Universität Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 1, ws2014-2015
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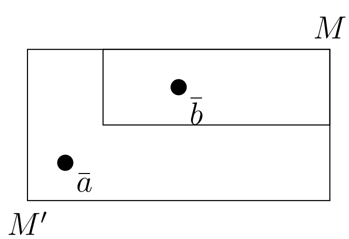
Blatt 13, elimination of the quantifier there are infinitely many

2 out of 5!

Aufgabe 1. Assume that T eliminates $\exists^\infty$. Prove the statement below:
for every formula $\phi(x_1, \ldots, x_n, \bar{y})$ there is a formula $\theta(\bar{y})$ such that for all $M \models T$ and $\bar{b} \in M$, we have $M \models \theta(\bar{b})$ if and only there is an $M' \succ M$ and elements $a_1, \ldots, a_n \in M' - M$ such that $M' \models \phi(a_1, \ldots, a_n, b)$.

$M \models \theta(b)$ if and only if $M' \models \phi(\bar{a}, \bar{b})$
for some $M' \succ M$
$\bar{a} \notin M$ ($a_i \notin M$ for each i)

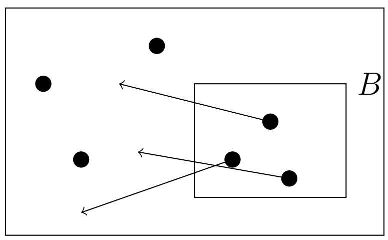


Aufgabe 2. Suppose that T_1 and T_2 are model complete theories in disjoint languages L_1 and L_2. Suppose that both T_1 and T_2 eliminate $\exists^\infty$. Show that then $T_1 \cup T_2$ has a model companion.

Bemerkung. T is model complete if whenever $M, N \models T$ then $M \subseteq N$ implies $M \preceq N$; and T^* is called a model-companion of T if T^* is model-complete and $T_\forall = T^*_\forall$

Hinweis. Use Aufgabe 1.

Aufgabe 3 (P.M. Neuman). Let B be a subset of M and $(c_0, \dots, c_n)$ be a sequence of elements all non-algebraic over $\emptyset$. Show that If M is $|B|^+$-saturated, then $\mathrm{tp}(c_0, \dots, c_n)$ has a realisation in M disjoint from B (in other words there are $b_0 \dots b_n \equiv c_0 \dots c_n$ with $b_i \notin B$ for each i).



Hinweis. Let us do the proof by induction on n. For $n = 0$, use the fact that M is $|B|^+$ saturated to find an element not in B that realises $\mathrm{tp}(c_0)$.
Let us assume the statement true for $< n$. Consider $\mathrm{tp}(c_0, \dots, c_n)$. We have two cases:
case 1: one (or more than one) of $c_0, \dots, c_{n-1}$ is in the algebraic closure of c_n. Let's say c_1 is in the algebraic closure of c_n. Again using the fact that M is saturated prove that there are $a_0, \dots, a_{n-1}, a_n$ such that $a_0, \dots, a_n \models \mathrm{tp}(c_0, \dots, c_n)$ and $a_0, \dots, a_{n-1} \notin \mathrm{acl}(B)$. Now it is clear that also $a_n \notin B$ because otherwise since c_1 is in $\mathrm{acl}(c_n)$, we have $a_1 \in \mathrm{acl}(a_n) \subseteq \mathrm{acl}(B)$, contradiction with the choice of a_i's.
case 2: non of $c_1, \dots, c_n$ are in $\mathrm{acl}(c_n)$. In this case first find a_n not in B realising $\mathrm{tp}(c_n/c_0, \dots, c_{n-1})$ and then find $a_0, \dots, a_{n-1}$ not in B realising $\mathrm{tp}(c_0, \dots, c_{n-1}/a_n)$ and then prove that $\mathrm{tp}(a_0, \dots, a_n) = \mathrm{tp}(c_1, \dots, c_n)$

Aufgabe 4. Suppose that M is $|A|^+$-saturated for A a subset of M. Show that then $p \in S(A)$ is algebraic if and only if $p(M)$ is finite.

Aufgabe 5. Let B be a subset of an L-structure A. Show that the **restriction map** $S_{m+n}(B) \to S_n(B)$ is open, continuous, and surjecctive. Let a be an n-tuple in A. Show that the fibre over $\mathrm{tp}(a/B)$ is canonically homeomorphic to $S_m(aB)$ (note that aB means $\{a\} \cup B$).

Bemerkung. An open map is one that maps open sets to open sets, that is if X and Y are topological spaces, then $f : X \to Y$ is open if for each open set O, $f(O)$ is open. Note that O is open with the topology of X and $f(O)$ with the topology of Y. Also, f is continuous if for every open subset O of Y, $f^{-1}(O)$ is open in X. A homeomorphism between two topological spaces is a map f which is continuous and has a continuous inverse. In our case basis open sets are $[\phi]$'s (=types containing ϕ).