Theorem. Suppose that T has the following property: for every $M, N \models T$ and every $\bar{a} \in M$ and $\bar{b} \in N$,

$$\mathrm{tp}_{qf}^{M}(\bar{a}) = \mathrm{tp}_{qf}^{N}(\bar{b}) \text{ implies } \mathrm{tp}^{M}(\bar{a}) = \mathrm{tp}^{N}(\bar{b}) \tag{1}$$

where $tp_{qf}^{M}(\bar{a})$ is the quantifier-free type of $\bar{a}$ in M. Then T has quantifier elimination.

Proof. Let $\phi(\bar{x})$ be an L-formula for which we want to find a quantifier-free equivalent modulo T. Let $\Sigma(\bar{x})$ be the set of all quantifier-free implications of ϕ; that is

$$\Sigma(\bar{x}) := \{\psi(\bar{x}) | T \models \phi(\bar{x}) \to \psi(\bar{x})\}.$$

We claim that $T \cup \Sigma(\bar{x}) \models \phi(\bar{x})$. Note that if this claim is proved, then so is the theorem because then there are finitely many formulas in Σ that imply ϕ and they, themselves are implication of ϕ. So their conjunction is the quantifier-free formula we are looking for.

Let us prove the claim. If the claim does not hold, then there is a $K \models T$ and $\bar{k} \in K$ such that

$$K \models \Sigma(\bar{k}) \text{ and } K \models \neg\phi(\bar{k}). \tag{2}$$

Then $T \cup \mathrm{tp}_{qf}^{K}(\bar{k}) \cup \{\phi(\bar{x})\}$ is consistent since otherwise, there is a $\gamma \in \mathrm{tp}_{qf}^{K}(\bar{k})$ such that

$$T \cup \{\phi(\bar{x})\} \models \neg\gamma(\bar{x})$$

which implies $\neg\gamma \in \Sigma$ and this is a contradiction because then $K \models \gamma(\bar{k})$ and $K \models \neg\gamma(\bar{k})$.

Since $T \cup \mathrm{tp}_{qf}^{K}(\bar{k}) \cup \{\phi(\bar{x})\}$ is consistent, there is a model K' of T and a tuple $\bar{k}'$ in it such that

$$\mathrm{tp}_{qf}^{M}(\bar{k}) = \mathrm{tp}_{qf}^{N}(\bar{k}') \text{ and } K' \models \phi(\bar{k}')$$

and this is contradictory with condition 1 in the statement of theorem. □

The proof above is from Pillay in `http://www.math.uiuc.edu/People/pillay/lecturenotes_modeltheory.pdf`.