Universität Freiburg
Abteilung für Mathematische Logik
Exercises in Model Theory

September 14, 2015

Abstract

The following are the exercises we tried to solve with our students of Model Theory in the University of Freiburg during WS 2014-2015 and SS 2015. The exercises are mostly from the Model Theory book of David Maker [2], and the Model Theory book of Ziegler and Tent [3]. The language switches between German and English. The language switches between English and German!

Contents

Chapter 1

Model Theory 1

1.1 Strukturen

Aufgabe 1. Seien $\mathcal{M}, \mathcal{N}$, L-Strukturen, $\mathcal{M} \subseteq \mathcal{N}$ ($\mathcal{M}$ eine Substruktur von $\mathcal{N}$), $\bar{a} \in M$, und $\phi(\bar{x})$ ein quantorenfreie Formel. Zeigen Sie, dass $\mathcal{M} \models \phi(\bar{a})$, genau dann, wenn $\mathcal{N} \models \phi(\bar{a})$.

Aufgabe 2. (Satz von Tarski) Sei $\mathcal{M} \subseteq \mathcal{N}$. Zeigen Sie, dass $\mathcal{M} \preceq \mathcal{N}$ genau dann, wenn für alle $\phi(x, \bar{y}) \in L$ und $\bar{a} \in M$, gilt: $\mathcal{N} \models \exists x \phi(x, \bar{a})$ genau dann, wenn es ein $c \in M$ gibt, so dass $\mathcal{N} \models \phi(c, \bar{a})$.

Definition. Seien $(I, <)$ eine Totalordnung und $\mathcal{M}_i$ eine L-Struktur für jedes $i \in I$. Wir sagen, dass $(M_i : i \in I)$ eine Kette von L-Strukturen ist, wenn $\mathcal{M}_i \subseteq \mathcal{M}_j$ für alle $i < j$. Wir sagen, dass $(\mathcal{M}_i : i \in I)$ eine elementare Kette ist, wenn $\mathcal{M}_i \preceq \mathcal{M}_j$ für alle $i < j$.

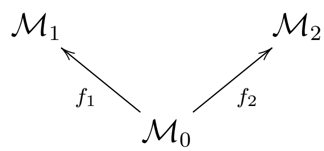
Aufgabe 3.

1. Sei $(\mathcal{M}_i : i \in I)$ eine Kette von L-Strukturen. Geben Sie eine L-Struktur $\mathcal{M}$ mit Grundmenge $\bigcup_{i \in I} \mathcal{M}_i$, so dass $\mathcal{M}_i \subseteq \mathcal{M}$ für alle i.

2. Sei $(\mathcal{M}_i : i \in I)$ eine elementare Kette von L-Strukturen. Zeigen Sie, dass $\mathcal{M}$ eine elementare Erweiterung von jede $\mathcal{M}_i$ ist.

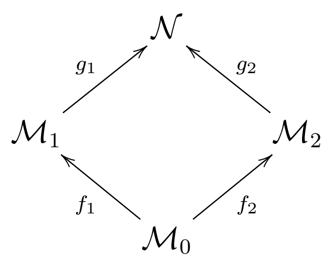
Aufgabe 4. Wir nehmen an, dass im folgenden Diagramm $\mathcal{M}_0, \mathcal{M}_1$ und

$\mathcal{M}_2$, L-Strukturen sind und dass f_1, f_2 elementare Abbildungen sind.



(1.1)

Zeigen Sie, dass es eine L-Struktur $\mathcal{N}$ und elementare Abbildungen g_1 und g_2 gibt, so dass im folgenden Diagramm $g_2 \circ f_2 = g_1 \circ f_1$ gilt.



(1.2)

Hinweis. Sei $\mathcal{N}$ eine $L(M)$-Struktur. Dann, $\mathcal{N}$ eine elementare Erweiterung von $\mathcal{M}$ ist genau dann, wenn $N \models \mathrm{Diag}_{el}(\mathcal{M}) := \{\phi(m_1, \ldots, m_n) : \phi$ eine L- Formel ist und $\mathcal{M} \models \phi(m_1, \ldots, m_n)$.

Aufgabe 5. Ist der vorherige Aufgabe wahr, wenn man elementare Abbildungen mit L-Einbettungen ersetzt?

1.2 Definierbarkeit

Definition. Sei $\mathcal{M}$ eine L-Struktur. Eine Teilmenge $X \subseteq M^n$ heißt (mit Parametern) definierbar, wenn es eine $m \in \mathbb{N}$, eine Formel $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m) \in L$, und $\bar{b} \in M^m$ gibt, sodass $X = \{\bar{a} \in M^n | \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. Wenn $\bar{b} \in A^m$ (A eine Teilmenge von M), dann heißt X auch A-definierbar; oder wir sagen, dass X mit Parametern in A definierbar ist.

Aufgabe 6. Sei $\mathcal{Z} = (\mathbb{Z}, +, \cdot, 0, 1)$. Zeigen Sie, dass die "Ordnung" in $\mathbb{Z}$ definierbar ist; das heißt, dass es eine Formel $\phi(x, y)$ gibt, sodass $\{(m, n) : m, n \in \mathbb{Z}, m < n\} = \{(m, n) : \mathcal{Z} \models \phi(m, n)\}$.

Aufgabe 7. Sei $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$. Wir nehmen an, dass $X \subseteq \mathbb{R}^n$ eine A-definierbare Menge ist. Zeigen Sie, dass der topologische Abschluss von X, auch A-definierbar ist.

Aufgabe 8. Seien $\mathcal{M}$ eine L-Struktur und $X \subseteq M^n$, A-definierbar. Zeigen Sie, dass jeder Automorphismus von $\mathcal{M}$, der A punktweise fest lässt, lässt X mengenweise fest (das heißt, wenn σ eine Automorphismus von M ist und für alle $a \in A$, $\sigma(a) = a$, dann $\sigma(X) = X$).

Aufgabe 9. Zeigen Sie (mit Hilfe der Aufgabe 2), dass $\mathbb{R}$ in $\mathcal{C} = (\mathbb{C}, +, \cdot, 0, 1)$ nicht definierbar ist.

Aufgabe 10. Ist $\{i\}$ in $\mathcal{C}$ (ohne Parameter) definierbar?

Aufgabe 11. Ist $\mathbb{Z}$ in $\mathcal{R}$ definierbar? Ist es in $\mathcal{C}$ definierbar?

Fact. Wenn $X \subseteq \mathbb{C}$ in $\mathcal{C}$ definierbar ist, dann ist entweder X oder $\mathbb{C} - X$ endlich. Wenn $X \subseteq \mathbb{R}$ in $\mathcal{R}$ definierbar ist, dann besteht X exakt aus endlich vielen von Punkten und Intervallen (den Beweis werden wir später sehen).

1.3 Vollständigkeitssatz

Bemerkung.

1. (der Gödelsche Vollständigkeitssatz) $T \models \phi$ genau dann, wenn $T \vdash \phi$.
2. (der Gödelsche Unvollständigkeitssatz) es gibt eine Aussage ϕ die in $(\mathbb{N}, +, ., 0, 1)$ wahr und in PA nicht beweisbar ist.
3. (Kompaktheitssatz) T ist genau dann erfüllbar, wenn jede endliche $T' \subseteq T$ erfüllbar ist.
4. (Satz von Löwenheim-Skolem) Eine Theorie T habe ein unendliche Modell, dann hat T Modelle in jeder unendlichen Kardinalität.

Aufgabe 12. Eine Totalordnung $(G, +, <)$ heißt archimedisch, wenn es für alle $x, z \in G$ ein $m \in \mathbb{N}$ gibt, do dass $|x| < m|y|$. Zeigen Sie, dass es nicht-archimedische zu $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$ elementare äquivalente Körper gibt.

Aufgabe 13.

1. Geben Sie die Axiome für torsionsfreie abelsche Gruppen.
2. Zeigen Sie, dass jede torsionsfreie abelsche Gruppe, geordnet werden kann (man kann eine total $<$ definieren, sodass $a + c < b + d$ für alle $a < b, c \leq d$).

Hinweis. Zeigen Sie es erst für der endlich erzeugte Fall.

1.4 Ultraprodukten

Sei I eine Menge und $P(I) = \{X | X \subseteq I\}$. Ein **Filter** auf I ist eine Menge $D \subseteq P(I)$ mit den folgenden Eigenschaften

1. $I \in D$, $\emptyset \notin D$,
2. wenn $A, B \in D$, dann $A \cap B \in D$,
3. wenn $A \in D$ und $A \subseteq B \subseteq I$, dann $B \in D$.

D ist ein **Ultrafilter**, wenn für alle $X \subseteq I$ entweder $X \in D$ oder $I - X \in D$. Jeder Filter kann zu einem Ultrafilter erweitert werden (Beweis mit Hilfe von Lemma von Zorn).
Seien

- L unsere Sprache,
- I eine unendliche Menge,
- für $i \in I$ sei M_i eine L-Struktur,
- $\prod_{i \in I} M_i = \{(a_i)_{i \in I} | a_i \in M_i\}$, und
- D ein Ultrafilter auf I.

Wir definieren: $(a_i)_{i \in I} \sim (b_i)_{i \in I} \Leftrightarrow \{i | a_i = b_i\} \in D$. Man rechtet nach, dass $\sim$ eine Äquivalenzrelation ist. Sei $M = \prod_{i \in I} M_i / \sim$. Im Folgenden finden wir eine L-Struktur $\mathcal{M}$ deren Grundmenge M ist. Wir nennen sie das Ultraprodukt von M_i (modulo D) und bezeichnen sie mit $\mathcal{M} = \prod_{i \in I}(M_i)/D$.

Interpretation der Konstanten Sei $c \in L$ eine Konstante. c ist in allen $\mathcal{M}_i$ als $c^{\mathcal{M}_i}$ interpretiert. Sei $c^{\mathcal{M}} = [(c_i^{\mathcal{M}_i})_{i \in I}]/\sim$.

Interpretation der Funktionen Sei $f(x_1, \ldots, x_n)$ ein Funktion-Zeichnen, das in $\mathcal{M}_i$ als $f^{\mathcal{M}_i}$ interpretiert ist ($i \in I$). Dann definieren wir:

$$f^{\mathcal{M}}\big([(a_{1i})_{i \in I}]/\sim, \ldots, [(a_{ni})_{i \in I}]/\sim\big) = [(b_i)_{i \in I}]/\sim \Leftrightarrow$$
$$\{i | f^{\mathcal{M}_i}(a_{1i}, \ldots, a_{ni}) = b_i\} \in D.$$

Interpretation der Relationen

$$R^{\mathcal{M}}([(a_{1i})_{i\in I}]/\sim, \ldots, [(a_{ni})_{i\in I}]/\sim) \Leftrightarrow$$
$$\{i | R^{\mathcal{M}_i}(a_{1i}, \ldots, a_{ni})\} \in D$$

Aufgabe 14 (Satz von Łoś)**.** Für alle L-Formeln $\phi(x_1, \ldots, x_n)$ und $[(a_{1i})_{i\in I}]/\sim, \ldots, [(a_{ni})_{i\in I}]/\sim) \in M$:

$$\mathcal{M} \models \phi\big([(a_{1i})_{i\in I}]/\sim, \ldots, [(a_{ni})_{i\in I}]/\sim\big) \text{ genau dann, wenn}$$
$$\{i | \mathcal{M}_i \models \phi(a_{1i}, \ldots, a_{ni})\} \in D.$$

Aufgabe 15. Beweisen Sie den Kompaktheitsatz mit Hilfe der Ultraprodukten.

Hinweis. Nehmen wir an, dass T endlich erfüllbar ist. Sei

- $I = \{\Delta \subseteq T | \Delta \text{ endlich}\}$,
- für alle ϕ, sei $X_\phi = \{\Delta | \Delta \subseteq T, \Delta \text{ endlich }, \phi \in \Delta\}$,
- $D = \{X_\phi | \phi \in T\}$.

Man zeigt, dass

1. D hat die 'Finite Intersection Property' und ist daher in einem Ultrafilter U enthalten.
2. $\prod_{\Delta\in I} \mathcal{M}_\Delta / U \models T$, wobei $\mathcal{M}_\Delta \models \Delta$ für alle $\Delta \in I$.

Aufgabe 16. Für jede L-Struktur $\mathcal{A}$ definieren wir: $\mathrm{Th}(\mathcal{A}) = \{\phi | \phi \text{ eine Aussage und } \mathcal{A} \models \phi\}$. Sei C eine Klasse von L-Strukturen. Dann $\mathrm{Th}(\mathrm{C}) := \bigcap_{\mathcal{A}\in C} \mathrm{Th}(\mathcal{A})$. Zeigen Sie, dass für alle L-Strukturen $\mathcal{M}$:
$\mathcal{M} \models \mathrm{Th}(\mathrm{C})$ genau dann, wenn $\mathcal{M}$ elementar äquivalent zu einer Ultraprodukt der Elementen aus C ist.

Aufgabe 17. Zeigen Sie, dass C eine elementare Klasse ist genau dann, wenn es unter Ultraprodukten und elementarer Äquivalenz geschlossen ist. ($\mathcal{C}$ heißt eine elementare Klasse, wenn es eine T gibt, so dass $\mathrm{C} = \{\mathcal{M} | \mathcal{M} \models T\}$).

Aufgabe 18. Sei C eine Klasse von endliche L-Strukturen, so dass für alle $n \in \omega$, $\{\mathcal{A} \in \mathrm{C} : |A| = n\}$ endlich ist. Sei $\mathrm{Th}_a(\mathrm{C}) := \{\phi | \text{ nur endliche viele } \mathcal{A} \in \mathrm{C} \text{ erfüllen } \phi \text{ nicht}\}$. Zeigen Sie folgendes

$\mathcal{M}$ ist eine unendliche Modell von $\mathrm{Th}(\mathrm{C})$ genau dann, wenn $\mathcal{M} \models \mathrm{Th}_a(\mathrm{C})$.

1.5 Elimination of Quantifiers

Bemerkung. The exercises follow after a short note on quantifier elimination and criteria for checking whether or not a given theory admits elimination of quantifiers. My references are [2],[3],[4]. You can of course skip the note and begin with the exercises!

Insight: $(\mathbb{R}, +, \cdot, 0, 1, <) \models \forall a, b, c(\underbrace{\exists x ax^2 + bx + c = 0}_{\text{a formula with a quantifier}} \leftrightarrow \underbrace{[(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0))}_{\text{a formula without quantifiers}}])$

Definition. T eliminates quantifiers if for every ϕ there is a quantifier free ψ such that

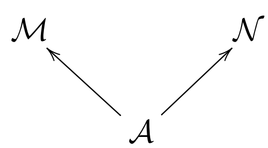
$$T \models \phi \leftrightarrow \psi.$$

We also say that T has/admits quantifier elimination, or it has qe.

Model theory is the study of definable sets. When T admits quantifier elimination, all definable sets can be obtained by Boolean combinations of solution-sets of equations. Quantifier elimination is an 'algebraic property' of a theory (or a structure).

Criteria

Criterion 1. $\phi(\bar{x})$ has a quantifier free equivalent (modulo T) if in all situations as in the diagram below, we have $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{a})$.



$$\mathcal{M}, \mathcal{N} \models T, \mathcal{A} \subseteq \mathcal{M}, \mathcal{N}, \bar{a} \in A. \qquad (1.3)$$

Criterion 2. T has qe if and only if it eliminates quantifiers from formulas of the form $\exists x \phi(x, \bar{y})$ where ϕ is quantifier free.
Combining 1 and 2: T has quantifier elimination if and only if in all situations as in diagram 1.3 we have $\mathcal{M} \models \exists x \phi(x, \bar{a}) \Leftrightarrow \mathcal{N} \models \exists x \phi(x, \bar{a})$.
To understand the next criterion better, we need more insight!
Insight. In every field F with characteristic zero we have a copy of $\mathbb{Z}$ because

$$(\underbrace{1 + 1 + \ldots}_{n \text{ times, any } n}) \in F.$$

If T is the theory of fields, then $\mathcal{Z} \models T_\forall$ ($\mathcal{Z} = (\mathbb{Z}, +, \cdot, 0, 1)$). $\mathbb{Z}$ is not a field, but it can be extended to $\mathbb{Q}$, which *is* a field and which embeds in all fields with characteristic zero.
Insight. Suppose that $F_1 \subseteq F_2$ are fields. It is important to know whether or not an equation with coefficient in F_1 solvable in F_2, has also a solution in F_1. For example $\mathbb{R} \subseteq \mathbb{C}$. In $\mathbb{C}$ the equation $x^2 + 1$ has a solution but in $\mathbb{R}$ it does not. For a theory with qunatifier elimination, this comes for free. For example, let $\mathcal{A} \subseteq \mathcal{M}$ be models of $\mathrm{Th}(\mathcal{R})$ (for $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$) and $\mathcal{A} \subseteq \mathcal{M}$. If $ax^2 + bx + c$ is a polynomial with coefficients in A that is solvable in M, then $b^2 - 4ac \geq 0$. So it also has a solution in A.

Definition. T has algebraically prime models if for any $\mathcal{A} \models T_\forall$, there is $\mathcal{M} \models T$ and an embedding $i : \mathcal{A} \to \mathcal{M}$ such that for all $\mathcal{N} \models T$ and all $j : \mathcal{A} \to \mathcal{N}$, there is an $f : \mathcal{M} \to \mathcal{N}$ to make the following diagram commute:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ i\ } & \mathcal{M} \\ & {}_{j}\searrow & \downarrow{\scriptstyle \exists f} \\ & & \mathcal{N} \end{array} \tag{1.4}$$

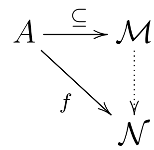
Criterion 3. Suppose that

1. T has algebraically prime models,
2. for every $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$, $\bar{a} \in M$ and quantifier free $\phi(x, \bar{y})$, we have $\mathcal{N} \models \exists x \phi(x, \bar{a}) \Leftrightarrow \mathcal{M} \models \exists x \phi(x, \bar{a})$.

Then T has quantifier elimination.
Criterion 4.(van den Dries) Suppose that T has at least one constant symbol. T has quantifier elimination if the two following algebraic conditions hold.

1. Every model $\mathcal{M}$ of $T_\forall$ has a T-closure $\overline{\mathcal{M}}$.
2. If $\mathcal{M} \subsetneqq \mathcal{N}$ are models of T, then **there is** a $b \in N - M$ such that $\mathcal{M}(b)$, the $T_\forall$-model generated by b over $\mathcal{M}$, can be embedded into an elementary extension of $\mathcal{M}$.

Criterion 5. T has quantifier elimination if whenever $\mathcal{M}, \mathcal{N} \models T$, $A \subseteq M$, $\mathcal{N}$ is $|M|^+$-saturated and $f : A \to \mathcal{N}$ is partial embedding, f extends to an embedding of $\mathcal{M}$ into $\mathcal{N}$.



(1.5)

Criterion 6. T has quantifier elimination if and only if T is model-complete and $T_\forall$ has amalgamation property.
Criterion 7. T has quantifier elimination if for every $M, N \models T$ and every $\bar{a} \in M$ and $\bar{b} \in N$,

$$\mathrm{tp}_{qf}^{M}(\bar{a}) = \mathrm{tp}_{qf}^{N}(\bar{b}) \text{ implies } \mathrm{tp}^{M}(\bar{a}) = \mathrm{tp}^{N}(\bar{b}). \tag{1.6}$$

Exercises

Bemerkung. The exercises of this week are from [2] and [3] (with slight changes).

Aufgabe 19. Two structures $\mathcal{A}$ and $\mathcal{B}$ are 'partially isomorphic' if there is a collection $(\mathcal{A}' \cong_{f'} \mathcal{B}' | \mathcal{A}' \subseteq \mathcal{A}, \mathcal{B}' \subseteq \mathcal{B}, f'$ an isomorphism) with the following properties.

1. For each $\mathcal{A}' \cong_f \mathcal{B}'$ in this collection and each $a \in A$ there is $\mathcal{A}^{''} \cong_{f''} \mathcal{B}^{''}$ in this collection such that $a \in A''$ and f'' extends f.

2. For each $\mathcal{A}' \cong_f \mathcal{B}'$ in this collection and each $b \in B$ there is $\mathcal{A}^{''} \cong_{f''} \mathcal{B}^{''}$ in this collection such that $b \in B''$ and f'' extends f.

Such a family is said to have the 'back and forth property' (or to be a back and forth system). Show that partially isomorphic structures are elementary equivalent.

Aufgabe 20. Let $\{E\}$ be a binary relation symbol. For each of the following theories, either prove that they have quantifier elimination or give an example showing that they do not have quantifier elimination and a natural extension $L' \supseteq L$ in which they do have quantifier elimination.

a) E is an equivalence relation with infinitely many classes of size 2.

b) E is an equivalence relation and it has infinitely many classes all of which are infinite.

c) E is an equivalence relation and it has infinitely many classes of size 2, infinitely many classes of size 3, and every class has size 2 or 3.

d) E is an equivalence relation and it has one class of size n for each $n < \omega$.

Aufgabe 21. Show that the theory of $(\mathbb{N}, <, 0, s)$ where $s(x) = x + 1$ has quantifier elimination and every definable subset of $\mathbb{N}$ is either finite or cofinite (=its complement is finite).

Aufgabe 22. Consider the theory of $(\mathbb{Z}, +, 0, 1)$ in the language where we add predicates p_n for the elements divisible by n. First axiomatise this theory and then prove that it has quantifier elimination. We call this the theory of $\mathbb{Z}$-groups.

Aufgabe 23. Show that in $(\mathbb{Z}, +, 0, 1)$ we cannot define the ordering (also discuss how we can, if we have a symbol for multiplication in the language; see the first Aufgabe, Blatt 1, after Definierbarkeit).

Hinweis. Remember that a definable set is preserved by automorphisms.

Aufgabe 24. Show that modulo the theory T of the structure $(\mathbb{Z}, +, 0, 1, <)$, the formula $\exists y 2y = x$ is not equivalent to a quantifier free formula (, or the set of even numbers is not defined by a quantifier free formula).

Aufgabe 25. We call $\mathcal{M} \models T$ existentially closed if for all quantifier free $\phi(\bar{x}, \bar{y})$, whenever $\mathcal{N} \models T$, $\mathcal{M} \subseteq \mathcal{N}$, $\bar{a} \in M$ and $\mathcal{N} \models \exists \bar{x} \phi(\bar{x}, \bar{a})$, we have $\mathcal{M} \models \exists \bar{x} \phi(\bar{x}, \bar{a})$.

a) Show that if T is $\forall\exists$-axiomatisable then is has existentially closed models. Indeed if $M \models T$ then there is $\mathcal{N} \supseteq \mathcal{M}$ existentially closed with $|N| = |M| + |L| + \aleph_0$.

b) Suppose that T has an infinite non-existentially closed model. Prove that T has non-existentially closed models of cardinality κ for any infinite $k \geq |L|$.

   Hinweis. Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are models of T and $\mathcal{N}$ satisfies an existential formula not satisfied in $\mathcal{M}$. Consider models of the theory of $\mathcal{N}$ where we add a unary predicate for M.

c) Show that if $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M}$ is existentially closed, then there is $\mathcal{M}_1 \models T$ such that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}_1$ with $\mathcal{M} \preceq \mathcal{M}_1$.

d) Show that T is model complete if and only if every model of T is existentially closed (we call T model complete if for all $\mathcal{M} \subseteq \mathcal{N}$, models of T, we have $\mathcal{M} \preceq \mathcal{N}$).

Hinweis. ($\Leftarrow$) Suppose that $\mathcal{M}_0 \subseteq \mathcal{N}_0$ are models of T. Use c) to build $\mathcal{M}_0 \subseteq \mathcal{N}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_2 \ldots$, a chain of models of T such that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ and $\mathcal{N}_i \preceq \mathcal{N}_{i+1}$.

1.6 Model Companion, Positive Quantifier Elimination

Aufgabe 26. Suppose that T and T' are L-theories. We say that T' is a **model companion** of T if

1. T' is model-complete (as defined in Blatt 3, Aufgabe 5d),
2. every model of T has an extension that is a model of T', and
3. every model of T' has an extension that is a model of T (2 and 3 together mean: $T_\forall = T'_\forall$).

a) Show that each theory has at most one model companion.

b) Show that DLO (dense linear order without endpoints) is the model companion of the theory of discrete linear orders.

c) Suppose that T is $\forall\exists$ axiomatisable. Show that if T' is a model companion of T, then T' is the theory of existentially closed models of T.

Definition. We say that an L-formula $\phi(\bar{x})$ is **positive** if it is in the smallest collection of L-formulas containing the atomic formulas and closed under $\wedge$, $\vee$, $\exists$ and $\forall$.

Definition. We say that $\eta : \mathcal{M} \to \mathcal{N}$ is an L-homomorphism if:

1. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constants;
2. $\eta(f^{\mathcal{M}}(\bar{x})) = f^{\mathcal{N}}(\eta(\bar{x}))$ for all $\bar{x} \in M$ and function symbols f;
3. if $\bar{x} \in R^{\mathcal{M}}$, then $\eta(\bar{x}) \in R^{\mathcal{N}}$ for all $\bar{x} \in M$ and relation symbols R.

Aufgabe 27. Let T be a complete L-theory and $\phi(\bar{x})$ be an L-formula such that $T \models \exists \bar{x} \phi(\bar{x})$. Show that the following are equivalent:

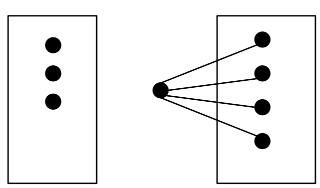
1. There is a positive quantifier-free formula $\psi(\bar{x})$ such that $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

2. For all $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{A} \subseteq \mathcal{M}$, if $f : \mathcal{A} \to \mathcal{N}$ is an L-homomorphism, $\bar{a} \in A$ and $\mathcal{M} \models \phi(\bar{a})$, then $\mathcal{N} \models \phi(f(\bar{a}))$.

Hinweis. For 2$\to$ 1, put $\Gamma = \{\psi(\bar{x}) : \psi$ is positive quantifie free and $T \models \psi(\bar{x}) \to \phi(\bar{x})\}$. Let $\Sigma = T \cup \{\neg\psi(\bar{c}) : \psi \in \Gamma\} \cup \{\phi(\bar{c})\}$. Show that Σ is unsatisfiable.

Definition. A random graph is a graph in which, given any sets $X = \{x_0, \ldots, x_m\}$ and $Y = \{y_0, \ldots, y_n\}$ of vertices with $X \cap Y = \emptyset$, there is a vertex z (with $z \notin Y$) such that there is an edge between z and all elements of X and there no edge between z and any element of Y. So the theory of random graphs is the union of the theory of the graphs with the following axiom scheme:

$$\forall x_1 \ldots x_m \forall y_1, \ldots y_n [\bigwedge_{i,j} \neg x_i = y_j \to$$
$$\exists z (\bigwedge_{i=1,\ldots,m} zRx_i) \wedge (\bigwedge_{j=1,\ldots,n} \neg zRy_j) \wedge \bigwedge_{j=1,\ldots,n} \neg(z = y_j)].$$



Aufgabe 28. 1. Show that the theory of random graphs has quantifier elimination and is complete;

2. show that it is the model companion of the theory of graphs.

Aufgabe 29. Let K be an algebraically closed field and $D \subseteq K^n$ be definable. Show that every injective polynomial map from D to D is surjective.

1.7 Real and Algebraically Closed Fields

Definition. Let $M \models T$ and A be a subset of M. By $\mathrm{acl}(A)$, algebraic closure of A in M, we mean the set of all y's in M for which there is a formula $\phi(x, \bar{a})$ with parameters $\bar{a}$ in A such that $\phi(M, \bar{a}) = \{y \in M | M \models \phi(y, \bar{a})\}$ is finite and $y \in \phi(M, \bar{a})$.
One can think of $\phi(x, \bar{a})$ as a 'polynomial' with coefficients in A and of y as a root of it. By $\mathrm{dcl}(A)$, definable closure of A in M, we mean the set of y's such that there is an $\bar{a} \in A$ such that y satisfies a formula $\phi(y, \bar{a})$ and y is the only element to satisfy this formula.

Aufgabe 30. Show that

1. in a model of DAG (the theory of torsion-free divisible abelian groups), algebraic closure and definable closure agree (= are the same thing!) and $\mathrm{acl}(A)$ is the $\mathbb{Q}$-vector space span of A.

2. Let $K \models$ ACF (the theory of algebraically closed fields) and A be a subset of K. Show that $a \in \mathrm{acl}(A)$ if and only if a is algebraic over the subfield of K generated by A. This means that the model theoretic 'algebraic closure' and the algebraic closure in the sense of Algebra coincide for models of ACF.

3. Let $R \models$ RCF (the theory of real closed fields) and A be a subset of R. Show that $\mathrm{acl}(A) = \mathrm{dcl}(A)$ and $\mathrm{acl}(A)$ is, similar to the previous item, the algebraic closure of the field generated by A in R.

Aufgabe 31. Show that the order on $\mathbb{R}$ is not quantifier-free definable in the language of rings.

Hinweis. Let c_1, c_2 be two algebraically independent elements over $\mathbb{R}$. First show that $\mathbb{R}(c_1, c_2)$, the field generated over $\mathbb{R}$ by c_1 and c_2, is formally real (that means -1 is not a sum of squares). Then note that if F is formally real and $a \in F$ is such that $-a$ is not a sum of squares, then there is an order $<$ on F such that $a > 0$. So there are two orders $<_1$ and $<_2$ on $\mathbb{R}(c_1, c_2)$ both extending the order of $\mathbb{R}$ such that $c_1 <_1 c_2$ and $c_2 <_2 c_1$. Now explain how this means that the order on $\mathbb{R}$ is not quantifier-free definable in the language of rings.

Aufgabe 32. (Real version of Nullstellensatz). Let F be a real closed field and I an ideal in $F[\bar{X}]$. Show that then, $v_F(I)$ is non-empty if and only if whenever $p_1, \ldots, p_m \in F[\bar{X}]$ and $\sum p_i^2 \in I$, then all p_i's are in I. By $v_F(I)$ we mean $\{\bar{a} | \bar{a} \in F$ and for all $f \in I \quad f(\bar{a}) = 0\}$.

Definition. We call an ordered structure $(M, <, \ldots)$, o-minimal (order minimal) if every definable subset of M can be defined using only $<$ and $=$; that is every definable subset of M is a finite union of points and intervals in M.

Aufgabe 33.

1. Show that $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$ is o-minimal.
2. Show that every model of $\mathrm{Th}(\mathcal{R})$ is o-minimal.
3. Show that whenever $(F, +, 0, \cdot, <)$ is an o-minimal field, F is real closed (note that a field is real closed if and only if it satisfies the intermediate value property).
4. Suppose that $M = (G, +, <, \ldots)$ is o-minimal and $(G, +, <)$ is an ordered group. Show that G is abelian.
5. In above show that G is also divisible.

Definition. If T' is a model companion (see Aufgabe 1 Blatt 4) of T and $T' \cup \mathrm{Diag}(M)$ is complete for any $M \models T$, then T' is a **model completion** of T. ($\mathrm{Diag}(M)$ is the set of quantifier-free formulas in the language $L(M)$ that hold in M.)

Definition. We say that T has **amalgamation property** if whenever M_0, M_1 and M_2 are models of T and $f_i : M_0 \to M_i$ are embeddings, there is $N \models T$ and $g_i : M_i \to N$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Aufgabe 34 (continued from Aufgabe 1 on Blatt 4)**.**

1. Suppose that T' is a model companion of T. Show that T' is a model completion of T if and only if T has the amalgamation property.
2. Suppose that T has a universal axiomaisation and T' is a model completion of T. Show that T' has quantifier elimination.

1.8 Complementary exercises

This sheet is intended for those who have interest in more involved algebraic exercises. While there is no need to hand in solutions, we will consider bonus points for those who do!

Aufgabe 35. Suppose that K is an algebraically closed field and $P \subseteq K[X_1, \ldots, X_n]$ is a maximal ideal. Show that P is generated by $X_1 - a_1, \ldots, X_n - a_n$ for some $a_1, \ldots, a_n \in K$.

Aufgabe 36 (completeness of projective varaeties)**.** Let K be a model of ACF. Suppose that $p_1, \ldots, p_k \in \mathbb{Z}[\bar{Y}, \bar{X}]$ are homogenious in $\bar{X}$ (i.e. $p_i(\bar{Y}, t\bar{X}) = t^d p_i(\bar{Y}, \bar{X})$ for some d). Let $\phi(\bar{y})$ be the formula that says that the system of equations $p_1(\bar{x}, \bar{y}) = \ldots = p_k(\bar{x}, \bar{y}) = 0$ has a nontrivial solution.

1. $\phi(\bar{y})$ is equivalent to a positive quantifier free-formula.

2. Let $\mathbb{P}^l$ be the projective l-space over K, and let $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ be the natural projecion map. Show that π is a closed map in the Zariski topology.

Aufgabe 37. Let $K \subseteq L$ be algebraically closed fields. Let $V, W \subseteq L^n$ be Zariski closed sets defined over K. Suppose that there is $f : V \to W$ a bijective polynomial map defined over L. Show that there is $g : V \cap K^n \to W \cap K^n$ a bijective polynomial map defined over K.

1.9 Types

Aufgabe 38 (quantifier elimination and types)**.** Show that a theory has quantifier elimination if and only if every type p is implied by the quantifier free formulas in p. Let us also express the 'if' condition this way: for every $M, N \models T$ and $\bar{a} \in M$ and $\bar{b} \in N$

$$tp^M_{qf}(\bar{a}) = \mathrm{tp}^N_{qf}(\bar{b}) \text{ implies } \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$$

where $\mathrm{tp}^M_{qf}(\bar{a})$ is the the class of quantifier free formulas satisfied by $\bar{a}$ in M.

Aufgabe 39 (describing types in RCF)**.**

1. Describe 1-types in models of RCF: let R be a real closed field. Show that 1-types over R (=types in $S^1(R)$) correspond to cuts in the ordering $(R, <)$. (This means, supposing that R is a model of RCF and $R \subseteq A \models$RCF and A is $|R|^+$-saturated and $x \in A - R$, then $\mathrm{tp}^A(x/R)$ is determined by the cut of x in R; that is if x and y in A are such that for all a_1, a_2 in R, $a_1 < x < a_2$ if and only if $a_1 < y < a_2$ then

$\mathrm{tp}^A(x/R) = \mathrm{tp}^A(y/R)$. By A being $|R|^+$-saturated we mean that all types in $S^1(R)$ are indeed types of elements in A: $p \in S(R)$ then $p = \mathrm{tp}^A(x/R)$ for some $x \in A$).

2. Show that RCF has no countable saturated models: T has a countable saturated model if and only if $|S_n(T)| \leq \aleph_0$ for all n. You need to characterise types in $S_1(\mathrm{RCF})$ and show that $|S_1(\mathrm{RCF})| = 2^{\aleph_0}$.

   (In case it is not yet covered in the lecture, a model M of T is called saturated if every consistent set of formulas in variables $\bar{x}$ is realised in M by some $\bar{a}$).

Aufgabe 40.

1. Show that a type p in $S_n(T)$ is isolated if $\{p\} = [\phi]$ for some ϕ; this means that p is an *isolated point* in the Stone topology.

2. Suppose that M is an L-structure, A is a subset of M and $b \in M$ is algebraic over A (= it is in $\mathrm{acl}^M(A)$, see Blatt 5 for the definition). Show that $\mathrm{tp}^M(b/A)$ is isolated.

Aufgabe 41.

1. In L the language of DLO, prove that if $a, b \in \mathbb{Q}$, then $\mathrm{tp}^{\mathbb{Q}}(a/\mathbb{N}) = \mathrm{tp}^{\mathbb{Q}}(b/\mathbb{N})$ if and only if there is an automorphism σ of $\mathbb{Q}$ that fixes $\mathbb{N}$ pointwise and sends a to b.

2. Let $A = \{1 - \frac{1}{n} : n = 1, \ldots\} \cup \{2 + \frac{1}{n} | n = 1, \ldots\}$. Show that 1 and 2 realise the same types over A, but there is no automorphism of $\mathbb{Q}$ fixing A pointwise sending 1 to 2.

3. Is the previous item contradictory with what you expect from elements of the same type? Perhaps, you can formulate a theorem that relates 'realising the same type' to 'one being sent to the other by an automorphism'.

Aufgabe 42 (describing types in ACF)**.** Suppose that $K \models \mathrm{ACF}$ and $k \subseteq K$ is a field. Show that n-types over k are determined by prime ideals in $k[X_1, \ldots, X_n]$: for every type p find a prime ideal I_p such that the map $p \mapsto I_p$ is a bijection between $S_n^K(k)$ and $\mathrm{Spec}\, k[X_1, \ldots, X_n] = \{\text{prime ideals of } k[X_1, \ldots, X_n]\}$. (if interested, prove that this map is continuous.)

Hinweis (Note that this hint spoils the exercise!)**.** To avoid confusion, let me first mention that the letter p is for a type and the letter P is for a prime ideal.
First, prove that given a type p in $S_n^K(k)$, the set $I_p = \{f|\text{the formula } f(\bar{X}) = 0 \text{ is in } p\}$ is a prime ideal of $K[\bar{X}]$.
For converse, Consider the prime ideal P in $k[\bar{X}]$. It is of the form

$$Q \cap k[\bar{X}]$$

for a prime ideal Q in $K[\bar{X}]$ (we take this for granted, but if you are interested in proving it, one of the main ingredients you may need is noetherianity of $K[\bar{X}]$).
Now, since Q is prime, $K[\bar{X}]/Q$ is an integral domain. Let F_1 be the fraction field of $K[\bar{X}]/Q$. I remind you quickly that D is called an integral domain if $ab \neq 0$ for all non-zero a and b. If D is an integral domain, then $F := \{\frac{a}{b}|a, b \in D\}$ with addition and multiplication of fractions is a field and is called the filed of fractions of D.
Let F_2 be the algebraic closure of F_1; luckily you have proved in Blatt 5 that 'algebraic closure' is the same thing in both model theoretic and algebraic senses.
Since ACF is model-complete, F_2 is an elementary extension of K:

$$k \subseteq K \preceq F_2 = \mathrm{acl}(\mathrm{Frac}(K[X_1, \dots X_n]/Q)).$$

Consider elements $X_1/Q, \dots, X_n/Q$ in F_2. Let p be the type of the tuple $(X_1/Q, \dots, X_n/Q)$ over k; that is

$$p := \mathrm{tp}^{F_2}((X_1/Q, \dots, X_n/Q)/k).$$

Show that $I_p = P$.
Now, using the information above and quantifier elimination of ACF, show that $p \mapsto I_p$ is a bijection.

1.10 $\aleph_0$-categoricity and ω-saturatedness

Aufgabe 43. Show that T is $\aleph_0$-categorical if and only if $S_n(T)$ is finite for each n.

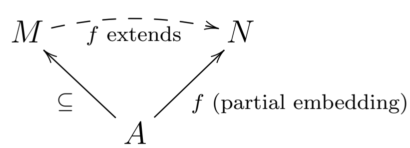
Aufgabe 44. Suppose that M is countable and it is a model of an $\aleph_0$-categorical theory. Show that if $X \subseteq M^n$ is invariant under all automorphisms of M, then X is definable (compare with Blatt 1, Definierbarkeit).

Aufgabe. Aufgabe 2 in above can be generalised: Let M be saturated and A be a subset of M with $|A| < |M|$. Let $X \subseteq M^n$ be definable with parameters in M. Then X is A-definable if and only if every automorphism of M that fixes A pointwise, fixes X setwise (the only if part of the statement does not require that M is saturated).

Aufgabe 45. Axiomatise a theory with exactly two countable models (also remember of Vaught's theorem that there is no countable complete theory with exactly two countable models).

Aufgabe 46. Suppose that M is ω-saturated. Show that N is partially isomorphic to M if and only if N is ω-saturated and elementarily equivalent to M (see Aufgabe 1 Blatt 3).

Aufgabe (a test for quantifier elimination)**.** Suppose that L is a language with at least one constant symbol and T is an L-theory. T has quantifier elimination if and only if whenever $M, N \models T$ and A is a subset of M and $f : A \to N$ is a partial embedding, f extends to an embedding of M into N.



$M, N \models T$

N saturated

A subset of M

$f : A \to N$ partial embedding

1.11 Fraïssé's Construction

Aufgabe 47 (from [1])**.** Let p be a prime number and let $\mathbf{K}$ be the class of all finite fields of characteristic p. Show that $\mathbf{K}$ has heredity property, joint embedding property and amalgamation property, and the Fraïssé's limit of

K is the algebraic closure of the prime field of characteristic p. (We need this observation that finite integral domain are fields because in Fraïssé's limit we talk about finitely generated **structures** not models.)

General preliminaries from Algebra

You may need some algebra of fields with finite characteristic. Let F be a field. Then $\operatorname{Char} F$ is the smallest n such that $n.1 = 0$. $\operatorname{Char} F$ is either a prime number, or it does not exist in which case we say $\operatorname{Char} F = 0$. Every field F with $\operatorname{Char} F = 0$ contains a copy of $\mathbb{Q}$ and every field F with $\operatorname{Char} F = p$ contains a copy of $\mathbb{Z}_p$.
If the filed F is finite, then $\operatorname{Char} F = p$ for some prime p. Also since F contains $\mathbb{Z}_p$, it is a vector space over $\mathbb{Z}_p$ and hence **as a vector space**,

$$F \cong \overbrace{\mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p}^{n \text{ times}}$$

for some n, that is the size of a finite field is always p^n for some p and n. Note that I haven't claimed that $\mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$ is a field! More interestingly, a finite field with p^n elements is the smallest field containing $\mathbb{Z}_p$ that includes all solutions of the equation

$$x^{p^n} = x$$

where $x^{p^n} - x \in \mathbb{Z}_p[X]$. This is also called the **splitting field** of the polynomial $x^{p^n} - x = 0$ over $\mathbb{Z}_p$.

Aufgabe 48 (from [1]). Let **K** be the class of finitely generated torsion-free abelian groups. Show that **K** has heredity property, joint embedding property and amalgamation property, and that the Fraïssé's limit of K is the direct sum of countably many copies of the additive group of rationals. Also discuss why countably many copies of additive group of integers is not the limit (it has to do with saturatedness).

Aufgabe 49. Let **K** be the class of all finite graphs. Show that the Fraïssé's limit of **K** is the countable random graph. Note that proving this, you will have also shown that the theory of random graphs has quantifier elimination.

Aufgabe 50.

1. Let **K** be the skeleton of M and M be **K**-saturated and countable. Show that M is ultrahomogeneous, meaning that each automorphism between finitely generated substructures of M extends to an automorphism of M.

2. Show that any two **K**-saturated structures are partially isomorphic (=there is a set of isomorphism between their substructures with the back and forth property).

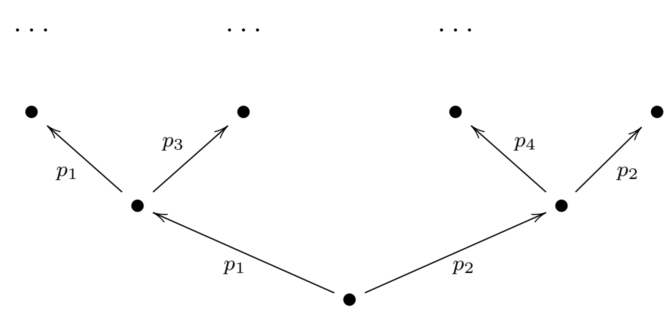
Aufgabe 51 (back to types and $\aleph_0$-categoricity!)**.**

1. Suppose that T is $\aleph_0$-categorical and $M \models T$ and A is a finite subset of M. Show that $\mathrm{acl}(A)$ is finite.

2. Show that the theory of $(\mathbb{R}, 0, +)$ has exactly two 1-types and $\aleph_0$-many 2-types.

1.12 Prime Models and Indiscernible Sequences

Aufgabe 52 (number of types and binary trees)**.** Suppose that T is a countable theory in which there is no binary tree of consistent formulae. Show that for each n, $|S_n(T)|$ is at most countable (the converse also holds and is a theorem in the script: if T is such that for each n, $|S_n(T)|$ is at most countable, then there is no binary tree of consistent formulae in T).



Aufgabe 53. Show that for any infinite L-structure M, we can find

$$N_0 \succ N_1 \succ N_2 \succ N_3 \succ \ldots,$$

a descending *elementary* chain of *elementary* extensions of M, such that $M = \bigcap_{i\in\mathbb{N}} N_i$.

Hinweis. Clues: L(Skolem), an indiscernible sequence $(a_i)_{i\in\mathbb{N}}$, N_0 being obtained from M and $(a_i)_{i\in\mathbb{N}}$.

Aufgabe 54.

1. Show that ACF (the theory of algebraically closed fields) has a prime model.
2. Show that RCF (the theory of real closed fields) has a prime model.
3. Show that Th($\mathbb{N}$) in the language $L = \{+, \cdot, <, 0, 1\}$ has a prime model.
4. Let T be the theory of $(\mathbb{R}, <, Q)$ where Q is a predicate for rational numbers. Does T have a prime model?

Aufgabe 55. Let (G, R) be an infinite graph. Use Ramsey's theorem to show that either G has an infinite complete subgraph (a subgraph in which there is an edge between any two vertices) or it has an infinite null subgraph (=there are infinitely many vertices in G with no edges in between).

Aufgabe 56. Show that if M is κ-saturated, then there is $I \subseteq M$, a sequence of order indiscernibles with $|I| = \kappa$.

Aufgabe 57. Suppose that $K \models$ACF and K has infinite transcendence degree. Let $I = \{a_1, a_2, \ldots, \}$ be an infinite algebraically independent set (its elements are algebraically independent over $\mathbb{Q}$). Show that I is an infinite set of indiscernibles in K.

Aufgabe 58. Show that there is no $\aleph_0$-categorical theory of fields. That is if T is a complete theory in the language of rings that contains the theory of fields, then T is not $\aleph_0$-categorical.

Hinweis. We have proved that if T is $\aleph_0$-categorical then the algebraic closure of a finite set is finite (Blatt 9 Auf 5)

1.13 Stability, Categoricity, Saturatedness

The first exercise of this week is a set theory exercise. I suggest you make yourself familiar with the statement and the proof of the following two.

Aufgabe 59.

1. Suppose that κ is an infinite cardinal. Show that $\kappa \cdot \kappa = \kappa$, where $\cdot$ denotes the multiplication of cardinals.

   Hinweis. First a quick reminder that $k \cdot k$ is by definition $|\kappa \times \kappa|$. The proof is by transfinite induction on κ. Assume that it holds for smaller cardinals. So if $\alpha < \kappa$ then $\alpha.\alpha = \alpha < \kappa$. Also it easy to see that $\kappa \leq \kappa.\kappa$. We need only to show that $\kappa \cdot \kappa \leq \kappa$. For this we need to define a well-ordering $\lhd$ on $\kappa \times \kappa$ in such a way that all initial segments of $\kappa \times \kappa$ with this well-ordering have size $\leq \kappa$. Define the following well-ordering $\lhd$ on $\kappa \times \kappa$:

   $$\langle \alpha, \beta \rangle \lhd \langle \alpha', \beta' \rangle \text{ if } \begin{cases} \max\{\alpha, \beta\} < \max\{\alpha', \beta'\} \text{ or} \\ \max\{\alpha, \beta\} = \max\{\alpha', \beta'\} \text{ and } \langle \alpha, \beta \rangle <_{\text{lex}} \langle \alpha', \beta' \rangle \end{cases}$$

   where $<_{\text{lex}}$ denotes the lexicographic order (with priority to the second coordinate). Now with the help of Figure 1 show that each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has no more than κ predecessors with the oredering $\lhd$. In Figure 1 the predecessors of an $\langle \alpha, \beta \rangle$ with $\beta < \alpha$ appear in grey. You may also think of an onion!

2. Show that for each infinite cardinal κ, there is a dense linear order $(A, <)$ and a $B \subset A$ such that B is dense in A and $|B| \leq \kappa < |A|$.

   Hinweis. Let $\lambda \leq \kappa$ be least such that $2^\lambda > \kappa$. Set

   $$A : \{\text{all functions from } \lambda \text{ to } \mathbb{Q}\}.$$

   Define the following order on A:

   $$f < g \text{ if } f(\alpha) < g(\alpha); \text{ where } \alpha \text{ is the least such that } f(\alpha) \neq g(\alpha).$$

   Let B be the set of sequences in A that are eventually zero. Show that B is the B!

Aufgabe 60. Let $L = \{E\}$ be the language with a single binary relation symbol. let T be the theory of an equivalence relation where for each $n \in \omega$ there is a unique equivalence class of size n. Show that T is ω-stable and not $\aleph_0$-categorical and not $\aleph_1$-categorical.

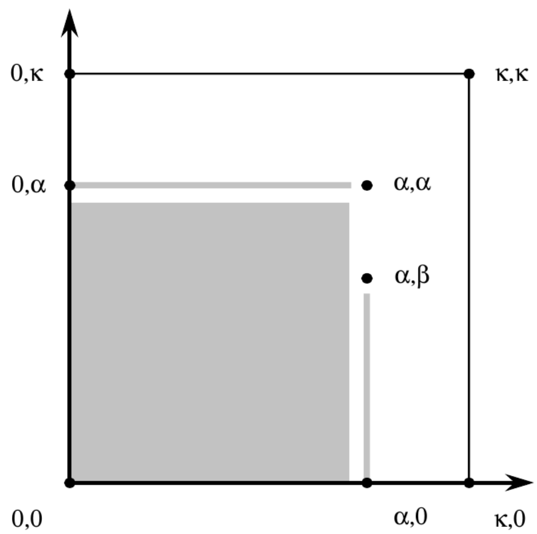


Figure 1.1: predecessors of $\langle \alpha, \beta \rangle$ where $\beta < \alpha$. The figure comes from `http://www.cl.cam.ac.uk/~lp15/papers/Sets/AC.pdf`

Bemerkung. Compare the above exercise with the following two facts:

1. The Categoricity Theorem says that if T is categorical in some **uncountable** cardinal, then it is κ-categorical in all uncountable κ's.
2. A countable theory that is categorical in some uncountable cardinal, is ω-stable.

Aufgabe 61. 1. Show that DLO is **not** κ-stable for any infinite κ.

2. (Generalisation of the previous item) we say that a theory T has the **order property** if there is a formula $\phi(\bar{x}, \bar{y})$ with $|x| = |y| = n$ and $M \models T$ and $(\bar{c}_i)_{i \in \omega}$ an infinite sequence in M^n such that

$$M \models \phi(\bar{c}_i, \bar{c}_j) \text{ if and only if } i < j.$$

Use part 2 of Aufgabe 1 to show that if T has the order property then it is **not** κ-stable for any infinite κ.

Hinweis (for part 2)**.** Use A and B in Aufgabe 1 part 2 as sets of indices of a suitable sequence (c_i) in such a way that $S_n(\{x_b | b \in B\}) > |B|$.

Aufgabe 62. If T is κ-stable, then (up to logical equivalences) $|T| \leq \kappa$.

Aufgabe 63.

1. If M is κ-saturated, then each definable subset of M is either finite or of cardinality at least κ.

2. Suppose that $|L| \leq \aleph_0$. Let $M_1, M_2, \ldots$ be a sequence of L-structures. Let F be a non-principle ultrafilter on ω. Show that $\prod_{i<\omega} M_i/F$ is $\aleph_1$-saturated. If we assume the Continuum Hypothesis, this implies that if M and N are countable L-structures and $M \equiv N$, then the $M^\omega/F \cong N^\omega/F$ where by M^ω/F we mean the ultrapower of M.

Ich wünsche Ihnen ein frohes neues Jahr!

1.14 Vaughtian Pairs, Prime Extensions, Indiscernibles

Aufgabe 64. Show that a sequence of elements in $(\mathbb{Q}, <)$ is indiscernible if and only if it is either constant, strictly increasing or strictly decreasing.

Aufgabe 65. Show that for a countable T the following are equivalent (show only $1 \to 2 \to 3$):

1. every parameter set has a prime extension;

2. the isolated types over any countable parameter set are dense;

3. the isolated types over any parameter set are dense.

Hinweis (Hinweise $1 \to 2$)**.** Let A be a countable parameter set and M its prime extension and ϕ a formula with parameters in A. We want to show that $[\phi]$ (open set in the space of types) contains an isolated type. In other words we want ϕ to belong to an isolated type. There is an element $a \in M$ such that $M \models \phi(a)$. Show that $\mathrm{tp}(a/A)$ is isolated (use the omitting type theorem).

Aufgabe 66. Solve only one item below (they are both solved with the same idea).

1. Suppose that T is countable and complete and with infinite models. Suppose that $M \models T$ and $\phi \in L(M)$ and $\phi(M)$ is infinite with smaller cardinality than the cardinality of M. Show that there is an elementary substructure N of M ($N \prec M$) such that (M, N) is Vaughtian pair for ϕ and the cardinality of N equals to the cardinality of $\phi(M)$ (the converse also holds and is a theorem in the script: if T has a Vaughtian pair, then it has a model M with cardinality $\aleph_1$ and there is a $\phi \in L(M)$ such that $\phi(M)$ is countable).

2. If T (as above) has a Vaughtian pair, then it has a Vaughtian pair (M, N) in which M is countable.

Aufgabe 67. Solve only two items.

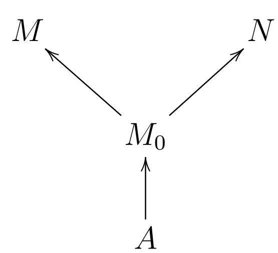
1. Show that the theory of the random graph has a Vaughtian pair.

2. Let $L = \{E\}$ be the language with a single binary relation symbol. Let T be the theory of an equivalence relation where for each $n \in \omega$ there is a unique equivalence class of size n. Exhibit a Vaughthian pair of models of T (remember that in Blatt 11 you have proved that T is ω-stable and not $\aleph_1$-categorical).

3. Show that there is no Vaughtian pair of real closed fields.

1.15 Strong Minimality

Aufgabe 68. Consider the following diagram:



$M_0 \prec M, M_0 \prec N$
A subset of M_0

1. let $\bar{a}$ be a tuple in A and $\phi(x, \bar{a})$ a formula. Then show that the fact that

$$\phi(x, \bar{a}) \text{ defines a strongly minimal set in } M$$

is an elementary property of $\bar{a}$ contained in the $\mathrm{tp}^M(\bar{a})$. It means that in the above diagram if $\phi(x, \bar{a})$ defines a strongly minimal set in M then it defines a strongly minimal set in N too.

2. Suppose that $a_1, \ldots, a_n$ in $\phi(M)$ are independent over A and $b_1, \ldots b_n \in \phi(N)$ are independent over A. Then show that $\mathrm{tp}^M(\bar{a}/A) = \mathrm{tp}^N(\bar{b}/A)$.

3. Let $B \subseteq \phi(M)$ be infinite and independent over A. Show that B is a set of indiscernibles over A (note that being a set of indiscernibles is a stronger property than being a sequence of indiscernibles).

4. Let $C \subseteq \phi(N)$ be infinite and independent over A. Show that C is a set of indiscernibles over A with the same type as type of B.

Aufgabe 69. 1. Let T be ω-stable. Show that if $M \models T$ then there is a minimal formula in M.

2. If $M \models T$ is $\aleph_0$-saturated and $\phi(\bar{x}, \bar{a})$ is minimal in M, then it is strongly minimal.

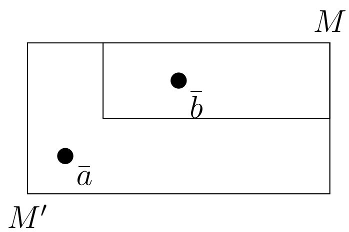
Aufgabe 70. 1. Show that the theory of K-vector spaces is κ-categorical for all $\kappa > |K|$.

2. Is ACF_p $\aleph_0$-categorical?

1.16 elimination of the quantifier 'there are infinitely many'

Aufgabe 71. Assume that T eliminates $\exists^\infty$. Prove the statement below: for every formula $\phi(x_1, \ldots, x_n, \bar{y})$ there is a formula $\theta(\bar{y})$ such that for all $M \models T$ and $\bar{b} \in M$, we have $M \models \theta(\bar{b})$ if and only there is an $M' \succ M$ and elements $a_1, \ldots, a_n \in M' - M$ such that $M' \models \phi(a_1, \ldots, a_n, b)$.

$M \models \theta(b)$ if and only if $M' \models \phi(\bar{a}, \bar{b})$
for some $M' \succ M$
$\bar{a} \notin M$ ($a_i \notin M$ for each i)

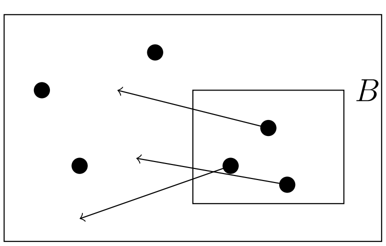


Aufgabe 72. Suppose that T_1 and T_2 are model complete theories in disjoint languages L_1 and L_2. Suppose that both T_1 and T_2 eliminate $\exists^\infty$. Show that then $T_1 \cup T_2$ has a model companion.

Bemerkung. T is model complete if whenever $M, N \models T$ then $M \subseteq N$ implies $M \preceq N$; and T^* is called a model-companion of T if T^* is model-complete and $T_\forall = T^*_\forall$

Hinweis. Use Aufgabe 71.

Aufgabe 73 (P.M. Neuman)**.** Let B be a subset of M and $(c_0, \ldots, c_n)$ be a sequence of elements all non-algebraic over $\emptyset$. Show that If M is $|B|^+$-saturated, then $\mathrm{tp}(c_0, \ldots, c_n)$ has a realisation in M disjoint from B (in other words there are $b_0 \ldots b_n \equiv c_0 \ldots c_n$ with $b_i \notin B$ for each i).



Hinweis. Let us do the proof by induction on n. For $n = 0$, use the fact that M is $|B|^+$ saturated to find an element not in B that realises $\mathrm{tp}(c_0)$.

Let us assume the statement true for $< n$. Consider $\text{tp}(c_0, \ldots, c_n)$. We have two cases:

case 1: one (or more than one) of $c_0, \ldots, c_{n-1}$ is in the algebraic closure of c_n. Let's say c_1 is in the algebraic closure of c_n. Again using the fact that M is saturated prove that there are $a_0, \ldots, a_{n-1}, a_n$ such that $a_0, \ldots, a_n \models \text{tp}(c_0, \ldots, c_n)$ and $a_0, \ldots, a_{n-1} \notin \text{acl}(B)$. Now it is clear that also $a_n \notin B$ because otherwise since c_1 is in $\text{acl}(c_n)$, we have $a_1 \in \text{acl}(a_n) \subseteq \text{acl}(B)$, contradiction with the choice of a_i's.

case 2: non of $c_1, \ldots, c_n$ are in $\text{acl}(c_n)$. In this case first find a_n not in B realising $\text{tp}(c_n/c_0, \ldots, c_{n-1})$ and then find $a_0, \ldots, a_{n-1}$ not in B realising $\text{tp}(c_0, \ldots, c_{n-1}/a_n)$ and then prove that $\text{tp}(a_0, \ldots, a_n) = \text{tp}(c_1, \ldots, c_n)$

Aufgabe 74. Suppose that M is $|A|^+$-saturated for A a subset of M. Show that then $p \in S(A)$ is algebraic if and only if $p(M)$ is finite.

Aufgabe 75. Let B be a subset of an L-structure A. Show that the **restriction map** $S_{m+n}(B) \to S_n(B)$ is open, continuous, and surjecctive. Let a be an n-tuple in A. Show that the fibre over $\text{tp}(a/B)$ is canonically homeomorphic to $S_m(aB)$ (note that aB means $\{a\} \cup B$).

Bemerkung. An open map is one that maps open sets to open sets, that is if X and Y are topological spaces, then $f : X \to Y$ is open if for each open set O, $f(O)$ is open. Note that O is open with the topology of X and $f(O)$ with the topology of Y. Also, f is continuous if for every open subset O of Y, $f^{-1}(O)$ is open in X. A homeomorphism between two topological spaces is a map f which is continuous and has a continuous inverse. In our case basis open sets are $[\phi]$'s (=types containing ϕ).

1.17 Blatt 16

Aufgabe 76. Show that the following theories are strongly minimal and in each case determine the closure of a given set X and make sense of the concepts of independence and bases:

1. The theory of ACF_0
2. The theory of K-vector spaces for a field K.

Aufgabe 77.

Aufgabe 78 (Robinson's Joint Consistency Lemma)**.** Extends the complete L-theory T to an L_1-theory T_1 and L_2-theory T_2 with $L_1 \cap L_2 = L$. Show that if T_1 and T_2 are both consistent, then so is $T_1 \cup T_2$.

Aufgabe 79. If M is κ-saturated then over every set of cardinality smaller than k every type in κ many variables is realised in M.

Aufgabe 80. Show that $\mathrm{acl}(A)$ is the intersection of all models containing A.

1.18 Morley Rank

Aufgabe 81.

1. It is mentioned in the script that 'the Morley rank of a formula $\phi(x, a)$ depends on $\phi(x, y)$ and the type of a'. Explain this. That is, show that if $\mathrm{tp}(a) = \mathrm{tp}(b)$ and $\phi(x, y)$ is a formula then $\mathrm{RM}\,\phi(x, a) = \mathrm{RM}\,\phi(x, b)$.

> Using the item above we can give an elementary definition for Morley rank of a formula in a structure M. That is given a structure M one can define $\mathrm{RM}^M\,\phi$ similarly, and then show that if $M \preceq N$ then $\mathrm{RM}^M\,\phi = \mathrm{RM}^N\,\phi$.

2. Show that if ψ implies ϕ then $\mathrm{RM}(\psi) \leq \mathrm{RM}(\phi)$.

Aufgabe 82 (examples of Morley rank)**.**

1. Let T be the theory of vector spaces over a field K.
   - (a) What is the Morley rank of a definable subset X of $\mathfrak{C}$?
   - (b) What is the Morley rank of a definable subset X of $\mathfrak{C}^n$? (any n).
   - (c) Prove that T is strongly minimal.
   - (d) Here is a confusing observation: $\mathbb{R}^2$ is a vector space over $\mathbb{R}$. A line is definable and is neither finite nor cofinite. What mistake am I making? Can you provide a better framework for $\mathbb{R}^2$ compatible with the notion of strong minimality?

(e) Is it true that whenever T is strongly minimal, then every subset of any power of $\mathfrak{C}$ is finite or co-finite?

(f) Justify the definition of Morley rank with the vector space-dimension for vector spaces over a given field. That is given a vector space of dimension α, give a formula with Morley rank α.

2. Is is true that if X is strongly minimal then $\mathrm{RM}(X) = \dim(X)$? (compare with item 1 Aufgabe 83).

3. Let X be a definable set in ACF_0. What is $\mathrm{RM}(X)$? what is the Morley rank of X^n (for a given n)?(compare with Aufgabe 84)

4. Remember that the Morley rank of a theory is by definition the Morley rank of the formula $x = x$. What is the Morley rank of a strongly minimal theory T?

5. Let $L = \{E\}$ where E is a binary relation symbol. Let T be the theory of an equivalence relation with infinitely many classes each of which is infinite. Show that $\mathrm{RM}(T) = 2$.

6. For every n give an example of a theory whose Morley rank is n.

7. Let $K \subset F$ be algebraically closed fields of characteristic zero. Let $L = \{+, \cdot, U, 0, 1\}$, where U is a unary predicate, and let T the theory of an L-structure M. Show that T is ω-stable with Morley rank ω.

Aufgabe 83.

1. Suppose that T is a strongly minimal theory. Show that then for all $\bar{a}$ in a power of $\mathfrak{C}$, $\mathrm{RM}(\bar{a}/A) = \dim(\bar{a}/A)$ (see the definition below).

$$\mathrm{RM}(\bar{a}) := \mathrm{RM}(\mathrm{tp}(\bar{a}/A)) = \inf\{\mathrm{RM}(\phi(\bar{x})) | \phi(\bar{x}) \in \mathrm{tp}(\bar{a}/A)\}$$

2. Suppose that $X \subseteq \mathfrak{C}^n$ is definable. Show that

$$\mathrm{RM}(X) = \sup\{\mathrm{RM}(\bar{a}/A) | \bar{a} \in X, A \subset \mathfrak{C}, |A| < |\mathfrak{C}|, X, A\text{-definable}\}.$$

Krull dimension and Morley rank

From Marker's 'Model Theory, an introduction'. Let K be an algebraically closed field. A set $V \subseteq K^n$ is called a variety if

$$V = \bigcap_{f \in S} \text{roots of } f$$

for some (finite) $S \subseteq K[\bar{X}]$. So V is a definable set in ACF_0.
Let $V \subseteq K^n$ be an irreducible algebraic variety. Let $I(V)$ be the prime ideal of polynomials in $K[X_1, \ldots, X_n]$ vanishing on V. The Krull dimension of V is the largest number m such that there is a chain of prime ideals

$$I(V) = P_0 \subset P_1 \subset \ldots \subset P_m \subset K[X_1, \ldots, X_n].$$

If V has Krull dimension 0 then $I(V)$ is maximal and hence generated by some $X_1 - a_1, \ldots, X_n - a_n$.
If $V \subseteq K^n$ is an algebraic variety, by $K(V)$ we mean the fraction field $K[X_1, \ldots, X_n]/I(V)$. It is known that the Krull dimension of an irreducible veriety V is equal to the transcendence degree of $K(V)$ over K. We will show in the following exercise that the Krull dimension of V is indeed equal to its Morley degree as a definable set in a model of ACF_0.

Aufgabe 84. Let K be an algebraic closed field and $V \subseteq K^n$ be an irreducible variety. Show that then $\mathrm{RM}(V)$–we mean the Morley rank of the formula that defines V– is equal to the Krull diemension of V.

Hinweis. We prove this by induction on the Krull dimension of V. Show that the exercise is the case when the Krull dimension of V is zero.
Suppose that V has Krull dimension $k > 0$. Suppose that V is defined by ϕ. For each $\bar{a} \in \phi(\mathfrak{C})$ define

$$V_{\bar{a}} := \bigcap_{f(\bar{a})=0} \text{roots of } f.$$

Another way of defining $V_{\bar{a}}$ is to write

$$V_{\bar{a}} = V(I_{\bar{a}})$$

where $I_{\bar{a}}$ is the set of polynomials vanishing at $\bar{a}$. Note that

$$\mathrm{RM}(V) = \max\{\mathrm{RM}(\bar{a}/K) | \bar{a} \in \phi(\mathfrak{C})\}.$$

If $\bar{a}$ is such that $V_{\bar{a}} \subset V$ then by induction hypothesis

$$\mathrm{RM}(\bar{a}/K) \leq \mathrm{RM}(V_{\bar{a}}) \leq k-1;$$

if $V_{\bar{a}} = V$, then $I_{\bar{a}} = I(V)$ and as a result $K(V) = K(\bar{a})$. We have also proved that

$$\mathrm{RM}(\bar{a}/K) = \dim(\bar{a}/K)$$

and the dimension mentioned above is exactly the transcendence degree of $K(V)$.

1.19 the Monster!

Fix a monster model $\mathfrak{C}$.

Aufgabe 85.

1. For sets A and B and elements a and b show that

   $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ if and only if there is an automorphism of $\mathfrak{C}$ that sends a to b and is the identity on A.

Concerning the next exercise: we know that if Σ is an infinite set of formulae and

$$\Sigma \vdash \phi$$

then because the proofs involve only finitely many assumptions, there is a finite subset Σ' of Σ such that

$$\Sigma' \vdash \phi.$$

I want to emphasise that when we are working in a monster model, we can (somehow) assume that the implications $\mathfrak{C} \models$ and $\vdash$ are equivalent. Of course $\models \phi$ and $\vdash \phi$ are equivalent, but $\models \phi$ means for all models M we have $M \models \phi$ (and not just for the monster model).

2. Suppose that Σ is an infinite consistent set of formulae and

$$\mathfrak{C} \models \bigwedge_{\phi \in \Sigma} \phi(x) \to \psi(x)$$

(more formally I mean $\Sigma(\mathfrak{C}) \subseteq \psi(\mathfrak{C})$) then show that there are indeed finitely many $\phi_1, \ldots, \phi_n$ in this infinite conjunction such that

$$\mathfrak{C} \models \phi_1(x) \wedge \ldots \wedge \phi_n(x) \to \psi(x).$$

3. Suppose that

$$\mathfrak{C} \models \bigwedge_{\phi \in \Sigma} \phi(x) \to \bigvee_{\psi \in \Sigma'} \psi(x)$$

that is

$$\bigcap_{\phi \in \Sigma'} \phi(\mathfrak{C}) \subseteq \bigcup_{\psi \in \Sigma'} \psi(\mathfrak{C})$$

where $\bigcap_{\phi \in \Sigma'} \phi(\mathfrak{C})$ and $\bigcup_{\psi \in \Sigma'} \psi(\mathfrak{C})$ are both non-empty. Show that there are finitely many ϕ's on the left-hand side and finitely many ψ's on the right-hand side so that

$$\mathfrak{C} \models \phi_1(x) \wedge \ldots \wedge \phi_n(x) \to \psi_1(x) \vee \ldots \vee \psi_m(x)$$

4. Suppose that X is definable. Show that following are equivalent:

   (a) we can define X with a formula whose parameters are in A.

   (b) for every x, y,

$$\mathrm{tp}(x/A) = \mathrm{tp}(y/A) \Rightarrow (x \in X \leftrightarrow y \in X)$$

5. Using the item above, show that a definable set X can be defined by a formula with parameters in A if and only if X is preserved by all automorphisms of $\mathfrak{C}$ that are identity on A.

6. Suppose that A and B are definable subsets of $\mathfrak{C}$ and

$$\mathfrak{C} \models \forall x \in A \quad \exists y \in B \quad \phi(x, y).$$

Show that for some $n \in \mathbb{N}$,

$$\mathfrak{C} \models \exists y_1, \ldots y_n \in B \quad \forall x \in A \quad [\phi(x, y_1) \vee \ldots, \vee \phi(x, y_n)].$$

Aufgabe 86. Let A be a subset of B. Show that

$B \subseteq \mathrm{dcl}(A)$ if and only if every type over A extends uniquely to a type over B

Is the statement the case if dcl is replaced by acl? What is the corresponding statement for that case?

Aufgabe 87.

1. Show that b is in the definable closure of a if and only if there is an $\emptyset$-definable class D with $a \in D$ and an $\emptyset$-definable map $D \to \mathfrak{C}$ that sends a to b.

2. Two elements a and b are interdefinable if there are $\emptyset$-definable classes C, D with $a \in C$ and $b \in D$ and an $\emptyset$-definable bijection between C and D mapping a to b.

Aufgabe 88. Suppose that $\mathrm{tp}(a) = \mathrm{tp}(b)$ and $\mathrm{tp}(c) = \mathrm{tp}(d)$. Does this imply that $\mathrm{tp}(ac) = \mathrm{tp}(bd)$? Give counterexamples and provide sufficient conditions under which this holds.

We will see in the next exercise that in stable theories indiscernible **sequences** and indiscernible **sets** are the same. A sequence $X = (a_i)$ is by definition a sequence of indiscernibles if each $\mathrm{tp}(a_{i1}, \ldots, a_{in})$ depends only on $\mathrm{tp}_{\{=,<\}}(i1, \ldots, in)$. We call X an indiscernible set, or a strongly indiscernible sequence if each $\mathrm{tp}(a_{i1}, \ldots, a_{in})$ depends only on $\mathrm{tp}_{\{=\}}(i1, \ldots, in)$.

Aufgabe 89. Assuming L to be countable and T to be κ-stable for an infinite cardinal κ, let $M \models T$ and $X = (a_i)$ be an infinite sequence of indiscernibles in M. Show that X is an infinite set of indiscernibles.

Hinweis. Suppose that $M \models \phi(a_1, \ldots, a_n)$. We want to prove that

$$M \models \phi(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

for all permutations σ in S_n. As you may remember from the Group-Theory course, every permutation $(i_1 \ldots i_n)$ is a composition of permutations of the form (ab)—called transpositions. So it is enough to prove that whenever $M \models \phi(a_1, \ldots, a_n)$ then

$$M \models \phi(a_1, \ldots, a_{m-1}, \underbrace{a_{m+1}, a_m}, \ldots, a_n).$$

So suppose that

$$M \models \phi(a_1, \ldots, a_n)$$

and

$$M \models \neg\phi(a_1, \ldots, a_{m-1}, a_{m+1}, a_m, \ldots, a_n).$$

We have proved in an earlier exercise that there exist A and B where B is dense in A and $|B| \leq \kappa < |A|$. Find an $N \models T$ and Y a sequence of indiscernibles in N with the order type of A such that $\text{tp}(Y) = \text{tp}(X)$ (standard lemma). Let Y_0 be the subsequence corresponding to B. Show that every two elements in Y realise distinct types over Y_0, contradicting k-stability.

Chapter 2

Model Theory 2

2.1 teilen und forken

Aufgabe 90.

1. Seien M ein $|A|^+$-saturiertes Modell und $A \subseteq M$. Zeigen Sie, dass es für jeder vollständige Typ $p \in S(M)$, gilt:

$$p \text{ forkt über } A \Leftrightarrow p \text{ teilt über } A.$$

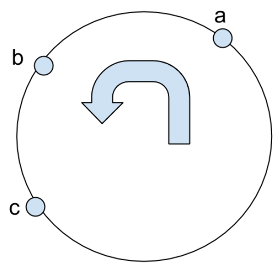
2. Angenommen, dass $p \in S(\mathfrak{C})$ ein A-invarianter Typ ist, zeigen Sie, dass p über A nicht forkt.

In der nächsten Aufgabe, sehen wir ein Beispiel für forken und nicht teilen.

Aufgabe 91 (dichte ziklische Ordnung)**.**

Definition. $cyc(a, b, c) \Leftrightarrow$ wenn man von a gegen den Uhrzeigersinn läuft, kommt c nach b.
Alternative Definition: definiere cyc auf $\mathbb{Q}$:

$$cyc(a, b, c) \Leftrightarrow (a < b < c) \vee (c < a < b) \vee (b < c < a)$$



Sei T_{co} die Theorie von $(\mathbb{Q}, cyc)$.

1. Axiomatisire T_{co}.
2. Zeige, dass T_{co} die Quantoren eliminiert.
3. $a \neq b \Rightarrow cyc(a, x, b)$ teilt über $\emptyset$.
4. Der einzige 1=Typ über $\emptyset$ forkt über $\emptyset$ und teilt nicht. ($a \not\downarrow_{\emptyset} \emptyset$).

In der nächsten Aufgabe, sehen wir ein Beispiel für nicht-transitive nicht-forkende Erweiterungen.

Aufgabe 92.

1. Seien $T = T_{DLO}$, $A \models T$ und $|a| = |b| = 1$. Beschreibe $a \downarrow_A^d b$.
2. Was wäre es mit $|a| = 1, |b| = 2$?
3. Beschreibe Symmetrie und Transitivät.
4. Zeigen Sie, dass die folgende Transivität **nicht** gelt: wenn $p \subseteq q \subseteq r$ nichforkende Erweiterungen sind, dann r ist eine nichtforkende Erweiterung von p.

Aufgabe 93.

Wir haben in der Vorlesung gesehen dass, es in streng-minimale Theorien, gilt: $a \downarrow_A b$ gdw. $a \downarrow_A^{\text{pregeometry}} b$. Gilt das auch in o-minimale Theorien? (Diskussion über o-minimal Theorien im Tutorium)

2.2 einfache Theorien, Miterben

Aufgabe 94 (zwei Versionen der gleichen Aussage)**.**

1. Sei $\phi(x, b)$ eine Formel die über A teilt. Sei $C \supset A$. Dann es ein $b' \equiv_A b$ gibt, sodaß $\phi(x, b')$ über C teilt.

2. Nehmen wir an, dass $\phi(x, b)$ über A teilt und $A \subseteq C$. Zeigen Sie, dass es ein $C' \equiv_A C$ gibt, sodass $\phi(x, b)$ über C' teilt.

Aufgabe 95.

Definition. Wir sagen, dass $\phi(x, y)$ Ordnung Eigenschaft hat, wenn es Folgen $(a_i)_{i<\omega}, (b_i)_{i<\omega}$ geben, sodass $\models \phi(b_j, a_i) \Leftrightarrow j < i$; das heißt, $\phi(x, a_i)$ definiert auf $B = \{b_j | j < \omega\}$ eine Kette von Teilmengen. Wenn es eine Formel $\phi(x.y)$ gibt und eine Folge $(a_i)_{i<\omega}$, sodass $\phi(-, a_i)$ eine unendliche Kette von Teilmengen definiert, sagen wir dass ϕ SOP (=strict order property, streng Ordnung Eigenschaft) hat. Also:

$$\phi(\mathfrak{C}, a_0) \subset \phi(\mathfrak{C}, a_1) \subseteq \dots.$$

Zeige SOP$\Rightarrow$ nicht einfach.

Shelah: Stabil=NIP+$\neg$ SOP.

Aufgabe 96.

Definition. Seien M ein Modell, M Teilmenge von A, und $q \in S(A)$. q heißt das Miterbe (coheir auf englisch) von $q|_M$, wenn q endlich erfüllbar in M ist.

Beispiel. $A = M$, $q \in S(M)$ beliebig.

Seien $A \subseteq A'$ und $q \in S(A)$ Miterbe von $q|_M$. Dann es einen Typ $q' \in S(A')$ gibt mit $q \subseteq q'$, sodass q' auch Miterbe von $q|_M$ ist.

Aufgabe 97.

1. q ist ein Miterbe, genaue dann wenn $q = \{\phi(x,a) | \phi(M,a) \in U, a \in A\}$, wobei U ein Ultrafilter auf M ist.

2. Seien M ein Model, M Teilmenge von A und $p \in S(M)$. Alle Miterbe (Erweiterungen) von p kriegen wir auf folgende Weise. Sei p das F-Limit der Familie $\{\mathrm{tp}(m/M) | m \in M\}$, wobei F ein Ultrafilter auf M ist. Das F-Limit der Familie $\{\mathrm{tp}(m/A) | m \in M\}$ ist ein Miterbe von p. Sehe folgende Bemerkung und Definition.

Bemerkung. Sei X ein Kompakter topologische Raum und F ein Ultrafilter auf I. Jede Familie $\{x_i | i \in I\}$ hat ein eindeutiges F-Limit.

Definition. Sei X ein Kompakter topologischer Raum, und F ein Ultrafilter auf I. x heißt des F-Limit der Familie $\{x_i | i \in I\}$, wenn für jede offene Umgebung O von x, die folgende Menge in F bleibt:

$$\{i \in I | x_i \in O\}.$$

Aufgabe 98.

1. Der Beweis von:

   stabil$\Rightarrow$ einfach (Baum Eigenschaft $\Rightarrow$ Ordnung Eigenschaft).

2. Der Beweis von Erdös-Rado Lemma:

$$\beth_n^+(\mu) \to (\mu^+)_\mu^{n+1}.$$

2.3 einfache Theorien, Shelahs Lemma, dicke Formeln, $\mathrm{nc}(a,b)$

Wir haben schon gesehen, dass man kann Standard Lemma oder Shelahs Lemma anwenden um eine indiscernible Folge mit bestimmten Typ zu finden. In der folgenden Aufgabe haben wir einen Fall, in dem man eine indiscernible Folge nur bei der Anwendung des Kompaktheit Satzs findet.

Aufgabe 99. Seien $(a_i)_{i\in\omega}$ eine ununterscheidbare Folge über B und I eine lineare Ordnung. Beweisen Sie (ohne Anwendung des Ramseys Lemma), dass es eine Folge $(b_i)_{i\in I}$ (indiscernible über B) gibt, sofass

$$\mathrm{tp}(b_{i_0}, \ldots, b_{i_{n-1}}/B) = \mathrm{tp}(a_0, \ldots, a_{n-1}/B) \quad \text{für jede } i_0 < \ldots < i_{n-1}.$$

Aufgabe 100.

- Schreiben Sie den Beweis des Shelahs Lemma.
- Erzählen Sie den Unterschied zwischen die Ergebnisse von Standard Lemma und Shelahs Lemma.

In der nächsten Aufgabe, sehen wir, dass wie in NIP und NTP2, es für Einfachheit genügt, dass man die Formeln $\phi(x, y)$ mit $|x| = 1$ behandelt.

Aufgabe 101.

- T ist einfach, wenn es keine Formel $\phi(x, y)$ mit $|x| = 1$ gibt, die die Baum Eigenschaft hat.
- T ist einfach, wenn jede 1-Typ, über eine Menge der Mächtigkeit am meisten $|T|$ nicht teilt.

Aufgabe 102.

1. Die Konjunktion von zwei dicken Formeln ist dick.
2. Wenn $\theta(x, y)$ dick ist, dann ist $\tilde{\theta}(x, y) = \theta(y, x)$ auch dick.
3. Wenn θ dick ist, dann gibt es eine symmetrische Formel θ', sodass θ' dick ist und $\theta' \subseteq \theta$.

Aufgabe 103. Sei $\theta(x, y)$ eine Formel mit Parametern in A. Zeigen Sie, dass θ genau dann dick ist, wenn $\models \theta(a_0, a_1)$ für alle A-indiscernible Folgen $a_0, a_1, \ldots$.

Aufgabe 104. Zeigen Sie, dass nc_A symmetrisch ist:

$$\mathrm{nc}_A(a, b) \Leftrightarrow \mathrm{nc}_A(b, a).$$

Aufgabe 105. Wenn $\mathrm{nc}_A(a, b)$, dann gibt es ein Modell $M \supseteq A$, sodass $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$.

> Wenn I eine indiscernible Folge ist, dann gibt es ein Modell M, sodass I über M indiscernible ist.

2.4 Unabhängigkeitssatz

Aufgabe 106. Seien M ein Modell, B_i, $i \in I$, unabhängig über M, b_i sodass $b_i \mathop{\smile\!\!\!\!\!\mid\,}_M B_i$ und $\mathrm{tp}(b_i/M) = \mathrm{tp}(b_j/M)$ für jede i, j. Es gibt d, sodass

$$\begin{aligned} &\mathrm{tp}(d/B_i) = \mathrm{tp}(b_i/B_i) \quad \text{für jedes } i, \\ &d \mathop{\smile\!\!\!\!\!\mid\,}_M \{B_i | i \in I\}. \end{aligned}$$

Aufgabe 107. Beweis von Shelahs Lemma:
Sei A eine Parametermenge. Es gibt λ sodass für jede lineare Ordnung I mit $|I| = \lambda$ und jede Familie $(a_i)_{i \in I}$, es eine ununterscheidbare Folge $(b_i)_{i \in \omega}$ gibt, sodass

$$\forall j_1 < \ldots < j_n \in \omega \quad \exists i_1 < \ldots < i_n \in I \quad a_{i_1}, \ldots, a_{i_n} \equiv_A b_{j_1}, \ldots, b_{j_n}.$$

Aufgabe 108. Erklären Sie die Fortsetzung von Erben (im Vergleich zur Fortsetzung von Coerben, die in der Vorlesung beweisen wurde): Sei

$$M \subseteq B \subseteq C \quad M \text{ ein Modell} \quad p \in S(M) \quad q \in S(B) \quad q \text{ Erbe von } p$$

dann gibt es ein $q \subseteq r \in S(C)$, sodass r auch ein Erbe von p ist.

Aufgabe 109. Abschließen Sie den Beweis des Folgendes, ausweislich der Vorlesung:

$$a \in \mathrm{acl}(Ab) \Rightarrow \mathrm{RM}(a/A) \leq \mathrm{RM}(ab/A) \leq \mathrm{RM}(b/A).$$

Aufgabe 110. Erklären Sie warum es in Strengminimale Theorien gilt: Erbe = Coerbe.

Aufgabe 111. Zeigen Sie, dass wenn $a \in \mathrm{acl}(A)$, dann $a \not\mathop{\smile\hspace{-0.9em}|}^d_A a$; anders gesagt,

$$a \mathop{\smile\hspace{-0.9em}|}^d_A a \Rightarrow a \in \mathrm{acl}(A).$$

2.5 Stabilität und Ordnung Eigenschaft

Aufgabe 112. Die Theorie T ist genau dann stabil, wenn es für eine Kardinalzahl λ, λ-stabil ist; das heißt, wenn $|B| \leq \lambda$, dann $|S(B)| \leq \lambda$.

Aufgabe 113.

1. Die Formel $\phi(x, y)$ hat die Ordnung Eigenschaft, wenn es Folgen $(a_i)_{i\in\omega}$ und $(b_j)_{j\in\omega}$ gibt mit
$$\phi(a_i, b_j) \Leftrightarrow i \leq j$$
betrachten Sie $\leq$ statt $<$.

2. Die Formel $\phi(x, y)$ hat genau dann die Ordnung Eigenschaft, wenn $\phi(y, x)$ die Ordnung Eigenschaft hat.

3. Die Formel $\phi(x, y)$ hat genau dann die Ordnung Eigenschaft, wenn $\neg\phi(x, y)$ die Ordnung Eigenschaft hat.

4. Wenn weder ϕ noch ψ die Ordnung Eigenschaft hat, dann auch hat $\phi \wedge \psi$ die Ordnung Eigenschaft nicht:
$$\begin{aligned} &\phi \quad \text{nicht Ordnung Eigenschaft} \\ &\psi \quad \text{nicht Ordnung Eigenschaft} \\ \Rightarrow\ &\phi \wedge \psi \quad \text{nicht Ordnung Eigenschaft} \end{aligned}$$

2.6 Stabilität, Erbe=Coerbe=eindeutige nichtforkende Erweiterung

In this frame I have provided all material needed for solving the exercises in this sheet.

Bemerkung.

1. Let $M \subseteq B$ and M be a model. If $p \in S(M)$ is definable, then it has a unique heir $q \in S(B)$. The type q is also definable over M and
$$q = \{\phi(x, \bar{b}) | \phi(x, \bar{y}) \in L, \bar{b} \in B, \mathfrak{C} \models d_p x \phi(x, \bar{b})\}.$$
2. If T is stable and $p \in S(A)$ is a type, then p is definable over A.
3. In stable theories, every type $p \in S(M)$ has a unique heir $q \in S(B)$ (for B a paremeter set extending the model M). This unique heir, is also the unique coheir, and the unique non-forking extension of p.
4. Stable theories are simple, so forking = dividing.
5. If $A \subseteq B$ and π is a partial type over B not forking over A, then π extends to a complete type over B not forking over A.

Bemerkung (Harrington)**.** Let T be a stable theory and p and q be global types. For every formula $\phi(x, y)$ without parameters

$$d_p x \phi(x, y) \in q(y) \Leftrightarrow d_q y \phi(x, y) \in p(x).$$

Aufgabe 114. Let I be an indiscernible sequence over a parameter set A. Show that there exists a models M containing A such that I is indiscernible over M.

Use the above Aufgabe, to prove the following:

Aufgabe 115. Let A be a parameter set. if $\phi(x, b)$ is satisfiable in every model M containing A, then $\phi(x, b)$ does not divide over A (in stable theories, the converse holds too; the next Aufgabe).

Aufgabe 116. Suppose that T is a stable theory. A formula $\phi(x, b)$ does not fork over a parameter set A if and only if $\phi(x, b)$ is satisfiable in every model M containing A (use item 3 and 5 in the frame and the previous Aufgabe).

Aufgabe 117. We call q a **weak heir** of p if it satisfies the definition of heir for formulae without parameters. Precisely, let $M \models T$, $M \subseteq B$, $p \in S(M)$ and $q \in S(A)$. Call q a weak heir of p if for every formula $\phi(x, y)$ **without parameters** and each $b \in B$,

$$\phi(x, b) \in q \Rightarrow \exists m \in M \quad \phi(x, m) \in p.$$

Show that in stable theories, weak heir and heir are the same.

Aufgabe 118. Assumptions: T is stable, M is a model, A is a parameter set, $M \subseteq A$, $p \in S(M)$ and $q \in S(A)$. Use the theorem of Harrington stated in the frame to prove that the following are equivalent:

1. q is an (the) heir of p.
2. q is a (the) coheir of p.

2.7 Elimination der Imaginäre

Aufgabe 119. Describe what is meant by $\mathrm{acl}(a/E)$ and $\mathrm{dcl}^{eq}(a)$.

> The Pillay-Lascar Theorem says "if T is strongly minimal and $\mathrm{acl}(\emptyset)$ is infinite, then T has weak elimination of imaginaries". The next Aufgabe is a counterexample to the requirements and the statement.

Aufgabe 120. Define a relation R over $\mathbb{Q}$ by

$$R(a, b, c, d) \Leftrightarrow a + b + c + d = 0.$$

Notice that $(\mathbb{Q}, R)$ is equivalent to $(\mathbb{Q}, +)$ (no zero in the language), and R determines the affine lines over $\mathbb{Q}$. Show that $(\mathbb{Q}, R)$ is strongly minimal, but it does not have weak elimination of imaginaries.

> We saw in the lecture that if T is a totally transcendental theory in which each global type has a canonical base in $\mathfrak{C}$, then T has weak elimination of imaginaries. The proof went as follows: if $e = c/E$ is imaginary and $\mathrm{RM}(c/E) = \alpha$, then we let $\mathbb{P}$ be the global type with $\mathrm{RM} = \alpha$ containing the formula xEc. This type has a canonical base d, and d is the canonical parameter we are looking for. It was left as an exercise to prove that d is finite. This assumption is justified in Aufgaben 3,4,5.

Aufgabe 121. Let $\mathbb{D}$ be a definable class and D be a set such that for each automorphism α

$$\underbrace{\alpha(D) = D}_{\text{pointwise}} \Leftrightarrow \underbrace{\alpha(\mathbb{D}) = \mathbb{D}}_{\text{setwise}}$$

show that D contains a canonical parameter of $\mathbb{D}$.

Aufgabe 122. Let T be totally transcendental and $\mathbb{P}$ be a global type. Show that $\mathbb{P}$ has a finite canonical base in $\mathfrak{C}^{eq}$.

Aufgabe 123. Using the previous two Aufgaben, show that if $\mathbb{P}$ has a canonical base $D \subseteq \mathfrak{C}$, then it has a finite base $d \subseteq \mathfrak{C}$.

We identified the canonical base for global types in ACF_p as follows: if $\mathbb{P}(\bar{x})$ is a global type, then it is given by an irreducible variety over $\mathfrak{C}^n$ via $\mathrm{tp}(\bar{c}/\mathfrak{C})$ (inaccurate) where $\bar{c}$ is the generic point of V. Also, we can say that $\mathbb{P}(\bar{x})$ is the type whose Morley rank is equal to the Morley rank of V and "$\bar{x} \in V$" $\in \mathbb{P}(\bar{x})$. Let I be the corresponding ideal of V. Then $cb(\mathbb{P}) = [V] = [I]$. Also $[I] = \bigcup_{k=0}^{\infty} [I_k]$ where $I_k = \{p \in I | \deg p \geq k\}$ is considered as a sub-$\mathbb{C}$-vector space of $\mathfrak{C}^{N(k)}$, where $N(k)$ is the number of all monomials of deg $\leq k$ in $X_1, \ldots X_n$, and we have $I = \bigcup_{k=0}^{\infty} I_k$. We are now supposed to apply Andre Weil's theorem over "the field of definition of variety" to prove:

Aufgabe 124. Let $\mathfrak{C}$ be a field and $U \leq \mathfrak{C}^n$ (as vector spaces). Then $[U]$ exists in $\mathfrak{C}$ (for example if $U = \mathfrak{C}.(a_1, \ldots, a_n)$ then $[U] = (a_2/a_1, \ldots, a_n/a_1)$.

Let T be stable. We proved that $p \in S(B)$ does not fork over $A \subseteq B$ if and only if p has a good definition over $\mathrm{acl}^{eq}(A)$. In the proof of $\Leftarrow$ we said if p has a good definition over $\mathrm{acl}^{eq}(A)$, then it defines a a global type $\mathbb{P}$ extending p, which does not fork over $\mathrm{acl}^{eq}(A)$, and hence over A. The last claim is to be justified below:

Aufgabe 125.

1. If $b_0, b_1 \ldots$ is an indiscernible sequence over A, then it is indiscernible over $\mathrm{acl}(A)$.

2. (Hence:) If $\phi(x, b)$ divides over A, then it divides over $\mathrm{acl}(A)$.

3. Give another proof for item 2, using transitivity.

The other direction of the proof went as follows: if p does not fork over A, then it has a global non-forking extension $\mathbb{P}$. If M is a model containing A, then $\mathbb{P}$ does not fork over M. Since T is stable, $\mathbb{P}$ is defined over M, and hence $cb(\mathbb{P}) \in M^{eq}$. Since M is arbitrary, $cb(\mathbb{P}) \in \mathrm{acl}(A)$. The last sentence is justified below:

Aufgabe 126. Show that

$$\mathrm{acl}(A) = \bigcap_{M \text{model containing } A} M.$$

2.8 stabile Theorien

Let T be stable.

Theorem. A type $p \in S(B)$ does not fork over $A \subseteq B$ if and only if it has a good definition over $\mathrm{acl}^{eq}(A)$.

Theorem. Types over $\mathrm{acl}^{eq}(A)$ are stationary.

By "p having good definition over a set" we mean that the definition of p with parameters in $\mathfrak{C}$ defines a global type. We wanted to use the above theorems to prove that types over models are stationary, that is each type over a model has a unique non-forking extension to any bigger set. To prove this, we need to show that if M is a model then $\mathrm{acl}^{eq}(M) = M^{eq}$.

Aufgabe 127.

1. Zeigen Sie, dass $\mathfrak{C}^{eq}$ der Monster Modell von T^{eq} ist (also $M^{eq} = \mathrm{dcl}^{eq}(M)$ eine elementare Unterstruktur von $\mathfrak{C}^{eq}$ ist).

2. Zeigen Sie, dass $\mathrm{acl}^{eq}(M) = \mathrm{dcl}^{eq}(M) = M^{eq}$.

A theory T is called κ-homogeneous if whenever $A \subseteq M$, M is a model, $f : A \to M$ is partial elementary, and $a \in M$, then there is a partial elementary map $f' : A \cup \{a\} \to M$ that extends f. In particular if M is homogeneous (that is $|M|$-homogeneous) then

$$\bar{a} \equiv \bar{b} \Leftrightarrow \exists \sigma \in Aut(M) \quad \sigma(a) = b.$$

If T is κ-saturated, then it is κ-homogeneous.

Theorem. Let T be a stable theory.

- Let $A \subseteq M$ and M be a sufficiently homogeneous model and $p \in S(A)$. Then all non-forking extensions of p to M are conjugates over A.
- If $A \subseteq B$ and $p \in S(A)$, then p has at most $2^{|T|}$ non-forking extensions to B.

The above two items do not hold in random graphs; next exercise.

Aufgabe 128. Sei T die Theorie von zufällige Graphen.

1. Beschreiben Sie die indiscernible Folgen.
2. Beschreiben Sie $A \perp_C B$ (und nicht forkende Erweiterungen) in T.
3. Zeigen Sie, dass T nicht stabil ist.
4. Zeigen Sie, dass T einfach ist.
5. Zeigen Sie, dass die beiden Aussagen des Satz 2.8 sind falsch in zufällige Graphen.

Define

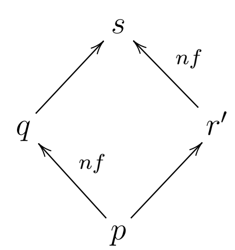
$$N(B/A) := \{q \in S(B) | q \text{ does not fork over } A\}$$

Let $\pi : S(B) \to S(A)$ be the restriction map. The **open mapping theorem** says that whenever T is stable, $\pi \restriction N(B/A)$ is an open map (it sends open sets to open sets).

Aufgabe 129. 1. Zeigen Sie, dass $\pi : S(\mathrm{acl}(A)) \to S(A)$ ist immer offen.

2. Sei T stabil. Ist $\pi : S(B) \to S(A)$ immer offen?

Theorem (Diamond lemma)**.** Let T be simple, $p \in S(A)$, q a non-forking extension of p and r any extension of p. Then there is an A-conjugate r' of r with non-forking extension s that extends q:



$r' = r^\alpha$ $\quad \alpha$ an automorphism

Assume that T eliminates imaginaries. The following are equivalent:

1. $\mathbb{D}$ is $\mathrm{acl}(A)$-definable.

2. There exists an A-definable equivalence relation E with finitely many classes $a_1/E, \ldots, a_n/E$ such that

$$\mathbb{D} = [a_1] \cup \ldots \cup [a_i] \text{ for some } i \leq n$$

where by $[a_i]$ we mean $\{b | E(b, a_i)\}$.

Aufgabe 130 (finite equivalence relation theorem)**.** Nehemen wir an, dass T die Imagninäre eliminiert. Seien $A \subseteq B$ und $p, q \in S(B)$ forken über A nicht. Dann es eine endliche A-definierbare äquivanlence Relation E gibt, sodass

$$p(x) \cup q(y) \vdash \neg E(x, y).$$

We have seen in the lecture that

1. If $p \subseteq q$ then $SU(p) \leq SU(q)$.
2. If $p \sqsubset_{nf} q$ then $SU(p) = SU(q)$.
3. If $SU(p) = SU(q) < \infty$ then $p \sqsubset_{nf} q$.

Aufgabe 131. Warum ist $< \infty$ in 3 notwendig?

2.9 stabile und superstabile Theorien

T is stable.

Aufgabe 132. Assume that $B \mathop{\smile\!\!\!\!\!\mid}_A a_1 a_1$. Show that

$$a_1 \underset{A}{\mathop{\smile\!\!\!\!\!\mid}} a_2 \Leftrightarrow a_1 \underset{AB}{\mathop{\smile\!\!\!\!\!\mid}} a_2.$$

Aufgabe 133. Let $p \in S(A)$ be stationary and I be a Morley sequence of p. Show that

1. If
$$B \supset A \quad I_0 \subseteq I \quad B \underset{AI_0}{\mathop{\smile\!\!\!\!\!\mid}} I$$
then $I - I_0$ is a Morley sequence of the non-forking extension of p to B.

2. The type
$$Av(I) = \{\phi(x, \bar{b}) | \bar{b} \in \mathfrak{C}, \{i | \models \neg\phi(a_i, \bar{b})\} \text{ finite}\}$$
is the non-forking global extension of p.

Aufgabe 134. Let I be a Morley sequence of a stationary type p over A and $B \mathop{\smile\!\!\!\!\!\mid}_A I$. Show that I is then a Morley sequence of the non-forking extension of p to AB.

Aufgabe 135. Correct the proof of Lemma 9.2.2: let I be indiscernible over A and B a countable set. Then I contains a countable I_0 such that $I - I_0$ is indiscernible over ABI_0.

Aufgabe 136. Show that in stable theories, U-rank=SU-rank.

Aufgabe 137. A simple theory is super-simple if and only if every **1-type** has SU-rank $< \infty$.

Aufgabe 138. T is stable if and only if every indiscernible sequence is totally indiscernible.

Aufgabe 139. Show that if $SU(p) = \infty$ then there is $q \sqsupset_{fork} p$ with $SU(q) = \infty$.

2.10 prime Erweiterungen

Aufgabe 140. Let T be totally transcendental. Show that the prime extensions are unique.

Aufgabe 141.

1. Let $M = (b_\alpha)$ be a construction over A and $C \subseteq M$ be construction closed. Show that for each $b_\alpha \in C$,

$$\mathrm{tp}(b_\alpha/A(b_{<\alpha} \cap C)) \vdash \mathrm{tp}(b_\alpha/AB_\alpha).$$

with the definition in the next item, this means that b_α and $Ab_{<\alpha}$ are weakly orthogonal over $A(b_{<\alpha} \cap C)$:

$$b_\alpha \underset{A(b_{<\alpha}\cap C)}{\overset{w}{\perp}} Ab_{<\alpha}$$

2. Two types $p(x)$ and $q(y)$, both in $S(A)$, are called **weakly orthogonal** if $p(x) \cup q(y)$ determines a complete type in x, y. Show that p and q are weakly orthogonal if for every $a \models p$, the type q has a unique extension to aA.

Aufgabe 142. Let T be countable. Show that the following are equivalent:

1. Every parameter set has a prime extension.

2. Over every countable parameter set, the isolated types are dense.

3. Over every parameter set the isolated types are dense.

Aufgabe 143. Show that the following two versions of Fodor's theorem are equivalent.

1. If $\{C_\alpha\}_{\alpha<\omega_1}$ are clubs, then

$$C := \{\alpha | \alpha \in \bigcap_{\beta<\alpha} C_\beta\} \text{ (the diagonal intersection of the } C_\alpha)$$

   is also a club.

2. If D is a club and $f : D \to \omega_1$ is regressive, then there is an ordinal β such that $\{\alpha \in D | f(\alpha) = \beta\}$ is stationary.

Aufgabe 144. What do you think is a better option for the next Monday?

1. Solving random exercises.

2. Beginning with Hrushovski's constructions.

3. Neither!

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