Abteilung für mathematische Logik
Mathematische Logik (SS 2014)
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Übung zur Vorlesung
Aufgabe 4, Kompaktheitssatz
Abgabe am 26.5 vor 16:00 Uhr

For your ease of reference, I first state the 'compactness' theorem in the first order and propositional logic.

(First order logic). Let Σ be an infinite set of sentences in a first order language L. Then if for every finite subset Δ of Σ, there is an L-structure $\mathfrak{M}_\Delta$ such that $\mathfrak{M}_\Delta \models \Delta$, then there is an L-structure $\mathfrak{M}_\Sigma$ such that $\mathfrak{M}_\Sigma \models \Sigma$.

(Propositional logic). Let Σ be an infinite set of formulae. Suppose that for every finite subset Δ of Σ there is an assignment β_Δ of variables such that all formulas in Δ are true with respect to β_Δ. Then there is an assignment β_Σ of variables, for which all formulas in Σ are true.

Aufgabe (4-1)**.**

a) Is it possible to have a set Σ of sentences in a first order language L such that the following two happen at the same time:

   (a) For every natural number n, there is some $\mathfrak{M} \models \Sigma$ with $|M| = n$ (the universe of $\mathfrak{M}$ has n elements).

   (b) There is no $\mathfrak{M} \models \Sigma$ with an infinite universe. (2)

   Remark. The above question can be better put in the following two ways:

   - Is the class of all finite L-structures elementary (i.e is it exactly the class of models of a theory)?
   - Is there any theory with arbitrarily large finite models and no infinite model?

b) Is it possible to write a set of axioms Σ such that for every L-structure $\mathfrak{M}$ we have $\mathfrak{M} \models \Sigma$ if and only if M is infinite? (put differently, is the class of infinite L-structures elementary?) (1)

c) Let L_{or} be the language of ordered rings and $\mathfrak{R}$ the ordered field of real numbers. Show that there exists an L_{or}-structure $\mathfrak{R}'$ with the following two properties:

   - $\mathfrak{R}'$ satisfies all first order sentences that $\mathfrak{R}$ satisfies[1].

[1] The set of all first order sentences that $\mathfrak{R}$ satisfies is called the theory of $\mathfrak{R}$ and is denoted by $Th(\mathfrak{R})$.

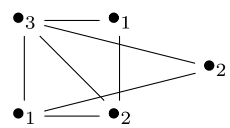
- $(R', <)$ is not archimedean (that is there is an element in R' which is greater than every natural number[2].(You may also have a look at exercise 2-3)). (2)

(Tutors: limits of functions in non-standard analysis should be explained).

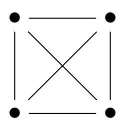
Definition. A (simple and undirected) graph G is called N-colourable if it is possible to assign a colour from the set C of colours: $\{\text{Colour } 1, \text{Colour } 2, \ldots, \text{Colour } N\}$ to each of its vertices in a way that:

1. Every vertex gets a (unique) colour.
2. No adjacent vertices have the same colour.

For example the following graph is coloured by three colours $\{1, 2, 3\}$:



But the following cannot get three colours:



Theorem (De Bruijn–Erdős)**.** Let $n \in \mathbb{N}$. An infinite graph G is N colourable if and only if every finite subgraph of G is N-colourable.

Aufgabe (4-2)**.**

1. Prove the above theorem using the compactness theorem in the first order logic. (2)
2. Prove the above theorem using the compactness theorem in the propositional logic. (2)

Aufgabe (4-3)**.** Are the following formulae logically valid (=allgemeingülig)?

(a) $\phi_1 = (\forall x\, Rxy \to (\exists z\, Pz \to Px)) \leftrightarrow ((\forall x\, Rxy \wedge \exists z\, Pz) \to Px)$ (1)

(b) $\phi_2 = (\exists x \forall y\, Rxy \to \forall y \exists x\, Rxy)$ (1)

(c) $\phi_3 = (\forall z\, Rzfxz \to \exists x \forall z Rzx)$ (1)

(d) As you have proved, (c) is not valid. Does this not violate $\exists$-quantifier axiom? (1)

[2]Such an element is called "infinitely large"

Aufgabe (4-4)**.** Prove the following in Hilbert Calculus and without employing Gödel's completeness theorem.

(a) $\exists v_0\, Rv_0v_1 \to \exists v_2\, Rv_2v_1$ (2)

(b) $\exists v_0\, \neg Rv_0 f v_0 \vee \exists\, v_1 Rcv_1$ (2)