Universität Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 1, ws2014-2015
Prof. Dr. Heike Mildenberger
Dr. Mohsen Khani

Blatt 15, the Monster!

Three items out of five exercises!
Fix a monster model $\mathfrak{C}$.

Aufgabe 1.

1. For sets A and B and elements a and b show that

$$\text{tp}(a/A) = \text{tp}(b/A) \text{ if and only if there is an automorphism of } \mathfrak{C} \text{ that sends } a \text{ to } b \text{ and is the identity on } A.$$

Concerning the next exercise: we know that if Σ is an infinite set of formulae and

$$\Sigma \vdash \phi$$

then because the proofs involve only finitely many assumptions, there is a finite subset Σ' of Σ such that

$$\Sigma' \vdash \phi.$$

I want to emphasise that when we are working in a monster model, we can (somehow) assume that the implications $\mathfrak{C} \models$ and $\vdash$ are equivalent. Of course $\models \phi$ and $\vdash \phi$ are equivalent, but $\models \phi$ means for all models M we have $M \models \phi$ (and not just for the monster model).

2. Suppose that Σ is an infinite consistent set of formulae and

$$\mathfrak{C} \models \bigwedge_{\phi \in \Sigma} \phi(x) \to \psi(x)$$

(more formally I mean $\Sigma(\mathfrak{C}) \subseteq \psi(\mathfrak{C})$) then show that there are indeed finitely many $\phi_1, \ldots, \phi_n$ in this infinite conjunction such that

$$\mathfrak{C} \models \phi_1(x) \wedge \ldots \wedge \phi_n(x) \to \psi(x).$$

3. Suppose that

$$\mathfrak{C} \models \bigwedge_{\phi \in \Sigma} \phi(x) \to \bigvee_{\psi \in \Sigma'} \psi(x)$$

that is

$$\bigcap_{\phi \in \Sigma'} \phi(\mathfrak{C}) \subseteq \bigcup_{\psi \in \Sigma'} \psi(\mathfrak{C})$$

where $\bigcap_{\phi \in \Sigma'} \phi(\mathfrak{C})$ and $\bigcup_{\psi \in \Sigma'} \psi(\mathfrak{C})$ are both non-empty. Show that there are finitely many ϕ's on the left-hand side and finitely many ψ's on the right-hand side so that

$$\mathfrak{C} \models \phi_1(x) \wedge \ldots \wedge \phi_n(x) \to \psi_1(x) \vee \ldots \vee \psi_m(x)$$

4. Suppose that X is definable. Show that following are equivalent:

   (a) we can define X with a formula whose parameters are in A.

   (b) for every x, y,

$$\mathrm{tp}(x/A) = \mathrm{tp}(y/A) \Rightarrow (x \in X \leftrightarrow y \in X)$$

5. Using the item above, show that a definable set X can be defined by a formula with parameters in A if and only if X is preserved by all automorphisms of $\mathfrak{C}$ that are identity on A.

6. Suppose that A and B are definable subsets of $\mathfrak{C}$ and

$$\mathfrak{C} \models \forall x \in A \quad \exists y \in B \quad \phi(x, y).$$

Show that for some $n \in \mathbb{N}$,

$$\mathfrak{C} \models \exists y_1, \ldots y_n \in B \quad \forall x \in A \quad [\phi(x, y_1) \vee \ldots, \vee \phi(x, y_n)].$$

Aufgabe 2. Let A be a subset of B. Show that

$B \subseteq \mathrm{dcl}(A)$ if and only if every type over A extends uniquely to a type over B

Is the statement the case if dcl is replaced by acl? What is the corresponding statement for that case?

Aufgabe 3.

1. Show that b is in the definable closure of a if and only if there is an $\emptyset$-definable class D with $a \in D$ and an $\emptyset$-definable map $D \to \mathfrak{C}$ that sends a to b.

2. Two elements a and b are interdefinable if there are $\emptyset$-definable classes C, D with $a \in C$ and $b \in D$ and an $\emptyset$-definable bijection between C and D mapping a to b.

Aufgabe 4. Suppose that $\mathrm{tp}(a) = \mathrm{tp}(b)$ and $\mathrm{tp}(c) = \mathrm{tp}(d)$. Does this imply that $\mathrm{tp}(ac) = \mathrm{tp}(bd)$? Give counterexamples and provide sufficient conditions under which this holds.

> We will see in the next exercise that in stable theories indiscernible **sequences** and indiscernible **sets** are the same. A sequence $X = (a_i)$ is by definition a sequence of indiscernibles if each $\mathrm{tp}(a_{i1}, \ldots, a_{in})$ depends only on $\mathrm{tp}_{\{=,<\}}(i1, \ldots, in)$. We call X an indiscernible set, or a strongly indiscernible sequence if each $\mathrm{tp}(a_{i1}, \ldots, a_{in})$ depends only on $\mathrm{tp}_{\{=\}}(i1, \ldots, in)$.

Aufgabe 5. Assuming L to be countable and T to be κ-stable for an infinite cardinal κ, let $M \models T$ and $X = (a_i)$ be an infinite sequence of indiscernibles in M. Show that X is an infinite set of indiscernibles.

Hinweis. Suppose that $M \models \phi(a_1, \ldots, a_n)$. We want to prove that

$$M \models \phi(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

for all permutations σ in S_n. As you may remember from the Group-Theory course, every permutation $(i_1 \ldots i_n)$ is a composition of permutations of the form (ab)—called transpositions. So it is enough to prove that whenever $M \models \phi(a_1, \ldots, a_n)$ then

$$M \models \phi(a_1, \ldots, a_{m-1}, \underbrace{a_{m+1}, a_m}, \ldots, a_n).$$

So suppose that

$$M \models \phi(a_1, \ldots, a_n)$$

and

$$M \models \neg\phi(a_1, \ldots, a_{m-1}, a_{m+1}, a_m, \ldots, a_n).$$

We have proved in an earlier exercise that there exist A and B where B is dense in A and $|B| \leq \kappa < |A|$. Find an $N \models T$ and Y a sequence of indiscernibles in N with the order type of A such that $\mathrm{tp}(Y) = \mathrm{tp}(X)$ (standard lemma). Let Y_0 be the subsequence corresponding to B. Show that every two elements in Y realise distinct types over Y_0, contradicting k-stability.