a 1-Parameter family of geodesics.

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Proof. Consider $\alpha : [0,T]\times(-\epsilon,\epsilon) \to M$ with $\alpha(t,\tau) = \exp_{\gamma(\tau)}(tX(\tau))$. α is smooth. Then $\frac{\partial}{\partial t}\alpha(t,0) = c'(t)$ and $\frac{\partial}{\partial\tau}\Big|_{\tau=0}\alpha(t,0) = \widetilde{Y}(t)$ is a Jacobi field along c since α is a smooth 1-parameter family of geodesics.

We have also observed that $\nabla_t \frac{\partial\alpha}{\partial\tau} = \nabla_\tau \frac{\partial\alpha}{\partial t}$. At $t=0$ it follows $Y(0) = \frac{d}{d\tau}|_{\tau=0} \exp_{\gamma(\tau)}(0) = \gamma'(0) = w$. Moreover

$$\nabla_t \frac{\partial\alpha}{\partial\tau}_{(0,0)} = \nabla_\tau \frac{\partial\alpha}{\partial t}\Big|_{(0,0)} = \nabla_\tau\Big(\frac{d}{dt}\Big|_{t=0} \exp_{\gamma(t)}(tX(\tau))\Big)\Big|_{\tau=0} = \nabla_\tau X(\tau)\Big|_{\tau=0} = z.$$

Because of uniqueness of the initial value problem it follows $Y = \widetilde{Y}$. □

6.9 Definition. Let (M,g) be a Riemannian manifold and $p\in M$. A point $q\in M$ is called conjugated to p if q is a singular value of the map $\exp_p$, i.e. $\exists v\in T_pM$ with $\exp_p(v)$ such that $D(\exp_p)_v : T_pM \to T_qM$ is degenerated (does not have full rank).

The next corollary follows directly from the previous lemma.

6.10 Corollary. *A point $q\in M$ is conjugated to p if and only if there exists a geodesic $c : [0,T]\to M$ with $c(0) = p$ and $c(T) = q$, and a Jacobi field $Y\neq 0$ along c such that $Y(0) = Y(T) = 0$. One also says q is conjugated to p along c (via Y).*

6.11 Remark. If q is conjugated to p along $c : [0,T]\to M$ via Y, then it follows $\langle Y(0), c'(0)\rangle = 0 = \langle Y(T), c'(T)\rangle$. But $Y^1(t) = \langle Y(t), c'(t)\rangle$ is linear. Hence $\langle Y(t), c'(t)\rangle = 0$ for all $t\in[0,T]$. It follows that Y is orthogonal to c'.

Let $U\subset T_pM$ be open and star-shaped w.r.t. 0_p, and let $\exp_p|_U : U\to V\subset M$ be a diffeomorphism. We want to analyze the pull-back metric $\widetilde{g} = (\exp_p)^*g$ on T_pM. $\widetilde{g}$ is essentially g in normal coordinates.

In the following we use the notation $R(u,v,w,z) = \langle R(u,v)w, z\rangle$. Note that $\widetilde{g}_{0_p}(w,z) = g_p(D(\exp_p)_{0_p}w, D(\exp_p)_{0_p}) = g_p(w,z) =: \langle w,z\rangle_p$.

6.12 Theorem. $\forall u\in U$ *and* $\forall w,z\in T_pM \simeq T_v(T_pM)$*:*

$$\widetilde{g}_v(w,z) = \langle w,z\rangle_p - \frac{1}{3}R(w,v,v,z) - \frac{1}{6}(\nabla_v R)(w,v,v,z) + \underbrace{o(\langle v,v\rangle_p^2)}_{|v|^4 f(v,w,z) \text{ with } f \text{ bounded}}.$$

(Taylor expansion of $\widetilde{g}$ at $v\in T_pM$)

Proof. Let $c(t) = \exp_p(tv)$ and let $P_t = P^c_{0,t} : T_pM \to T_{c(t)}M$ be the parallel transport along c. We define

$$R(t) = P_t^{-1}(R(P_t(\cdot), c'(t))c'(t)) \in \mathrm{End}(T_pM).$$

Since P_t is orthogonal and because of the symmetries of R it follows

$$\langle R(t)w, z\rangle_p = \langle w, R(t)z\rangle_p.$$

Hence $R(t)$ is symmetric linear map (self-adjoint). Also note the following: $R'(0)w = (\nabla_v R)(w,v)v$. Moreover $\nabla_v R$ satisfies the same symmetries as R and therefore $R'(0)$ is also symmetric.

Let $y : [0,1] \to \mathrm{End}(T_pM)$ be the solution of the following system of ODEs:

$$y''(t) + R(t)y(t) = 0 \text{ with } y(0) = 0 \text{ and } y'(0) = \mathrm{id}_{T_pM}.$$

Then it is straightforward to check that y solves this equation if and only if $Y(t) = P_t(y(t)w)$ is a Jacobi field along c with $Y(0) = 0$ and $Y'(0) = w$. Indeed we have

$$Y''(t) = P_t(y''(t)w) = -P_t(R(t)y(t)w) = -R(P_ty(t)w, c'(t))c'(t) = -R(Y(t), c'(t))c'(t).$$

Note that $D(\exp_p)_{tv}w = \frac{1}{t}P_ty(t)w$ because of the previous lemma. Hence

$$\widetilde{g}_{tv}(w,z) = g_{c(t)}(D(\exp_p)_{tv}w, D(\exp_p)_{tv}z) = \frac{1}{t^2}\langle y(t)w, y(t)z\rangle_p$$

where we used again that P_t is orthogonal.

Now we compute the Taylor expansion of $y(t)$. We have $y(0) = 0$ and $y'(0) = \mathrm{id}_{T_pM}$. Also

$$y''(0) = -R(0)y(0) = 0, \; y'''(0) = -R'(0)y(0) - R(0)y'(0) = -R(0)$$

and

$$y^{(4)}(0) = -R''(0)y(0) - R'(0)y'(0) - R'(0)y'(0) - R(0)y''(0) = -2R'(0).$$

It follows

$$y(t) = y(0) + ty'(0) + \frac{1}{2}t^2y''(0) + \frac{1}{6}t^3y'''(0) + \frac{1}{24}t^4y''''(0) + o(t^4) = t\mathrm{id}_{T_pM} - \frac{1}{6}t^3R(0) + \frac{1}{12}t^4R'(0) + o(t^4).$$

Inserting this into the formula for $\widetilde{g}$ yields (after rearranging the terms)

$$\begin{aligned}\widetilde{g}_{tv}(w,z) &= \langle w,z\rangle_p - \frac{1}{6}t^2\langle R(0)w,z\rangle - \frac{1}{6}t^2\langle w,R(0)z\rangle - \frac{1}{12}t^3\langle R'(0)w,z\rangle - \frac{1}{12}t^3\langle w,R'(0)z\rangle + o(t^4)\\ &= \langle w,z\rangle_p - \frac{1}{3}t^2\langle R(0)w,z\rangle - \frac{1}{6}t^3\langle R'(0)w,z\rangle + o(t^4)\end{aligned}$$

Now we can choose $\widetilde{v} = v/|v|$ for v and $|v|$ for t. It follows that $\frac{1}{6}|v|^2\langle R(0)w,z\rangle = \frac{1}{6}R(w,v,v,z)$ and $\frac{1}{6}|v|^3\langle R'(0)w,z\rangle = \frac{1}{6}(\nabla_vR)(w,v,v,z)$. □

6.13 Corollary. *Let (M,g) be Riem. mfd. and let $p \in M$. Assume $\exp_p|_{B_\rho(p)}$ is a diffeomorphism. If the sectional curvature $K(E) > 0$ $\forall E \in G_2(T_pM)$ ($K(E) < 0$), then there exists $\rho_1 \in (0,\rho)$ such that*

$$d(\exp_p(v), \exp_p(w)) \leq |v-w|_p \quad (\geq |v-w|_p).$$

" $=$ " holds only if $v = \lambda w$ with $\lambda \geq 0$.

$\exp_p$ is locally distance decreasing (increasing).

Proof. Let $K(E) > 0$ and $E = \mathrm{span}(v, z)$ with $v, z \in T_pM$ and $\langle v, z\rangle = 0$. Then $\langle R(z, v), v, z\rangle = K(E)|v|^2|z|^2$.

We fix $v \in T_pM$. For all E with $v \in E$ we find $z \in T_pM$ with $z \perp v$ and $|z| = 1$ such that $\mathrm{span}(v, z) = E$. Then we can consider K as a function on $\{z \in T_pM : z \perp v, |z| = 1\}$ that is continuous and hence $\exists \epsilon > 0$ s.t. $\min_{E\in G_2(T_pM),v\in E} K(E) = 6\epsilon > 0$.

The Taylor expansion formula of the previous Theorem implies

$$\widetilde{g}_v(z, z) = |z|^2 - \frac{1}{3}K(E)|v|^2|z|^2 - \frac{1}{6}(\nabla_v R)(z, v, v, z) + |v|_p^4 f(z, v, z)$$

where $(z, w) \mapsto f(z, v, w)$ is bilinear. Let $|v|$ be small enough such that $\frac{1}{6}\nabla_v R(z, v, v, z) + |v|_p^4 f(z, v, z) \leq \epsilon|v|_p^2|z|_p^4$. Then it follows that

$$\widetilde{g}_v(z, z) \leq (1 - \epsilon|v|_p^2)|z|_p^2.$$

Now let $w \in T_pM$ be arbitrary and decompose w as $w = w^\top + w^\perp$ where $w^\top||v$ and $w^\perp \perp v$. Then

$$\widetilde{g}_v(w, w) = |w|_v^2 = |w^\top|_v^2+|w^\perp|_v^2 \leq |D(\exp_p)_v(w^\top)|^2_{\exp_p(v)}+(1-\epsilon|v|_p)|w^\perp|_p^2 \leq |w^\top|_p^2+|w^\perp|_p^2.$$

In the seconde equality and in the second inequality for the first term, we also used the Gauß Lemma.

Now, we choose $\delta_1 \in (0, \delta)$ small enough and $v, w \in B_{\delta_1}(0_p)$. For $\gamma(t) = (1 - t)v + tw$ we have $\gamma(t) \in B_{\delta_1}(0_p)$ (δ_1 should be small enough such that the previous estimates hold).

Then $\widetilde{g}_v(\gamma'(t), \gamma'(t)) \leq |\gamma'(t)|_p^2$. Taking the square root and integrating from 0 to 1 yields

$$L^g(\exp_p \circ\gamma) = L^{\widetilde{g}}(\gamma) \leq L^{g_p}(\gamma) = |v - w|_p.$$

Taking the infimum w.r.t. γ on the left hand side yields $d^g(\exp_p v, \exp_p w) \leq |v - w|_p$.

Assume equality. Then $\widetilde{g}_v(\gamma'(t), \gamma'(t)) = |\gamma'(t)|_p^2\ \forall t \in [0, 1]$. Hence, the Taylor expansion, with v and $\gamma'(t)^\perp$ inserted, equals $|\gamma'(t)^\perp|_p$. Hence $K(\mathrm{span}(\gamma'(t)^\perp, v)) = 0$ and therefore $\gamma'(t)^\perp = 0$. Hence $\gamma'(t) = \gamma'(t)^\top\ \forall t \in [0, 1]$. $\square$

6.14 Remark. One can even show the following refined statement. Let $v \perp w \in T_pM$ with $|v|_p = |w|_p = 1$ and let $E \in G_2(T_pM)$ be the plane generated by v and w. Then

$$d_g(c_v(t), c_w(t)) = \sqrt{2}t\left(1 - K(E)\frac{1}{6}t^2 + o(t^4)\right).$$

Hence, the sectional curvature only depends on the distance d_g.

6.15 Theorem (E. Cartan)**.** *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds with* $\dim_M = \dim_{\widetilde{M}}$*, $p \in M$, $\widetilde{p} \in \widetilde{M}$, and there exists an orthogonal map $I : (T_pM, g_p) \to (T_p\widetilde{M}, \widetilde{g}_p)$ such that*

$$R|_{c_v(1)}(Pw, c_v'(1), c_v'(1), Pw) = \widetilde{R}_{c_{\widetilde{v}}(1)}(\widetilde{P}\widetilde{w}, c_{\widetilde{v}}'(1), c_{\widetilde{v}}'(1), \widetilde{P}w) \quad (**)$$

for all $v, w \in T_pM$ with $\widetilde{v} = Iv, \widetilde{w} = Iw$. Here $P = P^{c_v}_{0,1}$ and $\widetilde{P} = P^{c_{\widetilde{v}}}_{0,1}$.

Let $U \subset T_pM$ be open and starshaped w.r.t. 0_p such that $\exp_p|_U$ is a diffeomorphism and let $\widetilde{U} = I(U)$ such that $\widetilde{\exp}_{\widetilde{p}}|_{\widetilde{U}}$ is a diffeomorphism.

Then $\widetilde{\exp}_{\widetilde{p}}\circ I\circ(\exp_p|_U)^{-1} = F$ is an isometry between $V = \exp_p(U)$ and $\widetilde{V} = \widetilde{\exp}_{\widetilde{p}}(\widetilde{U})$.

Proof. Define $R_v(t) = P_t^{-1} R(P_t w, c_v'(t)) c_v'(t)$. By 4.15 Lemma (∗∗) implies

$$R|_{c_v(1)}(Pw, c_v'(1), c_v'(1), Pz) = \widetilde{R}_{c_{\widetilde{v}}(1)}(\widetilde{P}\widetilde{w}, c_{\widetilde{v}}'(1), c_{\widetilde{v}}'(1), \widetilde{P}z) \ \ \forall v, w, z \in T_pM.$$

Then we have

$$I^{-1} \circ \widetilde{R}_{\widetilde{v}}(t) \circ I = R_v(t) \ \ \forall v \in T_pM \ \ (*).$$

This follows since we have $\forall v, w, z \in T_pM$

$$\langle R_v w, z\rangle_p = \langle (I \circ R_v(t))w, \widetilde{z}\rangle_{\widetilde{p}} = \langle (\widetilde{R}_{\widetilde{v}}(t) \circ I)w, \widetilde{z}\rangle_{\widetilde{p}} = \langle (I^{-1} \circ \widetilde{R}_{\widetilde{v}}(t) \circ I)w, z\rangle_p.$$

Let $y_v(t) \in \mathrm{End}(T_pM)$ be the solution of

$$y_v''(t) + R_v y_v = 0 \text{ and } y_v(0) = 0, \ y_v'(0) = \mathrm{id}_{T_pM}.$$

Because of (∗) we have $y_{\widetilde{v}} \circ I = I \circ y_v(t)$, and $P_{0,t}^{c_v} y_v(t)w$ as well as $P_{0,t}^{c_{\widetilde{v}}} y_{\widetilde{v}}(t)\widetilde{w}$ are Jacobi fields. In 6.8 Lemma we showed that

$$D(\exp_p)_{tv}(tw) = P_{0,t}^{c_v}(y_v(t)(tw)), \quad D(\widetilde{\exp}_{\widetilde{p}})_{t\widetilde{v}}(t\widetilde{w}) = P_{0,t}^{c_{\widetilde{v}}}(y_{\widetilde{v}}(t)t\widetilde{w}).$$

Hence $\forall v \in U$ and $\forall w \in T_pM$, setting $t = 1$,

$$|D(\exp_p)_v w|_{\exp_p(v)} = |P_{0,1}^{c_{\widetilde{v}}} y_v(1)w|_{\exp_p(v)} = |y_v(1)w|_p = |I \circ y_v(1)w|_{\widetilde{p}} = |y_{\widetilde{v}}(1)\widetilde{w}|_{\widetilde{p}} = |D(\exp_p)_{\widetilde{v}} \circ Iw|_{\widetilde{\exp}_{\widetilde{p}}(\widetilde{v})}.$$

Similarly one can show, using that the parallel transport is orthogonal, that

$$g_{\exp_p(v)}(D(\exp_p)_v w, D(\exp_p)_v z) = \widetilde{g}_{\widetilde{\exp}_{\widetilde{p}}(\widetilde{v})}(D(\widetilde{\exp}_{\widetilde{p}})_{\widetilde{v}} Iw, D(\widetilde{\exp}_{\widetilde{p}})_{\widetilde{v}} Iz).$$

Hence $D(\widetilde{\exp}_{\widetilde{p}})_{\widetilde{v}} \circ I \circ D(\exp_p)_v^{-1}$ is an orthogonal map for all $v \in U$, and consequently $\widetilde{\exp}_{\widetilde{p}} \circ I \circ (\exp_p|_U)^{-1}$ is an isometry. □

6.16 Remark. (1) Note also the following: if $\widetilde{\exp}_{\widetilde{p}}$ is defined on $\widetilde{U}$ then F is a local isometry.

(2) If $\exp_p$ and $\widetilde{\exp}_{\widetilde{p}}$ are diffeomorphisms on T_pM and $T_{\widetilde{p}}\widetilde{M}$ respectively, then the map $\widetilde{\exp}_{\widetilde{p}}^{-1} \circ I \circ \exp_p$ is an isometry between M and $\widetilde{M}$.
But: if the assumptions in the previous theorem hold at any point (the curvature tensors of g and $\widetilde{g}$ are locally the same), then in general this does Not imply that (M, g) and $(\widetilde{M}, \widetilde{g})$ are isometric.

6.17 Corollary. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ with the same constant sectional curvature K_0 and with $\dim_M = \dim_{\widetilde{M}}$. Consider $p \in M$ and $\widetilde{p} \in \widetilde{M}$ and $I : (T_pM, g_p) \to (T_{\widetilde{p}}\widetilde{M}, \widetilde{g}_{\widetilde{p}})$ orthogonal. Then there exist open neighborhoods V of p and $\widetilde{V}$ of $\widetilde{p}$ and an isometry $F : (V, g|_V) \to (\widetilde{V}, \widetilde{g}|_{\widetilde{V}})$ such that $F(p) = \widetilde{p}$ and $DF_p = I$.*

Consider

$$\mathbb{M}_K^m = \begin{cases} (\mathbb{S}_\rho^m, i^*\langle\cdot,\cdot\rangle_{eucl}) & K = \frac{1}{\rho^2} > 0, \\ (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{eucl}) & K = 0, \\ (\mathbb{H}_\rho^m, i^*g_1) & K = \frac{1}{-\rho^2} < 0. \end{cases}$$

Recall that $\mathbb{S}_\rho^m = \{x \in \mathbb{R}^{m+1} : \langle x, x\rangle_{eucl} = \rho^2\}$ and $\mathbb{H}_\rho^m = \{x \in \mathbb{R}^{m+1} : g_1(x, x) = \rho^2\}$.

Remark. If a Riemannian manifold (M, g) is frame homogeneous ("Raum freier Beweglichkeit"), then the sectional curvature is constant $K = K_0 \in \mathbb{R}$.

Proof. $\forall E, \widetilde{E} \subset G_2(TM)$ $\exists$ isometry $F : (M, g) \to (M, g)$ such that $DF(E) = \widetilde{E}$. Then it follows $K(\widetilde{E}) = K(DF(E)) = K(E)$. $\square$

Hence $\mathbb{M}^m_K$ has constant sectional curvature.

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6.18 Lemma. *Let $(\widetilde{M}, \widetilde{g})$ be a Riem. mfd. and $\Gamma \subset Isom(\widetilde{M}, \widetilde{g})$ a subgroup that operates free and properly discontinuous on $\widetilde{M}$. Then there exists exactly one Riem. metric g on M/Γ such that $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry (for every point $\widetilde{p} \in \widetilde{M}$ there exists an open neighborhood V such that $(\pi|_V)^* g = \widetilde{g}$).*

Proof. We know that M/Γ is a smooth manifold and π is a covering map, i.e. a local diffeomorphism.

Uniqueness of g: Let $\widetilde{p} \in \pi^{-1}(\{p\})$. For $v \in T_pM$ there exsits exactly one $\widetilde{v} \in T_{\widetilde{p}}\widetilde{M}$ such that $D\pi_{\widetilde{p}}\widetilde{v} = v$. Hence g has to satify $g_p(v, v) = g_{\pi(\widetilde{p})}(D\pi_{\widetilde{p}}\widetilde{v}, D\pi_{\widetilde{p}}\widetilde{v}) = \widetilde{g}_{\widetilde{p}}(\widetilde{v}, \widetilde{v})$.

Existence of g: Define $g_p(v, v)$ via $\widetilde{g}_{\widetilde{p}}(\widetilde{v}, \widetilde{v})$. This is well-defined, since for any other $\hat{p} \in \pi^{-1}(\{p\})$ and $\hat{v} \in T_{\hat{p}}\widetilde{M}$ as above, it follows that there exists an isometry $F \in \Gamma$ such that $F(\hat{p}) = \widetilde{p}$ and $\pi \circ F = \pi$. Hence $D\pi_{\widetilde{p}} DF_{\hat{p}}\hat{v} = D\pi_{\hat{p}}\hat{v}$. This implies $DF_{\hat{p}}\hat{v} = \widetilde{v}$. Therefore $\widetilde{g}_{\widetilde{p}}(\widetilde{v}, \widetilde{v}) = g_{F(\hat{p})}(DF_{\hat{p}}\hat{v}, DF_{\hat{p}}\hat{v}) = g_{\hat{p}}(\hat{v}, \hat{v})$. ☐

6.19 Theorem. *Let (M^m, g) be a complete, connected Riem. mfd. with constant sectional curvature K. (M^m, g) is called a space form. Then $\exists \Gamma \subset Isom(\mathbb{M}^m_K)$ subgroup, such that (M, g) is isometric to $\mathbb{M}^m_K/\Gamma$.*

Proof of 6.19 Theorem. We set

$$\rho = \begin{cases} \infty & K \leq 0, \\ \frac{\pi}{\sqrt{K}} & K > 0. \end{cases}$$

Choose $p_0 \in \mathbb{M}^m_K$, $p \in M$ and $I : T_{p_0}\mathbb{M}^m_K \to T_pM$ orthogonal and define

$$F := \exp_p \circ I \circ (\exp_{p_0}|_{B_\rho(0_{p_0})})^{-1}.$$

Since M is complete, $\exp_p$ is defined on T_pM and F therefore well-defined and a local isometry because of Cartan's theorem. It follows

$$\begin{cases} K \leq 0: & F : \mathbb{M}^m_K \to M \text{ is a local isometry,} \\ K > 0: & F : \mathbb{S}^m_\rho \backslash \{-p_0\} \to M \text{ is a local isometry.} \end{cases}$$

In the case of $K > 0$ we pick $q_0 \in \mathbb{S}^m_\rho \backslash \{p_0, -p_0\}$ and consider $F(q_0) = q \in M$ as well as $\widetilde{I} := DF_{q_0} : T_{q_0}\mathbb{S}^m_\rho \to T_qM$. $\widetilde{I}$ is orthogonal.

We repeat the construction and define $\widetilde{F} = \exp_q \circ \widetilde{I} \circ (\exp_{q_0}|_{\mathbb{S}^m_\rho \backslash \{-q_0\}})$ that is again a local isometry.

It holds $D\widetilde{F}_{q_0} = \widetilde{I} = DF_{q_0}$ by construction.

6.20 Lemma (Rigidity of Isometries)**.** *Let (M, g) and (N, h) be Riemannian manifolds with $\dim_M = \dim_N$, and let $F : (M, g) \to (N, h)$ be an isometry. Let $p \in M$ and $f(p) = q \in N$.*

(1) Let $U_p \subset T_pM$ and $U_q \subset T_qN$ be the open domains where $\exp^M_p|_{U_p}$ and $\exp^N_q|_{U_q}$ are defined. Then $DF_p(U_p) \subset U_q$ and $F \circ \exp^M_p = \exp^N_q \circ DF_p$ on U_p.

(2) *If M is connected and $L : (T_pM, g_p) \to (T_qN, h_q)$ orthogonal, then there is at most one isometry $F : (M, g) \to (N, h)$ such that $DF_p = L$.*

Proof of the lemma. Let $v \in U_p$ and $c_v(t) = \exp_p^M(tv)$. Since F is an isometry also $f \circ \exp_p^M(tv)$ is a geodesic in N. It follows $\frac{d}{dt}|_{t=0}(F \circ \exp_p^M(tv)) = DF_pv$. Hence $F \circ \exp_p^M(tv) = \exp_q^N(tDF_pv)$ and $DF_pU_p \subset U_q$. For $t = 1$ we also get $F \circ \exp_p^M(v) = \exp_q^N \circ DF_pv \; \forall v \in U_p$. This proves (1).

Let $F, \widetilde{F}$ be isometries with $DF_p = D\widetilde{F}_p = L$. Define

$$A = \{x \in M : F(x) = \widetilde{F}(x), DF_x = D\widetilde{F}_x\} \neq \emptyset.$$

Note that $x \mapsto DF_x$ is continuous. Hence A is closed. We show A is also open. Indeed $F \circ \exp_p^M = \exp_q^N \circ DF_p = \exp_p^N \circ D\widetilde{F}_p = \widetilde{F} \circ \exp_p^M$ on U_p. In particular $F = \widetilde{F}$ in a neighborhood of p. Since M is connected, it follows $A = M$. This proves the lemma. □

It follows $F = \widetilde{F}$ in a neighborhood of q_0, and we can extend F to $-p_0$ via $F(-p_0) = \widetilde{F}(-p_0)$.

In any case $F : \mathbb{M}_K^m \to M$ is a local isometry and hence a covering map. Since $\mathbb{M}_K^m$ is simply connected, it follows that $F : \mathbb{M}_K^m =: \widetilde{M} \to M$ is in fact the universal cover of M.

Consider subgroup $\Gamma \subset \mathrm{Diff}(\widetilde{M})$ of deck transformations of the covering, i.e. $f \in \Gamma$ satisfies $F \circ f = F$. In particular $DF_{\hat{p}} Df_{\widetilde{p}} = DF_{\widetilde{p}}$. Therefore Γ is even a group of isometries on $(\widetilde{M}, \widetilde{g})$. Hence there exists exactly one Riemannian metric h on $\widetilde{M}/\Gamma$ (up to isometries) such that $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$ is a local isometry where π is the quotient map. But by construction $F = \pi$ and $M = \widetilde{M}/\Gamma$. Hence (M, g) is isometric to $(\widetilde{M}/\Gamma, h)$. □

6.21 Remark. The space form problem is about to determine all compact Riem. Manifolds with constant sectional curvature K up to isometries. The previous theorem tell us that this becomes the problem of finding all subgroups (up to conjugation) of $\mathrm{Isom}(\mathbb{M}_K^m)$ that act freely and properly discontinuous with compact quotient.

(a) $K = 0$: Every space form (M, g) is finitely covered by a flat torus (Bieberbach Theorem).

(b) $K > 0$: $\mathbb{M}_K^m$ is compact. Hence, the cardinality $\#\Gamma$ of the subgroup in $\mathrm{Isom}(\mathbb{M}_K^m)$ is finite. A full classification was given by J.A. Wolf: *Spaces of constant curvature.*

   Note that, if $m = 2n$ for $n \in \mathbb{N}$, then Γ can only be $\{\mathrm{id}_{\mathbb{S}^m_{\frac{1}{\sqrt{K}}}}\}$ or $\{\mathrm{id}_{\mathbb{S}^m_{\frac{1}{\sqrt{K}}}}, -\mathrm{id}_{\mathbb{S}^m_{\frac{1}{\sqrt{K}}}}\}$. Hence, only $\mathbb{S}^m_{\frac{1}{\sqrt{K}}}$ and $\frac{1}{\sqrt{K}}\mathbb{R}P^m$ can appear as space forms if $m = 2n$.

   If $m = 3$, one obtains so-called lense spaces $L_{p,q} = \mathbb{S}^3/\Gamma_{p,q}$ that depend on two real parameters.

(c) $K < 0$: For $m = 2$ the problem is solved ("Teichmüllerraum"). For $m \geq 3$ the construction of subgroups Γ is difficult and the full classification is open. (Theorem of Mostow: $\pi_1(M^m) \simeq \pi_1(\widetilde{M}^m)$, then the corresponding group isomorphism induces a unique isometry.

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6.22 Definition. Let (M,g) be a Riemannian manifold, $p \in M$ and $(e_i)_{i=1,\dots,m}$ an ONB of (T_pM, g_p).

(1) The Ricci curvature $\mathrm{ric}^M = \mathrm{ric}^g =: \mathrm{ric}$ of (M,g) is the symmetric $(0,2)$-Tensorfield

$$\mathrm{ric} : M \to T_2^0M,\ \ \mathrm{ric}_p(v,u) = \mathrm{trace}\Big(x \in T_pM \mapsto R(x,v)u \in T_pM\Big) = \sum_{i=1}^{m} \langle R(e_i,v)u, e_i\rangle_p.$$

Remark. If $v = u = e_m$, then $\mathrm{ric}_p(v,v) = \sum_{i=1}^{m-1}\langle R(e_i,v)v,e_i\rangle = \sum_{i=1}^{m-1} K(\mathrm{span}(v,e_i))$.

(2) The Scalar curvature $S : M \to \mathbb{R}$ of (M,g) is defined as the trace of ric via

$$S(p) = \sum_{i=1}^{m} \mathrm{ric}_p(e_i,e_i).$$

6.23 Example. Consider the unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ equipped with the induced metric $i^*\langle\cdot,\cdot\rangle_{eucl}$. Then $K \equiv 1$ and

$$\mathrm{ric}_p(v,v) = |v|^2 \sum_{i=1}^{m-1} 1 = |v|^2(m-1),\quad \mathrm{ric}_p(v,u) = (m-1)g_p(v,u).$$

and $S(p) = (m-1)n$.

A geometric Meaning of Ricci curvature. Let $\varphi : U \to V$ be a chart and $A \subset U$ be measurable. The m-dimensional volume of A w.r.t. g was defined as

$$\mathrm{vol}_m^g(A) = \int_{\varphi(A)} |\det g_{ij}|^{\frac{1}{2}} \circ \varphi^{-1}(x)dx.$$

In particular, if $\varphi = (\exp_p|_U)^{-1}$ and $\widetilde{A} = \varphi(A) \subset T_pM$, then

$$\mathrm{vol}_m^g(A) = \int_{\widetilde{A}} |\det \widetilde{g}_x(e_i,e_j)|^{\frac{1}{2}} dx$$

where $\widetilde{g}|_x = \varphi^* g|_x$. This is indeed clear since

$$g_{ij}\circ\varphi^{-1}(x) = g_{\varphi^{-1}(x)}(\frac{\partial}{\partial x^i}|_{\varphi^{-1}(x)}, \frac{\partial}{\partial x^j}|_{\varphi^{-1}(x)}) = g_{\varphi^{-1}(x)}(D(\varphi^{-1})_x e_i, D(\varphi^{-1})_x e_j) = \widetilde{g}_x(e_i,e_j).$$

Also note that

$$\widetilde{g}_x(e_i,e_j) = \langle y_x(1)e_i, y_x(1)e_j\rangle = \langle (y_x(1))^2 e_i, e_j\rangle$$

and hence $\det \widetilde{g}_x(e_i,e_j)^{\frac{1}{2}} = \det y_x(t)$.

6.24 Corollary. *If* $\mathrm{ric}(v,v) > 0$ *(or* $\mathrm{ric}(v,v) < 0$*)* $\forall v \in T_pM\backslash\{0\}$*, then it follows that* $\exp_p$ *is volume decreasing (increasing) in a neighborhood of* 0_p*. More precisely* $\exists U \subset T_pM$ *a neighborhood of* 0_p *such that*

$$\mathrm{vol}_m^g(\exp_p(\widetilde{A})) \le \mathrm{vol}_m^{g_p}(\widetilde{A}) \quad (\ge).$$

Proof. Let $v \in T_pM$ and set $a(t) := a_v(t) = \frac{1}{t} y_v(t)$. Recall the Taylor expansion

$$y_v(t) = t\mathrm{id}_{T_pM} - \frac{1}{6} t^3 R_v(0) + \frac{1}{12} t^4 R_v'(0) + o(t^4).$$

We can choose $\widetilde{v} = v/|v|$. Then one easily check from the defining ODE of $y_{\widetilde{v}}$ that $\frac{1}{|v|} y_{\widetilde{v}}(t|v|) = y_v(t)$. Hence $a_v(t) = \frac{1}{t} y_v(t) = \frac{1}{t|v|} y_{\widetilde{v}}(t|v|)$ and

$$a_v(t) = \frac{1}{t|v|} y_{\widetilde{v}}(t|v|) = \mathrm{id}_{T_pM} - \underbrace{\frac{1}{6} R_{\widetilde{v}}(0)(t|v|)^2}_{\frac{1}{6} R_v(0) t^2} + \frac{1}{12} R_{\widetilde{v}}'(0)(t|v|)^3 + o(t^3|v|^3).$$

In particular a satisfies $a(0) = \mathrm{id}_{T_pM}$, $a'(0) = 0$ and $a''(0) = -\frac{1}{3} R_v(0)$.

We compute the Taylor expansion of $\det a(t)$: One has

$$\begin{aligned}
(\det a(t))'(t) &= \det a(t) \mathrm{trace}(a'(t) a(t)^{-1}), \\
(\det a(t))''(t) &= (\det a)'(t) \mathrm{trace}(a'(t) a(t)^{-1}) \\
&\quad + \det a(t) \left(\mathrm{trace}(a''(t) a(t)^{-1} + \mathrm{trace}(a'(t)(-1) a(t)^{-1} a'(t) a(t)^{-1}) \right)
\end{aligned}$$

We observe that

$$\det a(0) = 1, \quad \det a(t)' = 0 \text{ and } (\det a)''(0) = \mathrm{trace} a''(0) = -\frac{1}{3} \underbrace{\mathrm{trace} R_v(0)}_{\mathrm{ric}_p(v,v)}.$$

Hence

$$\det a_v(t) = \det a(t) = 1 - \frac{1}{6} \mathrm{ric}_p(v,v) t^2 + o(t^2|v|^2).$$

Especially for $t = 1$ we get

$$\det a_v(1) = \det a_{\widetilde{v}}(|v|) = 1 - \frac{1}{6} \mathrm{ric}_p(v,v) + o(|v|^2).$$

More precisely, $o(|v|^2)$ has the form $f(|v|)|v|^2$ for a continuous function f with $f(0) = 0$. Our assumption $\mathrm{ric}_p(v,v) > 0 \ \forall v \in T_pM$ yields that we can choose a neighborhood U of 0_p such that

$$-\frac{1}{6} \mathrm{ric}_p(v,v) + f(|v|)|v|^2 < 0$$

for all $v \in U$. Then it follows for $\widetilde{A} \subset U$ and $A = \exp_p(A)$ that

$$\mathrm{vol}_m^g(A) = \int_{\widetilde{A}} |\det y_v(1)| dx = \int_{\widetilde{A}} |\det a_v(1)| dx \leq \int_{\widetilde{A}} 1 dx = \mathrm{vol}_m^{g_p}(\widetilde{A}).$$

$\square$

A geometric meaning of Scalarcurvature.

6.25 Corollary. *Let* (M, g) *be a Riem. manifold,* $p \in M$ *and* $\dim_M = m$*. Then*

$$\operatorname{vol}^g_m(B_\rho(p)) = \omega_m \rho^m \left(1 - \frac{1}{6(m+2)} S(p)\rho^2 + o(\rho^2)\right)$$

where $\omega_m = \operatorname{vol}^{eucl}_m(B_1(0))$.

Remark. $o(\rho^2)$ is in fact $o(\rho^3)$ (Reference: A. Gray "The volume of a small geodesic ball of a Riemannian manifolde").

Proof. We already saw that

$$\operatorname{vol}^g_m(B_\rho(p)) = \int_{B_\rho(0_p)} \det y_x(1) dx^1 \dots dx^m$$

where we have the Taylor expansion $\det y_x(1) = 1 - \frac{1}{6}\operatorname{ric}(x,x) + o(|x|^3)$. Moreover

$$\int_{\partial B_\rho(0_p)} \operatorname{ric}(x,x) d\operatorname{vol}^{\mathbb{S}^{m-1}_\rho}_{m-1} = \rho^{m-1}\int_{\partial B_1(0_p)} \operatorname{ric}(x,x) d\operatorname{vol}^{\mathbb{S}^{m-1}_1}_{m-1} = \frac{\rho^{m+1}}{m}\operatorname{vol}_{m-1}(\mathbb{S}^{m-1}_1)\underbrace{\operatorname{trace}\operatorname{ric}_p}_{S(p)}.$$

Hence

$$\operatorname{vol}^g_m(B_\rho(p)) = \omega_m\rho^m - \underbrace{\int_0^\rho \int_{\partial B_{\widetilde{\rho}}(0_p)} \frac{1}{6}\operatorname{ric}(x,x) d\operatorname{vol}^{\mathbb{S}^{m-1}_r}_{m-1} d\widetilde{\rho}}_{-\frac{1}{6(m+2)}\rho^{m+2}S(p)\omega_m} + o(\rho^{2+m})$$

□

5.27 Corollary. $\exists V \subset M \times M$ neighborhood of $\{(p,p) \in M \times M : p \in M\}$ and $V' \subset TM$ a neighborhood of $\{0_p \in T_pM : p \in M\}$ such that $\pi \times (\exp|_{V'}) : V' \to V$ diffeomorphismus.

12.07.2023

7 Second Variation of arclength and Bonnet-Myers theorem

Let (M, g) be a Riem. mfd.

7.1 Theorem (Second variation formula)**.** *Let $c : [a, b] \to M$ be a geodesic with $|c'| \equiv 1$ and let $\alpha : [a, b] \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M$ be smooth such that $c(t) = \alpha(t, 0, 0)$. ($c'(t) = \frac{\partial}{\partial t}\alpha|_{(t,0,0)}$.) Set $\alpha(t, \sigma, \tau) = \alpha_{\sigma,\tau}(t), t \in [a, b]$ and*

$$V(t) = \frac{\partial \alpha}{\partial \sigma}|_{(t,0,0)}, \quad W(t) = \frac{\partial \alpha}{\partial \tau}|_{(t,0,0)} \in \Gamma(c^*TM).$$

We define

$$L(\alpha_{\sigma,\tau}) = L(\sigma, \tau) = \int_a^b \left| \frac{\partial \alpha}{\partial t}(t, \sigma, \tau) \right|_g dt, \quad L(0, 0) = L(c).$$

Moreover $V' = \nabla_t V$ and $W' = \nabla_t W$. Then

$$\frac{\partial^2 \alpha}{\partial \alpha \partial \tau}|_{(0,0)} = \int_a^b \big(\langle V'(t), W'(t)\rangle - \langle R(V(t), c'(t))c'(t), W(t)\rangle - \langle V, c'\rangle'(t)\langle W, c'\rangle'(t)\big)\, dt$$
$$+ \langle \nabla_\sigma \frac{\partial \alpha}{\partial \tau}\Big|_{(t,0,0)}, c'(t)\rangle\Big|_a^b.$$

In particular, if $\langle V, c'\rangle = 0 = \langle W, c'\rangle$ and $\alpha(a, \sigma, \tau) = c(a), \alpha(b, \sigma, \tau) = c(b)\ \forall \sigma, \tau \in (-\epsilon, \epsilon)^2$ (hence V and W satisfy $V(a) = V(b) = W(a) = W(b) = 0$ and are therefore so-called proper variations), we get

$$\frac{\partial^2 L}{\partial \sigma \partial \tau}\Big|_{(0,0)} = \int_a^b \big(\langle V', W'\rangle - \langle R(V, c')c', W\rangle\big)\, dt = -\int_a^b \langle V'' + R(V, c')c', W\rangle dt$$

where $\langle V', W\rangle' = \langle V'', W\rangle + \langle V', W'\rangle$ and the boundary condition is used for the last equality.

Proof. We first compute

$$\frac{\partial L}{\partial \tau}_{(\sigma,\tau)} = \int_a^b \frac{\partial}{\partial \tau}\left|\frac{\partial \alpha}{\partial t}(t, \sigma, \tau)\right| dt = \int_a^b \left|\frac{\partial \alpha}{\partial t}(t, \sigma, \tau)\right|^{-1} \langle \nabla_\tau \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\rangle(t, \sigma, \tau) dt$$
$$= \int_a^b \left|\frac{\partial \alpha}{\partial t}(t, \sigma, \tau)\right|^{-1} \langle \nabla_t \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial t}\rangle(t, \sigma, \tau) dt$$

where we used the chain rule and that we can switch the partial derivative w.r.t. t and co-variant derivative w.r.t. τ. We further compute

$$\frac{\partial^2 L}{\partial \sigma \partial \tau}(\sigma, \tau) = \int_a^b \frac{\partial}{\partial \sigma}\left(\left|\frac{\partial \alpha}{\partial t}(t, \sigma, \tau)\right|^{-1} \langle \nabla_t \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial t}\rangle(t, \sigma, \tau)\right) dt$$
$$= \int_a^b \left\{ \left|\frac{\partial \alpha}{\partial t}(\dots)\right|^{-1} \left(\langle \nabla_\sigma \nabla_t \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial t}\rangle(t, \sigma, \tau) + \langle \nabla_t \frac{\partial \alpha}{\partial \tau}, \nabla_t \frac{\partial \alpha}{\partial \sigma}\rangle(t, \sigma, \tau)\right)\right.$$
$$\left. - \left|\frac{\partial \alpha}{\partial t}(\dots)\right|^{-2} \langle \nabla_t \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial t}\rangle(t, \sigma, \tau)\langle \nabla_t \frac{\partial \alpha}{\partial \sigma}, \frac{\partial \alpha}{\partial t}\rangle(t, \sigma, \tau)\right\} dt$$

Evaluation at $(\sigma, \tau) = (0, 0)$ yields

$$\frac{\partial^2 L}{\partial\sigma\partial\tau}(0,0)$$
$$= \int_a^b \left\{ \langle \nabla_\sigma \nabla_t \frac{\partial\alpha}{\partial\tau}\Big|_{(t,0,0)}, c'(t)\rangle + \langle V', W'\rangle(t) - \langle W'(t), c'(t)\rangle\langle V'(t), c'(t)\rangle \right\} dt$$
$$= \int_a^b \left\{ \underbrace{\langle \nabla_t \nabla_\sigma \frac{\partial\alpha}{\partial\tau}, c'(t)\rangle}_{\frac{d}{dt}\langle \nabla_\sigma \frac{\partial\alpha}{\partial\tau}|_{t,0,0}, c'(t)\rangle - 0} - \langle R(V(t), c'(t))c'(t), W(t)\rangle + \langle V', W'\rangle(t) + \langle W, c'\rangle'\langle V, c'\rangle'(t) \right\} dt$$

This is the claim. □

Notation.

$$\mathcal{M}_c = \{V \in \Gamma(c^*TM) : V \text{ orthogonal}, \langle V, c'\rangle = 0, V(b) = V(a) = 0 \text{ piecewise } C^2\}$$

where $c : [a, b] \to M$ Geodätische.

7.2 Definition. The symmetric bilinear form $I_c : \mathcal{M}_c \times \mathcal{M}_c \to \mathbb{R}$ defined via

$$I_c(V, W) = \int_a^b \left\{ \langle V', W'\rangle - \langle R(V, c')c', W\rangle \right\} dt$$

is called index form of c.

Remark. Let $\alpha(t, \sigma)$ be a proper variation of c such that the variation vectorfield $V(t) = \frac{\partial\alpha}{\partial\sigma}{}_{(t,0)}$ satisfies $V(t) \perp c'(t)$ $\forall t \in [a, b]$. Then it follows that $\frac{d^2}{d\sigma^2} L(\alpha_\sigma)|_{\sigma=0} = I_c(V, V)$.

7.3 Corollary. *Let $c : [a, b] \to M$ be a minimal geodesic, i.e. $L(c) = d^g(c(a), c(b))$, then I_c is positive semi-definit.*

Proof. Pick $V \in \mathcal{M}_c$ and let $\alpha(t, \sigma) = \exp_{c(t)}(\sigma V(t))$, i.e. $\frac{\partial\alpha}{\partial\sigma}(t, 0) = V(t)$. Since c is a minimal geodesic, $L(\alpha_\sigma)$ has a minimum at $\sigma = 0$. Hence $\frac{d}{d\sigma} L(\alpha_\sigma) = 0$ and $\frac{d^2}{d\sigma^2} L(\sigma) = I_c(V, V) \geq 0$. □

7.4 Theorem (Bonnet-Myers)**.** *Let (M, g) be a complete, connected Riem. mfd. with $\dim_M = m$ and $\forall v \in TM$ one has $\mathrm{ric}(v, v) \geq (m-1)K_0 > 0$. Then it follows that $\mathrm{diam}_M = \sup_{x,y\in M} d^g(x, y) \leq \frac{\pi}{\sqrt{K_0}}$.*
In particular: M is compact and the first fundamental group $\pi_1(M)$ is finite.

7.5 Remark. (1) $K(E) \geq K_0$ $\forall E \in G_2(TM)$, then

$$\mathrm{ric}(v, v) = \sum_{i=1}^{m} \langle R(e_i, v)v, e_i\rangle = \sum_{i=1}^{m-1} K(\mathrm{span}(e_i, v) \geq (m-1)K_0.$$

where $v \in T_pM$ for $p \in M$ and (e_i) is an ONB of (T_pM, g_p) with $e_m = v$.

(2) $\mathbb{S}_1^m$ satisfies $K_0 = K \equiv 1$ and $\mathrm{diam}\,\mathbb{S}_1^m = \pi$.

(3) (M,g) complete and connected with $\operatorname{diam}_M < \infty \Rightarrow M$ is compact (Hopf-Rinow-Theorem), since $\exp_p(\underbrace{\overline{B_R(0_p)}}_{\text{compact}}) = M$

(4) Let $\pi : \widetilde{M} \to M$ be the universal cover and define $\widetilde{g} = \pi^* g$ that is a Riemannian metric on $\widetilde{M}$ such that π is a local isometry. Hence $\operatorname{ric}_g \geq (m-1)K_0 \quad \Rightarrow \quad \operatorname{ric}_{\widetilde{g}} \geq (m-1)K_0$. The theorem of Bonnet-Myers implies $\widetilde{M}$ is compact, and therefore $\pi_1(M)$, that is isomorphic to the group of deck transformations, is finite.

Proof. Assume the statement is false and there exist points p, q such that $d^g(p,q) = L > \frac{\pi}{\sqrt{K_0}}$. Let c be the geodesic that connects p and q with $|c'| = 1$ and let $E_1, \ldots, E_{m-1}, E_m = c' \in \Gamma(c^*TM)$ ONB for every time t.

We define vectorfields $V_j(t) = \sin(\pi/Lt)E_i(t)$. It follows that $V_j(0) = 0$ and $v_j(L) = 0$. Moreover

$$I_c(V_j, V_j) = -\int_0^L \langle V_j, V_j'' + R(V_j, c')c', V_j \rangle dt = \int_0^L \sin(\pi/Lt)^2 \left(\pi^2/L^2 - K(\operatorname{span}(E_j, E_m))\right).$$

Summing up w.r.t. j yields

$$\sum I_c(V_j, V_j) = \int_0^L \sin(\pi/Lt)^2 \left((m-1)\pi^2/L^2 - \operatorname{ric}(E_m, E_m)\right) dt < 0.$$

Hence, there exists one index $j_0 \in \{1, \ldots, m-1\}$ such that $I_c(V_j, V_j) < 0$. This contradicts minimality of c. $\square$

17.07.2023

Let (M, g) be a Riemannian manifold and let $c : [a, b] \to M$ be a geodesic.
Recall

$$\mathcal{M}_c = \{V \in \Gamma(c^*TM) : V \text{ orthogonal}, \langle V, c' \rangle = 0, V(b) = V(a) = 0 \text{ piecewise } C^2\}$$

and

$$I_c(V, W) = \int_a^b \left\{ \langle V', W' \rangle - \langle R(V, c')c', W \rangle \right\} dt.$$

A point $t \in (a, b]$ is conjugated to $a \Leftrightarrow \exists$ Jacobi field Y that vanishes in a and $t \Leftrightarrow D(\exp_{c(0)})_{tc'(0)}$ is degenerated.

7.6 Lemma. *If $V \in \mathcal{M}_c$ with $I_c(V, W) = 0$ $\forall W \in \mathcal{M}_c$, then V is a Jacobi field.*

Proof. Let $a = t_0 < t_1 < \cdots < t_n = b$ such that $V|_{[t_i, t_{i+1}]}$ C^2.

$$0 = I_c(V, W) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\{ \langle V^i, W^i \rangle - \langle R(V, c')c', W \rangle \right\} dt = (*)$$

Recall $\langle V', W \rangle' = \langle V'', W \rangle + \langle V', W' \rangle$. Hence

$$\begin{aligned}
(*) &= \sum_{i=1}^{n} \big(\langle V', W \rangle(t_i) - \langle V', W \rangle(t_{i-1}) \big) - \int_a^b \langle V'' + R(V, c')c', W \rangle(t) dt \\
&= \sum_{i=1}^{n} \langle \underbrace{V'^{-}(t_i) - V'^{+}(t_i)}_{=:\Delta_{t_i} V'}, W(t_i) \rangle - \int_a^b \langle V'' + R(V, c')c', W \rangle(t) dt = (**)
\end{aligned}$$

Choose now $W \in \mathcal{M}_c$ with $\mathsf{spt} W \subset (t_{i-1}, t_i)$ $\forall i = 1, \ldots, n$. Then it follows

$$(**) = -\int_a^b \langle V'' + R(V, c')c', W \rangle(t) dt.$$

Hence $V'' + R(V, c')c' = 0$ on (t_{i-1}, t_i) $\forall i = 1, \ldots, n$. Therefore

$$0 = \sum_{i=1}^{n} \langle \Delta_{t_i} V', W(t_i) \rangle \quad \forall W \in \mathcal{M}_c.$$

Choose $W \in \mathcal{M}_c$ with $\mathsf{spt} W \subset (t_{i-1}, t_{i+1})$ and such that $W(t_i) = \Delta_{t_i} V'$ for one $i \in \{1, \ldots, n-1\}$. It follows

$$I_c(V, W) = |\Delta_{t_i}(V')|^2 = 0 \quad \Rightarrow \quad V'^{-}(t_i) = V'^{+}(t_i).$$

Hence V is C^1 in t_i $\forall i$, and since V satisfies the Jacobi equation in $[t_{i-1}, t_i]$ for all i, it is even smooth and satisfies the Jacobi equation on $[a, b]$. ☐

7.7 Theorem. *Let $c : [a, b] \to M$ be a geodesic and there is not $t \in (a, b]$ that is conjugated to a along c. Then*

(a) $I_c : \mathcal{M}_c \times \mathcal{M}_c \to \mathbb{R}$ *is positive definit.*

(b) *(Index Lemma) If* $V \in \Gamma(c^*TM)$ *with* $\langle V, c' \rangle = 0$ *and if* Y *is the Jabobi field with* $Y(a) = V(a)$ *and* $Y(b) = V(b)$*, it follows* $I_c(V,V) \geq I_c(Y,Y)$ *with " = " if and only if* $Y = V$*.*

Proof. (1) We show: I_c is positive semi-definit on $\mathcal{M}_c$. W.l.o.g. $a = 0$ and $c(0) = p$.

For $V \in \mathcal{M}_c$ we consider the variation $\alpha(t,\sigma) = \exp_{c(t)}(\sigma V(t))$, i.e. $\frac{\partial \alpha}{\partial \sigma}(t,0) = V(t)$ and $\alpha(t,0) = c(t)$. α is piecewise C^2 on $[t_{i-1}, t_i] \times (-\epsilon, \epsilon)$ for a decomposition of $0 = t_0 \leq t_1 \leq \cdots \leq t_n = b$ of $[0,b]$ and $\epsilon > 0$ small enough.

Recall: every $t \in [0,b]$ is not conjugated to $0 \Leftrightarrow D(\exp_p)_{tc'(0)}$ is not degenerated $\forall t \in [a,b]$.

Hence $\exists \epsilon > 0$ and $\widetilde{\alpha} : [0,b] \times (-\epsilon, \epsilon) \to T_pM$ such that $\exp_p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0,\sigma) = 0_p$ and $\widetilde{\alpha}(b,\sigma) = bc'(0)$.

More explicitly: since $c([0,b])$ is compact we can find $n \in \mathbb{N}$ and U_i, $i = 1, \ldots, n$, such that $\varphi_i := \exp_p|_{U_i} : U_i \to V_i = \exp_p(U_i)$ is a diffeomorphism $\forall i = 1, \ldots, n$ and such that $c([t_{i-1}, t_i]) \subset V_i$ (note that for $\exp_p$ is a diffeomorphism in a sufficiently small neighborhood of $c(t)$ for every $t \in [0,b]$). Moreover we can choose $\epsilon_i > 0$ such that $\alpha([t_{i-1}, t_i] \times (-\epsilon_i, \epsilon_i)) \subset V_i$. Set $\epsilon = \min_i \epsilon_i$. Set $\alpha|_{[t_{i-1},t_i] \times (-\epsilon,\epsilon)}$. Now we can define $\widetilde{\alpha}$ as follows

$$\widetilde{\alpha}(t,\sigma) = \varphi_i^{-1} \circ \alpha_i(t,\sigma) \text{ for } (t,\sigma) \in [t_{i-1}, t_i] \times (-\epsilon, \epsilon).$$

This is well-defined, since $\varphi_i^{-1} \circ \alpha_i(t_i) = \varphi_{i+1}^{-1} \circ \alpha_{i+1}(t_i)$.

The Gauss Lemma implies $L(\sigma) := L(\alpha(\cdot,\sigma)) = L(\widetilde{\alpha}(\cdot,\sigma))$ and $L(\sigma) \geq L(0)$.

More explicitely: Set $\gamma(t) = \alpha(t,\sigma)$ for $\sigma \in (-\epsilon,\epsilon)$ fixed, and $\widetilde{\gamma}(t,\sigma) = \widetilde{\alpha}(t,\sigma)$. Hence $\exp_p \circ \widetilde{\gamma} = \gamma$. $\exp_p$ is a local diffeomorphism on $\bigcup_{i=1}^n U_i =: U$. Hence $\widetilde{g} = (\exp_p)^* g$ is a well-defined Riemannian metric on U and by construction $L^{\widetilde{g}}(\widetilde{\gamma}) = L^g(\gamma)$. We can decompose $\widetilde{\gamma}'(t)$ as follows

$$\widetilde{\gamma}'(t) = \widetilde{\gamma}'(t)^{\perp} + \widetilde{g}(\widetilde{\gamma}'(t), \frac{\gamma(t)}{|\gamma(t)|_{\widetilde{g}}}) \frac{\gamma(t)}{|\gamma(t)|_{\widetilde{g}}}.$$

Note that $\widetilde{g}(\widetilde{\gamma}'(t), \gamma(t))/|\gamma(t)|_{\widetilde{g}} = (\widetilde{g}(\widetilde{\gamma}, \widetilde{\gamma})^{1/2})'$. Hence

$$L^{\widetilde{g}}(\gamma) = \int_0^b |\widetilde{\gamma}'(t)|_{\widetilde{g}} dt \geq \int_0^b |\widetilde{\gamma}(t)|_{\widetilde{g}}' dt = |\widetilde{\gamma}(b)|_{\widetilde{g}} - 0 = b|c'(0)|_{\widetilde{g}} = L^g(c).$$

$\Rightarrow L''(0) \geq 0 \Rightarrow L''(0) = I_c(V,V) \geq 0.$

(2) Is $V \in \mathcal{M}_c$ and $I_c(V,V) = 0$.

By (1) it follows $0 \leq I_c(V + \epsilon W, V + \epsilon W) = 2\epsilon I_v(V,W) + \epsilon^2 I_c(W,W)$ $\forall \epsilon > 0$ and $\forall W \in \mathcal{M}_c$.

It follows $I_c(V,W) \geq 0$. Replacing W with W implies $I_c(V,W) = 0$ $\forall W \in \mathcal{M}_c$. Hence V is a Jacobi field with $V(0) = V(b) = 0$ (since $V \in \mathcal{M}_c$). This is a contradiction with b not conjugated to 0.

Hence I_c positive definit. This proves a.

(3) Uniqueness of the Jacobi field Y in (b): Consider the space of $\mathcal{Y}$ of orthogonal Jacobi field along c (this is a $(2m-2)$-dimensional vector space).

We consider the linear map $Y \in \mathcal{Y} \mapsto (Y(a), Y(b)) \in (c'(a))^{\perp} \oplus (c'(b))^{\perp}$. The kernel of this map is 0. Otherwise a, b are conjugated to each other.

Now we consider $V - Y \in \mathcal{M}_c$. (a) implies $I_c(V,V) - 2I_c(V,Y) + I_c(Y,Y) = I_c(V - Y, V - Y) > 0$.

We show $-2I_c(V,Y) + I_c(Y,Y) = -I_c(Y,Y)$. Indeed

$$
\begin{aligned}
I_c(V,Y) &= \int_a^b \left\{\langle Y', V'\rangle - \langle R(Y,c')c', V\rangle\right\} dt \\
&= \langle Y', V\rangle\Big|_a^b - \int_a^b \langle Y'' + R(Y,c')c', V\rangle dt = \langle Y', V\rangle\Big|_a^b . \\
I_c(Y,Y) &= \int_a^b \left\{\langle Y', Y'\rangle - \langle R(Y,c')c', Y\rangle\right\} dt = \langle Y', Y\rangle\Big|_a^b
\end{aligned}
$$

Hence $I_c(V,Y) = I_c(Y,Y)$.

If $I_c(V,V) = I_c(Y,Y)$, then $I_c(V - Y, V - Y) = 0$. Since I_c positive definit, it follows $V - Y = 0$.

□

7.8 Theorem. *Let $c : [a,b] \to M$ be a geodesic and let $t_0 \in (a,b)$ be conjugated to a along c. Then $\exists X \in \mathcal{M}_c$ such that $I_c(X,X) < 0$. In particular c is not a minimal geodesic.*

Proof. $\exists Y \neq 0$ Jacobi field along c with $Y(a) = Y(t_0) = 0$. We define

$$
V(t) = \begin{cases} Y(t) & t \in [0, t_0] \\ 0 & t \in [t_0, b]. \end{cases}
$$

Hence $V \in \mathcal{M}_c$.

Moreover let $W \in \mathcal{M}_c$ be smooth such that $W(t_0) = -Y'(t_0)$. It holds $Y'(t) \neq 0$, since otherwise $Y \equiv 0$, and $Y'(t) \perp c'(t)$, since $0 = \langle Y, c'\rangle'(t) = \langle Y'(t), c'(t)\rangle$.

Next we define for $\epsilon > 0$, $X_\epsilon = V + \epsilon W \in \mathcal{M}_c$. Then

$$
I_c(X_\epsilon, X_\epsilon) = \underbrace{I_c(V,V)}_{=\langle V'^{-}(t_0) - V'^{+}(t_0), V(t_0)\rangle = 0} + 2\epsilon \underbrace{I(V,W)}_{=\langle V'^{-}(t_0) - V'^{+}(t_0), W(t_0)\rangle = -|Y'(t_0)|^2 = -|W(t_0)|^2} + \epsilon^2 I(W,W).
$$

Hence $I_c(X_\epsilon, X_\epsilon) = -2\epsilon |W(t_0)|^2 + \epsilon^2 I_c(W,W)$ and for $\epsilon > 0$ small enough it follows $I_c(X_\epsilon, X_\epsilon) < 0$. □

7.9 Remark.

- c minimal geodesic $\Rightarrow$ I_c positive semi-definit.
- I_c is not positive definit $\Rightarrow$ $\exists\, t_0 \in (a,b]$ conjugated with a.
- $\exists t_0 \in (a,b)$ conjugated with a $\Rightarrow$ I_c is Not positive semi-definit $\Rightarrow$ c is not a minimal geodesic.

19.07.2023

8 Rauch comparison theorem

8.1 Theorem (Rauch). *Let $(M,g),(\widetilde{M},\widetilde{g})$ be Riem. mfds. with $\dim_M \leq \dim_{\widetilde{M}}$. Let $c,\widetilde{c}:[0,L]\to M,\widetilde{M}$ be geodesics with $|c'|=|\widetilde{c}'|=1$. We assume*

1. *$\forall t\in[0,L]$ and $\forall E\in G_2(T_{c(t)}M)$ with $c'(t)\in E$ and $\forall \widetilde{E}\in G_2(T_{\widetilde{c}(t)}\widetilde{M})$ with $\widetilde{c}'(t)\in\widetilde{E}$ it holds $KE)\leq \widetilde{K}(\widetilde{E})$,*
2. *There is no $t\in(0,L)$ conjugated to 0 along $\widetilde{c}$.*

If $Y,\widetilde{Y}$ are Jacobi fields along $c,\widetilde{c}$ with $Y(0)=0=\widetilde{Y}(0)$, $\langle Y'(0),c'(0)\rangle=\langle \widetilde{Y}'(0),\widetilde{c}'(0)\rangle$ and $|Y'(0)|=|\widetilde{Y}'(0)|$, then it follows

$$|Y(t)|\geq|\widetilde{Y}(t)|\ \forall t\in[0,L].$$

Proof. W.l.o.g. we assume $\langle Y,c'\rangle=\langle\widetilde{Y},\widetilde{c}'\rangle=0$. Then, by the rule of L'Hospital,

$$\lim_{t\to0}\frac{|Y(t)|^2}{|\widetilde{Y}(t)|^2}=\lim_{t\to0}\frac{|Y'(t)|^2}{|\widetilde{Y}'(t)|^2}=\frac{|Y'(0)|}{|\widetilde{Y}'(0)|}=1.$$

For this also note that $\langle Y,Y\rangle'=2\langle Y',Y\rangle$ and $2\langle Y',Y\rangle'=2\langle Y'',Y\rangle+2\langle Y',Y'\rangle\to\langle Y'(0),Y'(0)\rangle$ as $t\to0$.

Then it is enough to show that $\frac{d}{dt}\frac{|Y(t)|^2}{|\widetilde{Y}(t)|^2}\geq0\ \forall t\in[0,L]$. Then $\frac{|Y(t)|^2}{|\widetilde{Y}(t)|^2}$ is monotone non-decreasing and hence $\frac{|Y(t)|^2}{|\widetilde{Y}(t)|^2}\geq1\ \forall t\in[0,L]$.

Moreover $\frac{d}{dt}\frac{|Y(t)|^2}{|\widetilde{Y}(t)|^2}\geq0\Leftrightarrow\langle Y',Y\rangle\langle\widetilde{Y},\widetilde{Y}\rangle\geq\langle\widetilde{Y}',\widetilde{Y}\rangle\langle Y,Y\rangle$.

If $t_0\in(0,L]$ with $Y(t_0)=0$, then the inequality follows trivally.

Let $t_1\in(0,L)$ be arbitrary such that $Y(t_1)\neq0$. By replacing Y with αY for $\alpha\in\mathbb{R}\backslash\{0\}$ we can assume that $|Y(t_1)|=|\widetilde{Y}(t_1)|=1$.

We show $\langle Y',Y\rangle(t_1)\geq\langle\widetilde{Y}',\widetilde{Y}\rangle(t_1)$. For this observe first

$$\begin{aligned}I_{c|_{[0,t_1]}}(Y,Y)&=\int_0^{t_1}\left\{\langle Y',Y'\rangle-\langle R(Y,c')c',Y\rangle\right\}ds\\&=\int_0^{t_1}\left\{\langle Y',Y'\rangle+\langle Y'',Y\rangle\right\}ds\\&=\int_0^{t_1}\langle Y',Y\rangle' ds=\langle Y'(t_1),Y(t_1)\rangle-\underbrace{\langle Y'(0),Y(0)\rangle}_{=0}.\end{aligned}$$

If $E(t)=\operatorname{span}\{c'(t),Y(t)\}$, then $\langle R(Y(t),c'(t))c'(t),Y(t)\rangle=K(E(t))\Big(|Y(t)|^2\underbrace{|c'(t)|^2}_{=1}-\underbrace{\langle Y,c'\rangle^2}_{=0}\Big)$.

Hence

$$I_{c|_{[0,t_1]}}(Y,Y)=\int_0^{t_1}|Y'|^2-K(E(t))|Y(t)|^2dt=\langle Y'(t_1),Y(t_1)\rangle.$$

Let $E_1, \ldots, E_m = c'$ be a parallel orthonormal frame along c such that $E_1(t_1) = Y(t_1)$. In particular, we can write $Y(t) = \sum_{i=1}^{m-1} Y^i(t) E_i(t)$.

We assume $\dim_M \leq \dim_{\widetilde{M}}$. $\Rightarrow \exists$ orthonormal, parallel fields $\widetilde{E}_1, \ldots, \widetilde{E}_m = \widetilde{c}'$ along $\widetilde{c}$ such that $E_1(t_1) = \widetilde{Y}(t_1)$.

Now we define
$$\widetilde{V}(t) = \sum_{i=1}^{m-1} Y^i(t) \widetilde{E}_i(t) \perp c'(t).$$

It follows $\widetilde{V}(0) = \sum_{i=1}^{m-1} Y^i(0)\widetilde{E}_i(0) = 0 = \widetilde{Y}(0)$, $\widetilde{V}(t_1) = \sum_{i=1}^{m-1} Y^i(t_1)\widetilde{E}_i(t_1) = \widetilde{Y}(t_1)$, $|\widetilde{V}| = |Y|$ and $|\widetilde{V}'| = |Y'|$.

Let $\widetilde{E}(t) = \operatorname{span}\{\widetilde{V}(t), \widetilde{c}'(t)\}$. With this we can compute

$$\begin{aligned}
\langle Y', Y\rangle(t_1) &= \int_0^{t_1} \left(|Y'|^2 - K(E(t))|Y|^2\right) dt \geq \int_0^{t_1} \left(|V'|^2 - \widetilde{K}(\widetilde{E}(t))|\widetilde{V}|^2\right) dt \\
&= \int_0^{t_1} \left(|\widetilde{V}'|^2 - R(\widetilde{V}, \widetilde{c}')\widetilde{c}', \widetilde{V}\rangle\right) dt = I_{c|_{[0,t_1]}}(\widetilde{V}, \widetilde{V}) \geq I_{c|_{[0,t_1]}}(Y, \widetilde{Y}) = \langle \widetilde{Y}', \widetilde{Y}\rangle(t_1).
\end{aligned}$$

where the last inequality is (b) of 7.7 Theorem (Index Lemma). This finishes the proof. $\square$

8.2 Corollary. *Let* $\dim_M = \dim_{\widetilde{M}}$, $p \in M$, $\widetilde{p} \in \widetilde{M}$, *and* $I : T_pM \to T_{\widetilde{p}}\widetilde{M}$ *orthogonal and*

1. $\sup K \leq \inf \widetilde{K}$,
2. $\widetilde{\exp}_{\widetilde{p}}|_{B_r(0_{\widetilde{p}})}$ *such that all differentials are non-degenerated, hence a local diffeomorphism.*

Then
$$L(\exp_p \circ \gamma) \geq L(\widetilde{\exp}_{\widetilde{p}} \circ \underbrace{I \circ \gamma}_{=:\widetilde{\gamma}}) \text{ for every } C^1 \text{ curve } \gamma : [a, b] \to B_r(0_p).$$

Proof. If we show that $|D(\exp_p)_{\gamma(\tau)}\gamma'(\tau)| \geq |D(\exp_{\widetilde{p}})_{\widetilde{\gamma}(\tau)}\widetilde{\gamma}'(\tau)|$, the claim follows.

We fix $\tau \in [a, b]$ such that $\gamma(\tau) \neq 0$ (w.l.o.g.) and let $Y, \widetilde{Y}$ be Jacobi fields along $t \mapsto \exp_p(t\gamma(\tau)) = \alpha(t, \tau)$ and along $t \mapsto \widetilde{\exp}_{\widetilde{p}}(t\widetilde{\gamma}(\tau)) = \widetilde{\alpha}(t, \tau)$ with $Y(0) = 0 = \widetilde{Y}(0)$, $Y'(0) = \gamma'(\tau)$ and $\widetilde{Y}'(0) = \widetilde{\gamma}'(\tau) = I \circ \gamma'(\tau)$. Hence $Y, \widetilde{Y}$ arise from the variations α, $\widetilde{\alpha}$.

I orthogonal $\Rightarrow$
$$\langle Y'(0), \gamma(\tau)\rangle = \langle I \circ \gamma'(\tau), I \circ \gamma(\tau)\rangle = \langle \widetilde{\gamma}'(\tau), \widetilde{\gamma}'(\tau)\rangle = \langle \widetilde{Y}'(0), \widetilde{\gamma}(\tau)\rangle$$

as well as $|Y'(0)| = |\widetilde{Y}'(0)|$.

Rauch's theorem yields $|Y(t)| \geq |\widetilde{Y}(t)|\ \forall t \in [0, 1]$. If $t = 1$ this is what was to show. $\square$

8.3 Corollary. *Let* (M, g) *be a Riem. mfd. such that* $\inf K_M \geq 0$, *consider* $p \in M$ *and* $r > 0$ *such that* $\exp_p|_{B_r(0_p)}$ *is a diffeomorphism, and let* $I : (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{eucl}) \to (T_pM, g_p)$. *Then*
$$|\widetilde{x} - \widetilde{y}|_{eucl} \geq d^g(\exp_p \circ I(\widetilde{x}), \exp_p \circ I(\widetilde{y})) \quad \forall \widetilde{x}, \widetilde{y} \in I^{-1}(B_r(0_p)).$$

8.4 Remark. Let $x, y \in B_r(p) \subset M$ and $v, w \in B_r(0_p)$ such that $\exp_p(v) = x, \exp_p(w) = y$. Consider $I^{-1}(v) = \widetilde{x}$ and $I^{-1}(w) = \widetilde{y}$ in $\mathbb{R}^m$. Since I is orthogonal, $g_p(v, w) = \langle \widetilde{x}, \widetilde{y} \rangle_{eucl}$. By the corollary we have

$$|\widetilde{x} - \widetilde{y}| \geq d^g(x, y).$$

Let γ_{px} be the geodesic between p and x, let γ_{px} be the geodesic between p and y and let γ_{xy} be the geodesic between x and y. All geodesics are parametrized by arclength. The geodesics $\gamma_{px}, \gamma_{py}, \gamma_{xy}$ form a geodesic triangle Δ in $B_r(p)$. Let $\angle_p(\gamma_{px}, \gamma_{py}) = \arccos g_p(\gamma'_{px}(0), \gamma'_{py}(0))$ be the angle between γ_{px} and γ_{py} and similar for the other corners of the geodesic triangle Δ. A comparison triangle for Δ in $\mathbb{R}^m$ are three points $\bar{p}, \bar{x}, \bar{y} \in \mathbb{R}^m$ such that $d^g(p, x) = |p - x|, d^g(p, y) = |p - y|, d^g(x, y) = |x - y|$.

The points $0, \widetilde{x}, \widetilde{y}$ from before are not yet a comparison triangle in general. But we can move the point $\widetilde{x}$ to decrease the angle $\arccos \frac{\langle \widetilde{x}, \widetilde{y} \rangle}{|\widetilde{x}||\widetilde{y}|} =: \angle(\widetilde{x} 0 \widetilde{y})$ (keeping the distance with 0 fixed) to obtain one.

As a consequence we get that a comparison triangle $\bar{p}, \bar{x}, \bar{y}$ for Δ satisfies

$$\angle_p(\gamma_1, \gamma_2) \geq \angle(\bar{x} 0 \bar{y}).$$

The choice of p where we measure this angle was arbitrary. Via the same procedure we can construct comparison triangles such that

$$\angle_x(\gamma_1, \gamma_3) \geq \angle(0 \bar{x} \bar{y}) \text{ and } \angle_y(\gamma_3, \gamma_2) \geq \angle(\bar{x} \bar{y} 0).$$

Since up to translation and rotatation comparison triangle are unique, we get that

$$\angle_p(\gamma_1, \gamma_2) \geq \angle(\bar{x} 0 \bar{y}),\ \angle_x(\gamma_1, \gamma_3) \geq \angle(0 \bar{x} \bar{y}) \text{ and } \angle_y(\gamma_3, \gamma_2) \geq \angle(\bar{x} \bar{y} 0)$$

for any comparison triangle $0, \bar{x}, \bar{y}$ in $\mathbb{R}^n$.

9 Second fundamental form and Gauß equations

9.1 Definition. Let $(\bar{M}, \bar{g})$ be a Riem. mfd. with $\dim_{\bar{M}} = \bar{m}$ and let $i : M \subset \bar{M}$ be an embedded m-dimensional submanifold. $\bar{g}$ induces a Riem. metric on M via $i^*\bar{g} = g$. (M, g) is called Riemannian submanifold of $(\bar{M}, \bar{g})$.

Remark. We usually write $i(p) = p$ and via $Di|_p$ we can identify the tangent space T_pM with a linear subspace in $T_{i(p)}\bar{M}$. In particular, one writes

$$g_p(v, w) = \bar{g}_{\pi(p)}(Di|_p v, Di|_p w) = \bar{g}_p(v, w).$$

9.2 Lemma. *The restriction of the tangent bundle $\bar{\pi} : T\bar{M} \to \bar{M}$ to M is*

$$i^*T\bar{M} = \{v \in T\bar{M} : \bar{p}(v) \in M\} \to M.$$

$i^*T\bar{M} \to M$ is an $\bar{m}$-dimensional vector bundle over M. Moreover

$$TM^\perp = \{v \in i^*T\bar{M} | v \in (T_{\bar{\pi}(v)}\bar{M})^\perp \subset T_{\bar{\pi}(v)}\bar{M}\} \to M$$

is a $\bar{m} - m$-dimensional subvector bundle of $i^*T\bar{M}$.

9.3 Remark. The LC connection $\bar{\nabla}$ of $(\bar{M}, \bar{g})$ induces a linear connection $\bar{\nabla}$ on $i^*T\bar{M} \to M$ via

$$X \in \Gamma(i^*T\bar{M}), v \in TM \mapsto \bar{\nabla}_v \bar{X} =: \nabla_v X \in T_{\pi(v)}\bar{M}$$

where $\bar{X} \in \Gamma(T\bar{M})$ with $\bar{X}|_M = X$. Each $\Gamma(i^*T\bar{M})$ admits a unique decomposition

$$X = X^\top + X^\perp, \; X^\top \in \Gamma(TM), X^\perp \in \Gamma(TM^\perp).$$

9.4 Theorem. *The LC connection of (M, g) is given through $X, Y \in \Gamma(TM) \mapsto (\bar{\nabla}_{\bar{X}}\bar{Y})^\top|_M = \nabla_X Y \in \Gamma(TM)$ where $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$ such that $\bar{X}|_M = X, \bar{Y}|_M = Y$. $\bar{X}|_M = X$ actually means $\bar{X} \circ i = DiX$.*

Proof. 1. ∇ is linear connection.

2. ∇ is symmetric. For this choose $\bar{X}, \bar{Y}$ for X, Y as in the theorem. We compute

$$\nabla_X Y - \nabla_Y X := \left(\bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{Y}\right)^\top|_M = [\bar{X}, \bar{Y}]^\top|_M = ([\bar{X}, \bar{Y}]|_M)^\top.$$

Note that $\bar{X} \circ i = DiX, \bar{Y} \circ i = DiY$ implies $[\bar{X}, \bar{Y}] \circ i = Di[X, Y]$, or in other words $[\bar{X}, \bar{Y}]|_M = [X, Y]$. Hence $([\bar{X}, \bar{Y}]|_M)^\top = [X, Y]$.

3. ∇ is a Riemannian connection. Let $Z \in \Gamma(TM)$ and $\bar{Z} \in \Gamma(T\bar{M})$ such that $\bar{Z}|_M = Z$. Let $p \in M$. We compute

$$\begin{aligned} Z_p g(X, Y) = (\bar{Z}_p)\bar{g}(\bar{X}|_M, \bar{Y}|_M) = \bar{Z}_p \bar{g}(\bar{X}, \bar{Y})|_M = \bar{g}(\bar{\nabla}_{\bar{Z}_p}\bar{X}, \bar{Y}) \circ i + \bar{g}(\bar{X}, \bar{\nabla}_{\bar{Z}_p}\bar{Y}) \circ i \\ = \bar{g}(\bar{\nabla}_{Z_p}X, Y) + \cdots = g((\bar{\nabla}_{Z_p}\bar{X})^\top, Y) + \cdots = g(\nabla_{Z_p}X, Y) + \ldots. \end{aligned}$$

□

9.5 Definition. The second fundamental form $h \in \Gamma(TM^* \otimes TM^* \otimes TM^\perp)$ is defined as

$$X, Y \in \Gamma(TM) \mapsto h(X, Y) : (\bar{\nabla}_X Y)^\perp \in \Gamma(TM^\perp).$$

9.6 Remark. It holds

(a) $h(X,Y) = h(Y,X)$, and h is a tensor field

(b) If $N \in \Gamma(TM^{\perp})$, then $\bar{g}(h(X,Y),N) = -\bar{g}(Y, \bar{\nabla}_X N)$.

(c) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$.

Proof. (a) $h(X,Y) - h(Y,X) = \big(\nabla_{\bar{X}}\bar{Y} - \nabla_{\bar{Y}}\bar{X}\big)^{\perp}|_M = ([\bar{X},\bar{Y}]|_M)^{\perp} = ([X,Y])^{\perp} = 0$. Moreover $h(fX,Y) = (\bar{\nabla}_{fX}Y)^{\perp} = (f\bar{\nabla}_X Y)^{\perp} = fh(X,Y) = fh(Y,X) = h(fY,X)$. Hence h is a tensor field and $h(X,Y)|_p$ only depends on X_p and Y_p.

(b) Note that $\bar{g}(Y,N) = 0$ on M. Hence $0 = X_p\bar{g}(Y,N)$ for $p \in M$. It follows

$$0 = X_p\bar{g}(Y,N) = X_p\bar{g}(\bar{Y},\bar{N}) = \underbrace{\bar{g}(\bar{\nabla}_{X_p}\bar{Y}, N(p))}_{g((\bar{\nabla}_{X_p}Y,N(p)))=g(h(X,Y),N)(p)} + \bar{g}(Y(p), \bar{\nabla}_{X_p}\bar{N}).$$

(c) This is clear from the definition.

□

9.7 Theorem (Gauß equation)**.** *Let (M,g) be a Riem. submfd. of $(\bar{M},\bar{g})$. Then*

$$\bar{g}(R(X,Y)Z,W) = \bar{g}(\bar{R}(X,Y)Z,W) + \bar{g}(h(X,W),h(Y,Z)) - \bar{g}(h(X,Z),h(Y,W))$$

$\forall X,Y,Z,W \in \Gamma(TM)$.

Proof. By definition: $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$. Then

$$\bar{R}(X,Y)Z = (\bar{R}(\bar{X},\bar{Y})\bar{Z})|_M = \big(\bar{\nabla}_{\bar{X}} \underbrace{\bar{\nabla}_{\bar{Y}}\bar{Z}}_{=\bar{\nabla}_X Z = \nabla_X Z + h(X,Z) \text{ on } M} - \bar{\nabla}_{\bar{Y}}\bar{\nabla}_{\bar{X}}\bar{Z} - \bar{\nabla}_{[\bar{X},\bar{Y}]}\bar{Z}\big)|_M.$$

Hence

$$\bar{R}(X,Y)Z = \bar{\nabla}_X(\nabla_Y Z + h(Y,Z)) - \bar{\nabla}_Y(\nabla_X Z + h(X,Z)) - \bar{\nabla}_{[X,Y]}Z.$$

Note that $\bar{\nabla}_X\nabla_Y Z = \nabla_X\nabla_Y Z + h(X,\nabla_Y Z)$ with $h(X,\nabla_Y Z) \in TM^{\perp}$. This yields

$$(\bar{R}(X,Y)Z)^{\top} = R(X,Y)Z + (\bar{\nabla}_X h(Y,Z))^{\top} - (\bar{\nabla}_Y h(X,Z))^{\top}.$$

Note that $\bar{g}(h(Y,Z),W) = 0 \;\forall W \in \Gamma(TM)$. Hence

$$\bar{g}(\bar{\nabla}_X h(Y,Z),W) = -\bar{g}(h(Y,Z),\bar{\nabla}_X W) = -\bar{g}(h(Y,Z),h(X,W)).$$

This gives the claim. □

9.8 Remark. Let $E \in G_2(TM)$ with $\mathrm{span}(u,v) = E$. Then

$$K(E) = \bar{K}(E) + \frac{\bar{g}(h(u,u),h(v,v)) - \bar{g}(h(u,v),h(u,v))}{Q(u,v)}.$$

Recall $Q(u,v) = g(u,u)g(v,v) - g(u,v)^2$. In particular, for $(\bar{M},\bar{g}) = (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ this is Gauß' Theorema Egregium.

9.9 Definition. A Riem. submfd. (M, g) is called totally geodesic if every geodesic in (M, g) is also a geodesic in $(\bar{M}, \bar{g})$.

9.10 Theorem. *(M, g) is totally geodesic in $(\bar{M}, \bar{g})$ if and only if $h \equiv 0$.*

Proof. Geodesic equation: $\bar{\nabla}_{c'}c' = \nabla_{c'}c' + h(c', c')$. □

9.11 Example. $M \subset \mathbb{R}^n$ totally geodesic if and only if M is affine subspace.

In general, there are not totally geodesic submanifolds. for $\dim_M < \dim_{\bar{M}}$.

9.12 Corollary. *Let $(\bar{M}, \bar{g})$ be Riem., $p \in \bar{M}$, and $U \subset T_p\bar{M}$ offen such that $O_p \in U$ and $\bar{\exp}_p|_U$ is a diffeomorphism. Falls $E \subset G_2(T_p\bar{M})$, then $M := \bar{\exp}_p(E \cap U)$ is a embedded submanifold. Then $h_p = 0$ and $\bar{K}(E) = K(E) = K_p$ where K_p is the Gauß curvature of $(M, \bar{g}|_M)$ in p.*

Remark. The corollary provides the original definition of sectional curvature by B. Riemann (Habilitationsvortrag 1854).

Proof. $v \in E$ and $\exp_p(v) = c_v$. $\Rightarrow 0 = \bar{\nabla}_t c_v = (\bar{\nabla}_t c_v)^\top = \nabla_t c_v$. Hence $h(c_v', c_v') = 0$. In particular for $t = 0$ we get $h_p(v, v) = 0$ $\forall v \in E$. Hence $h_p = 0$. Then the claim follows from the Theorema egregium. □