Universität Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 1, ws2014-2015
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Blatt 11, Stability, Categoricity, Saturatedness

The first exercise of this week is a set theory exercise. It is worth making yourself familiar with the statement and the proof of the following two.

Aufgabe 1.

1. Suppose that κ is an infinite cardinal. Show that $\kappa \cdot \kappa = \kappa$, where $\cdot$ denotes the multiplication of cardinals.

   Hinweis. First a quick reminder that $k \cdot k$ is by definition $|\kappa \times \kappa|$. The proof is by transfinite induction on κ. Assume that it holds for smaller cardinals. So if $\alpha < \kappa$ then $\alpha.\alpha = \alpha < \kappa$. Also it easy to see that $\kappa \leq \kappa.\kappa$. We need only to show that $\kappa \cdot \kappa \leq \kappa$. For this we need to define a well-ordering $\triangleleft$ on $\kappa \times \kappa$ in such a way that all initial segments of $\kappa \times \kappa$ with this well-ordering have size $\leq \kappa$. Define the following well-ordering $\triangleleft$ on $\kappa \times \kappa$:

   $$\langle \alpha, \beta \rangle \triangleleft \langle \alpha', \beta' \rangle \text{ if } \begin{cases} \max\{\alpha, \beta\} < \max\{\alpha', \beta'\} \text{ or} \\ \max\{\alpha, \beta\} = \max\{\alpha', \beta'\} \text{ and } \langle \alpha, \beta \rangle <_{\text{lex}} \langle \alpha', \beta' \rangle \end{cases}$$

   where $<_{\text{lex}}$ denotes the lexicographic order (with priority to the second coordinate). Now with the help of Figure 1 show that each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has no more than κ predecessors with the oredering $\triangleleft$. In Figure 1 the predecessors of an $\langle \alpha, \beta \rangle$ with $\beta < \alpha$ appear in grey. You may also think of an onion!

2. Show that for each infinite cardinal κ, there is a dense linear order $(A, <)$ and a $B \subset A$ such that B is dense in A and $|B| \leq \kappa < |A|$.

   Hinweis. Let $\lambda \leq \kappa$ be least such that $2^\lambda > \kappa$. Set

   $$A : \{\text{all functions from } \lambda \text{ to } \mathbb{Q}\}.$$

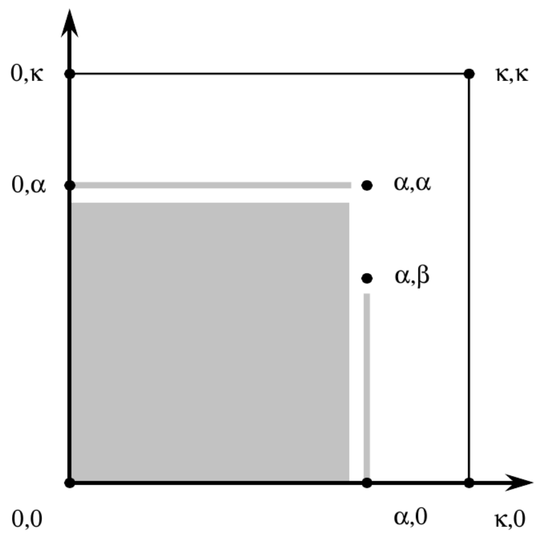

Figure 1: predecessors of $\langle\alpha,\beta\rangle$ where $\beta<\alpha$. The figure comes from `http://www.cl.cam.ac.uk/~lp15/papers/Sets/AC.pdf`

Define the following order on A:

$$f<g \text{ if } f(\alpha)<g(\alpha); \text{ where } \alpha \text{ is the least such that } f(\alpha)\neq g(\alpha).$$

Let B be the set of sequences in A that are eventually zero. Show that B is the B!

Aufgabe 2. Let $L=\{E\}$ be the language with a single binary relation symbol. let T be the theory of an equivalence relation where for each $n\in\omega$ there is a unique equivalence class of size n. Show that T is ω-stable and not $\aleph_0$-categorical and not $\aleph_1$-categorical.

Bemerkung. Compare the above exercise with the following two facts:

1. The Categoricity Theorem says that if T is categorical in some **uncountable** cardinal, then it is κ-categorical in all uncountable κ's.
2. A countable theory that is categorical in some uncountable cardinal, is ω-stable.

Aufgabe 3. 1. Show that DLO is **not** κ-stable for any infinite κ.

2. (Generalisation of the previous item) we say that a theory T has the **order property** if there is a formula $\phi(\bar{x}, \bar{y})$ with $|x| = |y| = n$ and $M \models T$ and $(\bar{c}_i)_{i \in \omega}$ an infinite sequence in M^n such that

$$M \models \phi(\bar{c}_i, \bar{c}_j) \text{ if and only if } i < j.$$

   Use part 2 of Aufgabe 1 to show that if T has the order property then it is **not** κ-stable for any infinite κ.

Hinweis (for part 2)**.** Use A and B in Aufgabe 1 part 2 as sets of indices of a suitable sequence (c_i) in such a way that $S_n(\{x_b | b \in B\}) > |B|$.

Aufgabe 4. If T is κ-stable, then (up to logical equivalences) $|T| \leq \kappa$.

Aufgabe 5.

1. If M is κ-saturated, then each definable subset of M is either finite or of cardinality at least κ.

2. Suppose that $|L| \leq \aleph_0$. Let $M_1, M_2, \ldots$ be a sequence of L-structures. Let F be a non-principle ultrafilter on ω. Show that $\prod_{i<\omega} M_i/F$ is $\aleph_1$-saturated. If we assume the Continuum Hypothesis, this implies that if M and N are countable L-structures and $M \equiv N$, then the $M^\omega/F \cong N^\omega/F$ where by M^ω/F we mean the ultrapower of M.

Ich wünsche Ihnen ein frohes neues Jahr!