Abteilung für mathematische Logik
Mathematische Logik (SS 2014)
Prof. Dr. Martin Ziegler
Dr. Mohsen Khani

Übung zur Vorlesung
Aufgabe 5, Der Sequenzenkalkül
Abgabe am 2.6 vor 16:00 Uhr

Definition. Fix a language L. Let $\mathfrak{A}$ be an L-substructure of $\mathfrak{B}$. Then $\mathfrak{A}$ is called an elementary substructure of $\mathfrak{B}$ if for every L-formula $\phi(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n$ all in A, we have

$$\mathfrak{A} \models \phi(a_1, \ldots, a_n) \text{ if and only if } \mathfrak{B} \models \phi(a_1, \ \ldots, a_n).$$

You can see in particular that in this case $\mathfrak{A}$ satisfies all sentences that $\mathfrak{B}$ satisfies. For example if the universe of $\mathfrak{B}$ with some operators in the language is a field or a group, the so is the universe of $\mathfrak{A}$ with those operations.

Theorem (Tarski)**.** Fix a language L. Let $\mathfrak{A}$ be an L-substructure of $\mathfrak{B}$. Then $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ if and only if for every formula $\phi(x, y_1, \ldots, y_n)$ in the language, and every $a_1, \ldots, a_n$ in A we have:

$$\text{if } \mathfrak{B} \models \exists x \phi(x, a_1, \ldots, a_n) \text{ then } \mathfrak{A} \models \exists x \phi(x, a_1, \ldots, a_n)$$

Aufgabe (5-1)**.** Consider the structure $\mathfrak{C} = \langle \mathbb{C}, +, ., 0, 1 \rangle$.

1. Give an example of a countable substructure $\langle A, +, ., 0, 1 \rangle$ of $\mathfrak{C}$ such that $\langle A, +, ., 0, 1 \rangle$ is not a field. (1)

2. Give an example of a countable substructure $\langle A, +, ., 0, 1 \rangle$ of $\mathfrak{C}$ in which the universe is a field (with the operations +,.) but the substructure is not elementary. (1)

3. By a Henkin construction (like the theory T^+ in the proof of completeness theorem) prove that there exists a countable elementary substructure of $\langle \mathbb{C}, +, ., 0, 1 \rangle$. (2)

4. Give an algebraic description of the above elementary substructure. (a mathematical guess suffices: begin with $\langle \mathbb{Q}, +, ., 0, 1 \rangle$. Note that by Tarski's theorem, and the fact that terms here are polynomials, the structure we are looking for should contain the roots of all polynomials with coefficient in $\mathbb{Q}$, because $\mathbb{C}$ does). (1)

   (Tutors: As a complementary remark, every substructure of $\mathfrak{C}$ which is an algebraically closed field, is an elementary substructure. More generally if $\mathfrak{A}$ and $\mathfrak{B}$ are algebraically closed fields and $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, then it is an elementary substructure).

5. Let $\mathfrak{R} = \langle \mathbb{R}, +, ., 0, 1, < \rangle$. Let $\mathfrak{R}'$ be a substructure of $\mathfrak{R}$ such that $\mathfrak{R}' \models \mathrm{Th}(\mathfrak{R})$. Let $ax^2 + bx + c = 0$ be an equation with coefficients a, b, c in R'. Show that this equation has a root in $\mathbb{R}$ if and only if it has a root in R'. (2)

6. Prove Tarski's theorem (stated above) (2)

Aufgabe (5-2)**.** Suppose that x is not free in ψ.

a) Prove in Hilbert Calculus that $\vdash \exists x(\phi(x) \vee \psi) \leftrightarrow \exists x\phi(x) \vee \psi$. (2)

b) Prove it in the sequence calculus. (2)

c) Explain why the condition "x is not free in ψ" is necessary. Examples in every-day language are also accepted. (1)

d) (Krivine's hat, see also Aufgabe 3-1) using a) prove the following in Hilbert Calculus:

$$\exists x(H(x) \to \forall y H(y))$$

(2)

Aufgabe (5-3)**.** Let L_1 and L_2 be two languages, T_1 a consistent L_1-theory and T_2 a consistent L_2- theory. Let $L = L_1 \cap L_2$. Suppose that T_1 and T_2 prove the same L-sentences. Show that $T_1 \cup T_2$ is consistent. (4)

Aufgabe (5-4)**.** We call a class $\mathcal{K}$ of L-structures, finitely axiomatisable if it is the class of models of a finite theory (= a finite set of sentences). Show that the class $\mathcal{K}$ of L-structures is finitely axiomatisable if and only if $\mathcal{K}$ and its complement are both axiomatisable. (=each of them is the set of models of a theory) (2)

Hinweis. First observe that if an infinite set of sentences is inconsistent, then there is a finite subset of it which is inconsistent (because if every finite subset is consistent, the by compactness the whole set is consistent). Suppose that T_1 axiomatises $\mathcal{K}$ and T_2 axiomatises $\mathcal{K}^c$. Then observe that $T_2 \cup T_1$ is inconsistent and continue.

Aufgabe (5-5, Skolem's paradox, complementary)**.** Let ZFC be the Zermelo-Frankel axioms for set theory with the axiom of choice. Show that there is a countable model $\mathcal{M}$ of ZFC. How do you explain: $\mathcal{M} \models$ "there is an uncountable set"?