Abteilung für Mathematische Logik
Mathematische Logik (SS 2014)
Prof. Dr. Martin Ziegler
Dr. Mohsen Khani

Übung zur Vorlesung
Aufgabe 1, Strukturen und Formeln
Abgabe am 5.5 vor 16 Uhr

Remark. *No need to answer the complementary exercises, unless you are interested.*

Aufgabe 1.1 Let $L = \{R\}$ be a language with R a binary relation symbol. Consider the set of the people of your class as the universe of an L-structure $\mathcal{M}$ in which R is interpreted as friendship relation, i.e.

$$R^{\mathcal{M}}(x, y) \Leftrightarrow x \neq y \text{ and } x \text{ and } y \text{ are friends.}$$

Write down the following as L-formulae.

a) There is person who is friends with every one who has at least one friend.(1)

b) If it is not the case that every one has a friend, then there is a person who has no friends. (1)

c) Every two people who have friends are friends. (1)

Aufgabe 1.2 Let $L = \{c_1, c_2, f(.,.), g(.,.), R(.,.), P\}$ where c_1, c_2 are constant symbols, f, g are function symbols, and $R(x, y)$ is a symbol for a binary relation. Consider the set of natural numbers $\mathbb{N}$ as the universe of an L-structure $\mathcal{N}$ by interpreting c_1 and c_2 respectively as 0 and 1, f and g receptively as addition and multiplication ($f^{\mathcal{N}} = +, g^{\mathcal{N}} = .$), and R as the order. Let $P^{\mathcal{N}}$ also be the set of prime numbers (i.e. $P^{\mathcal{N}}(x)$ if and only if x is a prime number). Write down the following statements as L-formulae.

a) x is a prime number if and only if it is not 1 and whenever it is written as a multiple of two smaller numbers then one of them is 1. (1)

b) There are infinitely many prime numbers (of course in a first order language you cannot have a formula with infinitely many quantifiers!) (1)

c) There is a unique even prime number. (1)

d) There are infinitely many prime twins (= pairs of primes (p, q) such that $q = p + 2$). (1)

Aufgabe 1.3

a) (The substructure generated by a set) Let $\mathfrak{A}$ be an L-structure. [1] Show that for every non-empty subset S of A there is a smallest substructure of $\mathfrak{A}$ containing S. This structure is called the *substructure generated by* S' (read the following definition if you are unfamiliar with 'substructures'). (3)

b) Consider the structure $\mathcal{R} = \langle \mathbb{R}, +, ., 0, 1 \rangle$. What it the substructure generated by $\{0, 1\}$?

Definition For $\mathfrak{A}$ as above, let B be a subset of A that contains $c^{\mathfrak{A}}$ for all constant symbols c and is closed under all functions $f^{\mathfrak{A}}$. Then one can interpret all symbols of L in B and make it into an L-structure $\mathfrak{B}$. We call $\mathfrak{B}$ a substructure of $\mathfrak{A}$ and this is usually denoted by $\mathfrak{B} \subseteq \mathfrak{A}$.
As for the answer you just need to show that the intersection of a family of substructures is a substructure (because then one considers the intersection of the substructures of $\mathfrak{A}$ whose universes include S).

Aufgabe 1.4 Zeigen Sie, daß sich jedes Endstück eines Terms eindeutig als eine Folge von Termen schreiben läßt. (4)

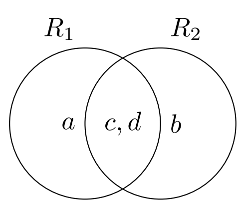
Aufgabe 1.5 Eine *Teilformel* von ϕ ist ein zusammenhängendes Teilstück von ϕ, das selbst eine Formel ist. Zeigen Sie, daß alle Teilformeln von ϕ im rekursiven Aufbau von ϕ vorkommen müssen. (4)

Aufgabe 1.6 Sei $\mathfrak{A}$ eine L–Struktur mit Grundmenge A. Ein *Automorphismus* ist ein Isomorphismus von $\mathfrak{A}$ mit sich selbst. Zeigen Sie: Wenn A endlich ist, gibt es auf der Grundmenge A genau

Anzahl der Permutationen von A/Anzahl der Automorphismen von $\mathfrak{A}$

viele L–Strukturen, die isomorph zu $\mathfrak{A}$ sind. (4)

Remark. *To have a clearer picture consider the following example. Let* $L = \{R_1, R_2\}$ *be a language with* R_1 *and* R_2 *unary predicate symbols. Consider an* L*-structure with the universe* $\{a, b, c, d\}$ *in which these relations are interpreted as in the following figure:*



In the above diagram automorphisms are obtained only when c is sent to d vice versa; but to have an isomrphic picture we only need to keep two elements in the middle and two in the left and right-hand-side circles.

[1] When $\mathcal{A}$ or $\mathfrak{A}$ is used to denote an L-structure, it is implicitly assumed that the underlying universe is A.

Hint Treat this as a group theory exercise: consider the group P of permutations of A, define a suitable action of G on the set of L-structures whose universe is A, use the orbit-stabiliser theorem.

Complementary exercises

C1. Let L be a countable first order language. Show that for any infinite cardinal κ there are at most 2^κ non-isomorphic L-structures of cardinality κ. (exercise from [2]).
C2. Try to write down the following simple sentence in a suitable first order language:

Every farmer who has a donkey beats it;

or the following:

Every police officer who arrested a murderer insulted him.

The reason that the first order equivalents of these sentences do not seem as easy as they are in English language is the difference between the conventional scope of a quantifier in a first order langugage and a natural language. This phenomenon is known as *donkey anaphora* (exercise from [1] with changes).

C3. Consider the the structure $\langle \mathbb{R}, +, ., <, f \rangle$ where $\mathbb{R}$ is the set of real numbers and f is a unary function. Which of the following can be expressed with a first order formula?

- The intermediate value theorem holds for all polynomials over $\mathbb{R}$.
- $\langle \mathbb{R}, < \rangle$ is complete (i.e. every bounded above subset of $\mathbb{R}$ has a supremum).
- $\lim_{x \to y} f(x) = z$.
- $\mathbb{R}$ with $+$ and $.$ is a field.
- $\langle \mathbb{R}, < \rangle$ is archimedean.

References

[1] I. Chiswell and W. Hodges, *Mathematical logic*, OXFORD TEXTS IN LOGIC, OUP Oxford, 2007.

[2] D. Marker, *Model theory: An introduction*, Graduate Texts in Mathematics, Springer, 2010.