Universität Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 2, ss2015
Prof. Dr. Martin Ziegler
Dr. Mohsen Khani

Blatt 9, stabile Theorien

Let T be stable.

Theorem 1. A type $p \in S(B)$ does not fork over $A \subseteq B$ if and only if it has a good definition over $\mathrm{acl}^{eq}(A)$.

Theorem 2. Types over $\mathrm{acl}^{eq}(A)$ are stationary.

By "p having good definition over a set" we mean that the definition of p with parameters in $\mathfrak{C}$ defines a global type. We wanted to use the above theorems to prove that types over models are stationary, that is each type over a model has a unique non-forking extension to any bigger set. To prove this, we need to show that if M is a model then $\mathrm{acl}^{eq}(M) = M^{eq}$.

Aufgabe 1.

1. Zeigen Sie, dass $\mathfrak{C}^{eq}$ der Monster Modell von T^{eq} ist (also $M^{eq} = \mathrm{dcl}^{eq}(M)$ eine elementare Unterstruktur von $\mathfrak{C}^{eq}$ ist).

2. Zeigen Sie, dass $\mathrm{acl}^{eq}(M) = \mathrm{dcl}^{eq}(M) = M^{eq}$.

A theory T is called κ-homogeneous if whenever $A \subseteq M$, M is a model, $f : A \to M$ is partial elementary, and $a \in M$, then there is a partial elementary map $f' : A \cup \{a\} \to M$ that extends f. In particular if M is homogeneous (that is $|M|$-homogeneous) then

$$\bar{a} \equiv \bar{b} \Leftrightarrow \exists \sigma \in Aut(M) \quad \sigma(a) = b.$$

If T is κ-saturated, then it is κ-homogeneous.

Theorem 3. Let T be a stable theory.

- Let $A \subseteq M$ and M be a sufficiently homogeneous model and $p \in S(A)$. Then all non-forking extensions of p to M are conjugates over A.
- If $A \subseteq B$ and $p \in S(A)$, then p has at most $2^{|T|}$ non-forking extensions to B.

The above two items do not hold in random graphs; next exercise.

Aufgabe 2. Sei T die Theorie von zufällige Graphen.

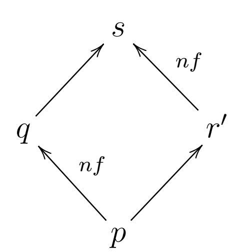
1. Beschreiben Sie die indiscernible Folgen.
2. Beschreiben Sie $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ (und nicht forkende Erweiterungen) in T.
3. Zeigen Sie, dass T nicht stabil ist.
4. Zeigen Sie, dass T einfach ist.
5. Zeigen Sie, dass die beiden Aussagen des Satz 3 sind falsch in zufällige Graphen.

Define
$$N(B/A) := \{q \in S(B) | q \text{ does not fork over } A\}$$
Let $\pi : S(B) \to S(A)$ be the restriction map. The **open mapping theorem** says that whenever T is stable, $\pi \restriction N(B/A)$ is an open map (it sends open sets to open sets).

Aufgabe 3. 1. Zeigen Sie, dass $\pi : S(\mathrm{acl}(A)) \to S(A)$ ist immer offen.

2. Sei T stabil. Ist $\pi : S(B) \to S(A)$ immer offen?

Theorem 4 (Diamond lemma)**.** Let T be simple, $p \in S(A)$, q a non-forking extension of p and r any extension of p. Then there is an A-conjugate r' of r with non-forking extension s that extends q:



$r' = r^\alpha$ α an automorphism

Assume that T eliminates imaginaries. The following are equivalent:

1. $\mathbb{D}$ is $\mathrm{acl}(A)$-definable.
2. There exists an A-definable equivalence relation E with finitely many classes $a_1/E, \ldots, a_n/E$ such that

$$\mathbb{D} = [a_1] \cup \ldots \cup [a_i] \text{ for some } i \leq n$$

where by $[a_i]$ we mean $\{b | E(b, a_i)\}$.

Aufgabe 4 (finite equivalence relation theorem)**.** Nehemen wir an, dass T die Imagninäre eliminiert. Seien $A \subseteq B$ und $p, q \in S(B)$ forken über A nicht. Dann es eine endliche A-definierbare äquivanlence Relation E gibt, sodass

$$p(x) \cup q(y) \vdash \neg E(x, y).$$

We have seen in the lecture that

1. If $p \subseteq q$ then $SU(p) \leq SU(q)$.
2. If $p \sqsubset_{nf} q$ then $SU(p) = SU(q)$.
3. If $SU(p) = SU(q) < \infty$ then $p \sqsubset_{nf} q$.

Aufgabe 5. Warum ist $< \infty$ in 3 notwendig?