Universität Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 1, ws2014-2015
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Blatt 16, Morley Rank

Answer only 4 items.

Aufgabe 1.

1. It is mentioned in the script that 'the Morley rank of a formula $\phi(x, a)$ depends on $\phi(x, y)$ and the type of a'. Explain this. That is, show that if $\mathrm{tp}(a) = \mathrm{tp}(b)$ and $\phi(x, y)$ is a formula then $\mathrm{RM}\,\phi(x, a) = \mathrm{RM}\,\phi(x, b)$.

> Using the item above we can give an elementary definition for Morley rank of a formula in a structure M. That is given a structure M one can define $\mathrm{RM}^M\,\phi$ similarly, and then show that if $M \preceq N$ then $\mathrm{RM}^M\,\phi = \mathrm{RM}^N\,\phi$.

2. Show that if ψ implies ϕ then $\mathrm{RM}(\psi) \leq \mathrm{RM}(\phi)$.

Aufgabe 2 (examples of Morley rank)**.**

1. Let T be the theory of vector spaces over a field K.
   (a) What is the Morley rank of a definable subset X of $\mathfrak{C}$?
   (b) What is the Morley rank of a definable subset X of $\mathfrak{C}^n$? (any n).
   (c) Prove that T is strongly minimal.
   (d) Here is a confusing observation: $\mathbb{R}^2$ is a vector space over $\mathbb{R}$. A line is definable and is neither finite nor cofinite. What mistake am I making? Can you provide a better framework for $\mathbb{R}^2$ compatible with the notion of strong minimality?
   (e) Is it true that whenever T is strongly minimal, then every subset of any power of $\mathfrak{C}$ is finite or co-finite?

(f) Justify the definition of Morley rank with the vector space-dimension for vector spaces over a given field. That is given a vector space of dimension α, give a formula with Morley rank α.

2. Is is true that if X is strongly minimal then $\mathrm{RM}(X) = \dim(X)$? (compare with item 1 Aufgabe 3).

3. Let X be a definable set in ACF_0. What is $\mathrm{RM}(X)$? what is the Morley rank of X^n (for a given n)?(compare with Aufgabe 4)

4. Remember that the Morley rank of a theory is by definition the Morley rank of the formula $x = x$. What is the Morley rank of a strongly minimal theory T?

5. Let $L = \{E\}$ where E is a binary relation symbol. Let T be the theory of an equivalence relation with infinitely many classes each of which is infinite. Show that $\mathrm{RM}(T) = 2$.

6. For every n give an example of a theory whose Morley rank is n.

7. Let $K \subset F$ be algebraically closed fields of characteristic zero. Let $L = \{+, \cdot, U, 0, 1\}$, where U is a unary predicate, and let T the theory of an L-structure M. Show that T is ω-stable with Morley rank ω.

Aufgabe 3.

1. Suppose that T is a strongly minimal theory. Show that then for all $\bar{a}$ in a power of $\mathfrak{C}$, $\mathrm{RM}(\bar{a}/A) = \dim(\bar{a}/A)$ (see the definition below).

$$\mathrm{RM}(\bar{a}) := \mathrm{RM}(\mathrm{tp}(\bar{a}/A)) = \inf\{\mathrm{RM}(\phi(\bar{x})) | \phi(\bar{x}) \in \mathrm{tp}(\bar{a}/A)\}$$

2. Suppose that $X \subseteq \mathfrak{C}^n$ is definable. Show that

$$\mathrm{RM}(X) = \sup\{\mathrm{RM}(\bar{a}/A) | \bar{a} \in X, A \subset \mathfrak{C}, |A| < |\mathfrak{C}|, X, A\text{-definable}\}.$$

Krull dimension and Morley rank

From Marker's 'Model Theory, an introduction'. Let K be an algebraically closed field. A set $V \subseteq K^n$ is called a variety if

$$V = \bigcap_{f \in S} \text{roots of } f$$

for some (finite) $S \subseteq K[\bar{X}]$. So V is a definable set in ACF_0.
Let $V \subseteq K^n$ be an irreducible algebraic variety. Let $I(V)$ be the prime ideal of polynomials in $K[X_1, \dots, X_n]$ vanishing on V. The Krull dimension of V is the largest number m such that there is a chain of prime ideals

$$I(V) = P_0 \subset P_1 \subset \dots \subset P_m \subset K[X_1, \dots, X_n].$$

If V has Krull dimension 0 then $I(V)$ is maximal and hence generated by some $X_1 - a_1, \dots, X_n - a_n$.
If $V \subseteq K^n$ is an algebraic variety, by $K(V)$ we mean the fraction field $K[X_1, \dots, X_n]/I(V)$. It is known that the Krull dimension of an ireducible veriety V is equal to the transcendence degree of $K(V)$ over K. We will show in the following exercise that the Krull dimension of V is indeed equal to its Morley degree as a definable set in a model of ACF_0.

Aufgabe 4. Let K be an algebraic closed field and $V \subseteq K^n$ be an irreducible variety. Show that then $\mathrm{RM}(V)$–we mean the Morley rank of the formula that defines V– is equal to the Krull diemension of V.

Hinweis. We prove this by induction on the Krull dimension of V. Show that the exercise is the case when the Krull dimension of V is zero.
Suppose that V has Krull dimension $k > 0$. Suppose that V is defined by ϕ. For each $\bar{a} \in \phi(\mathfrak{C})$ define

$$V_{\bar{a}} := \bigcap_{f(\bar{a})=0} \text{roots of } f.$$

Another way of defining $V_{\bar{a}}$ is to write

$$V_{\bar{a}} = V(I_{\bar{a}})$$

where $I_{\bar{a}}$ is the set of polynomials vanishing at $\bar{a}$. Note that

$$\mathrm{RM}(V) = \max\{\mathrm{RM}(\bar{a}/K) | \bar{a} \in \phi(\mathfrak{C})\}.$$

If $\bar{a}$ is such that $V_{\bar{a}} \subset V$ then by induction hypothesis

$$\mathrm{RM}(\bar{a}/K) \leq \mathrm{RM}(V_{\bar{a}}) \leq k - 1;$$

if $V_{\bar{a}} = V$, then $I_{\bar{a}} = I(V)$ and as a result $K(V) = K(\bar{a})$. We have also proved that

$$\mathrm{RM}(\bar{a}/K) = \dim(\bar{a}/K)$$

and the dimension mentioned above is exactly the transcendence degree of $K(V)$.