Universität Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 2, ss2015
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Blatt 8, Elimination der Imaginäre

Aufgabe 1. Describe what is meant by $\mathrm{acl}(a/E)$ and $\mathrm{dcl}^{eq}(a)$.

> The Pillay-Lascar Theorem says "if T is strongly minimal and $\mathrm{acl}(\emptyset)$ is infinite, then T has weak elimination of imaginaries". The next Aufgabe is a counterexample to the requirements and the statement.

Aufgabe 2. Define a relation R over $\mathbb{Q}$ by

$$R(a,b,c,d) \Leftrightarrow a+b+c+d=0.$$

Notice that $(\mathbb{Q}, R)$ is equivalent to $(\mathbb{Q}, +)$ (no zero in the language), and R determines the affine lines over $\mathbb{Q}$. Show that $(\mathbb{Q}, R)$ is strongly minimal, but it does not have weak elimination of imaginaries.

> We saw in the lecture that if T is a totally transcendental theory in which each global type has a canonical base in $\mathfrak{C}$, then T has weak elimination of imaginaries. The proof went as follows: if $e = c/E$ is imaginary and $\mathrm{RM}(c/E) = \alpha$, then we let $\mathbb{P}$ be the global type with $\mathrm{RM} = \alpha$ containing the formula xEc. This type has a canonical base d, and d is the canonical parameter we are looking for. It was left as an exercise to prove that d is finite. This assumption is justified in Aufgaben 3,4,5.

Aufgabe 3. Let $\mathbb{D}$ be a definable class and D be a set such that for each automorphism α

$$\underbrace{\alpha(D) = D}_{\text{pointwise}} \Leftrightarrow \underbrace{\alpha(\mathbb{D}) = \mathbb{D}}_{\text{setwise}}$$

show that D contains a canonical parameter of $\mathbb{D}$.

Aufgabe 4. Let T be totally transcendental and $\mathbb{P}$ be a global type. Show that $\mathbb{P}$ has a finite canonical base in $\mathfrak{C}^{eq}$.

Aufgabe 5. Using the previous two Aufgaben, show that if $\mathbb{P}$ has a canonical base $D \subseteq \mathfrak{C}$, then it has a finite base $d \subseteq \mathfrak{C}$.

> We identified the canonical base for global types in ACF_p as follows: if $\mathbb{P}(\bar{x})$ is a global type, then it is given by an irreducible variety over $\mathfrak{C}^n$ via $\mathrm{tp}(\bar{c}/\mathfrak{C})$ (inaccurate) where $\bar{c}$ is the generic point of V. Also, we can say that $\mathbb{P}(\bar{x})$ is the type whose Morley rank is equal to the Morley rank of V and "$\bar{x} \in V$" $\in \mathbb{P}(\bar{x})$. Let I be the corresponding ideal of V. Then $cb(\mathbb{P}) = [V] = [I]$. Also $[I] = \bigcup_{k=0}^{\infty} [I_k]$ where $I_k = \{p \in I | \deg p \geq k\}$ is considered as a sub-$\mathbb{C}$-vector space of $\mathfrak{C}^{N(k)}$, where $N(k)$ is the number of all monomials of deg $\leq k$ in $X_1, \ldots X_n$, and we have $I = \bigcup_{k=0}^{\infty} I_k$. We are now supposed to apply Andre Weil's theorem over "the field of definition of variety" to prove:

Aufgabe 6. Let $\mathfrak{C}$ be a field and $U \leq \mathfrak{C}^n$ (as vector spaces). Then $[U]$ exists in $\mathfrak{C}$ (for example if $U = \mathfrak{C}.(a_1, \ldots, a_n)$ then $[U] = (a_2/a_1, \ldots, a_n/a_1)$.

> Let T be stable. We proved that $p \in S(B)$ does not fork over $A \subseteq B$ if and only if p has a good definition over $\mathrm{acl}^{eq}(A)$. In the proof of $\Leftarrow$ we said if p has a good definition over $\mathrm{acl}^{eq}(A)$, then it defines a a global type $\mathbb{P}$ extending p, which does not fork over $\mathrm{acl}^{eq}(A)$, and hence over A. The last claim is to be justified below:

Aufgabe 7.

1. If $b_0, b_1 \ldots$ is an indiscernible sequence over A, then it is indiscernible over $\mathrm{acl}(A)$.
2. (Hence:) If $\phi(x, b)$ divides over A, then it divides over $\mathrm{acl}(A)$.
3. Give another proof for item 2, using transitivity.

> The other direction of the proof went as follows: if p does not fork over A, then it has a global non-forking extension $\mathbb{P}$. If M is a model containing A, then $\mathbb{P}$ does not fork over M. Since T is stable, $\mathbb{P}$ is defined over M, and hence $cb(\mathbb{P}) \in M^{eq}$. Since M is arbitrary, $cb(\mathbb{P}) \in \mathrm{acl}(A)$. The last sentence is justified below:

Aufgabe 8. Show that

$$\operatorname{acl}(A) = \bigcap_{M \text{model containing } A} M.$$