Universität Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 1, ws2014-2015
Prof. Dr. Heike Mildenberger
Dr. Mohsen Khani

Blatt 3, Elimination of Quantifiers

Bemerkung. The exercises follow after a short note on quantifier elimination and criteria for checking whether or not a given theory admits elimination of quantifiers. My references are [1], [2], [3]. You can of course skip the note and begin with the exercises!

Insight: $(\mathbb{R}, +, \cdot, 0, 1, <) \models \forall a, b, c(\underbrace{\exists x ax^2 + bx + c = 0}_{\text{a formula with a quantifier}} \leftrightarrow \underbrace{[(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0))]}_{\text{a formula without quantifiers}})$

Definition. T eliminates quantifiers if for every ϕ there is a quantifier free ψ such that

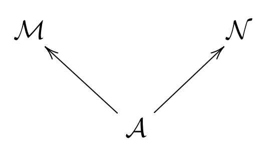
$$T \models \phi \leftrightarrow \psi.$$

We also say that T has/admits quantifier elimination, or it has qe.

Model theory is the study of definable sets. When T admits quantifier elimination, all definable sets can be obtained by Boolean combinations of solution-sets of equations. Quantifier elimination is an 'algebraic property' of a theory (or a structure).

Criteria

Criterion 1. $\phi(\bar{x})$ has a quantifier free equivalent (modulo T) if in all situations as in the diagram below, we have $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{a})$.



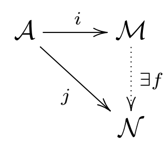
$$\mathcal{M}, \mathcal{N} \models T, \mathcal{A} \subseteq \mathcal{M}, \mathcal{N}, \bar{a} \in A. \quad (1)$$

Criterion 2. T has qe if and only if it eliminates quantifiers from formulas of the form $\exists x \phi(x, \bar{y})$ where ϕ is quantifier free.
Combining 1 and 2: T has quantifier elimination if and only if in all situations as in diagram 1 we have $\mathcal{M} \models \exists x \phi(x, \bar{a}) \Leftrightarrow \mathcal{N} \models \exists x \phi(x, \bar{a})$.
To understand the next criterion better, we need more insight!
Insight. In every field F with characteristic zero we have a copy of $\mathbb{Z}$ because

$$(\underbrace{1 + 1 + \dots}_{n \text{ times, any } n}) \in F.$$

If T is the theory of fields, then $\mathcal{Z} \models T_\forall$ ($\mathcal{Z} = (\mathbb{Z}, +, \cdot, 0, 1)$). $\mathbb{Z}$ is not a field, but it can be extended to $\mathbb{Q}$, which *is* a field and which embeds in all fields with characteristic zero.
Insight. Suppose that $F_1 \subseteq F_2$ are fields. It is important to know whether or not an equation with coefficient in F_1 solvable in F_2, has also a solution in F_1. For example $\mathbb{R} \subseteq \mathbb{C}$. In $\mathbb{C}$ the equation $x^2 + 1$ has a solution but in $\mathbb{R}$ it does not. For a theory with qunatifier elimination, this comes for free. For example, let $\mathcal{A} \subseteq \mathcal{M}$ be models of $\mathrm{Th}(\mathcal{R})$ (for $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$) and $\mathcal{A} \subseteq \mathcal{M}$. If $ax^2 + bx + c$ is a polynomial with coefficients in A that is solvable in M, then $b^2 - 4ac \geq 0$. So it also has a solution in A.

Definition. T has algebraically prime models if for any $\mathcal{A} \models T_\forall$, there is $\mathcal{M} \models T$ and an embedding $i : \mathcal{A} \to \mathcal{M}$ such that for all $\mathcal{N} \models T$ and all $j : \mathcal{A} \to \mathcal{N}$, there is an $f : \mathcal{M} \to \mathcal{N}$ to make the following diagram commute:



(2)

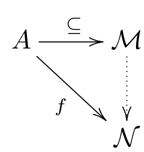
Criterion 3. Suppose that

1. T has algebraically prime models,
2. for every $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$, $\bar{a} \in M$ and quantifier free $\phi(x, \bar{y})$, we have $\mathcal{N} \models \exists x \phi(x, \bar{a}) \Leftrightarrow \mathcal{M} \models \exists x \phi(x, \bar{a})$.

Then T has quantifier elimination.
Criterion 4.(by van den Dries) Suppose that T has at least one constant symbol. T has quantifier elimination if the two following algebraic conditions hold.

1. Every model $\mathcal{M}$ of $T_\forall$ has a T-closure $\overline{\mathcal{M}}$.
2. If $\mathcal{M} \subsetneqq \mathcal{N}$ are models of T, then **there is** a $b \in N - M$ such that $\mathcal{M}(b)$, the $T_\forall$-model generated by b over $\mathcal{M}$, can be embedded into an elementary extension of $\mathcal{M}$.

Criterion 5. T has quantifier elimination if whenever $\mathcal{M}, \mathcal{N} \models T$, $A \subseteq M$, $\mathcal{N}$ is $|M|^+$-saturated and $f : A \to \mathcal{N}$ is partial embedding, f extends to an embedding of $\mathcal{M}$ into $\mathcal{N}$.



(3)

Criterion 6. T has quantifier elimination if and only if T is model-complete and $T_\forall$ has amalgamation property.
Criterion 7. T has quantifier elimination if for every $M, N \models T$ and every $\bar{a} \in M$ and $\bar{b} \in N$,

$$\mathrm{tp}^M_{qf}(\bar{a}) = \mathrm{tp}^N_{qf}(\bar{b}) \text{ implies } \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b}). \tag{4}$$

Exercises

Nur nummerierte Aufgaben sind abzugeben.

Bemerkung. The exercises of this week are from [1] and [2] (with slight changes).

Aufgabe 1. Two structures $\mathcal{A}$ and $\mathcal{B}$ are 'partially isomorphic' if there is a collection $(\mathcal{A}' \cong_{f'} \mathcal{B}' | \mathcal{A}' \subseteq \mathcal{A}, \mathcal{B}' \subseteq \mathcal{B}, f'$ an isomorphism) with the following properties.

1. For each $\mathcal{A}' \cong_f \mathcal{B}'$ in this collection and each $a \in A$ there is $\mathcal{A}'' \cong_{f''} \mathcal{B}''$ in this collection such that $a \in A''$ and f'' extends f.
2. For each $\mathcal{A}' \cong_f \mathcal{B}'$ in this collection and each $b \in B$ there is $\mathcal{A}'' \cong_{f''} \mathcal{B}''$ in this collection such that $b \in B''$ and f'' extends f.

Such a collection is said to have the 'back and forth property' (or to be a back and forth system). Show that partially isomorphic structures are elementary equivalent.

Aufgabe. Let $\{E\}$ be a binary relation symbol. For each of the following theories, either prove that they have quantifier elimination or give an example showing that they do not have quantifier elimination and a natural extension $L' \supseteq L$ in which they do have quantifier elimination.

a) E is an equivalence relation with infinitely many classes of size 2.

b) E is an equivalence relation and it has infinitely many classes all of which are infinite.

c) E is an equivalence relation and it has infinitely many classes of size 2, infinitely many classes of size 3, and every class has size 2 or 3.

d) E is an equivalence relation and it has one class of size n for each $n < \omega$.

Aufgabe. Show that the theory of $(\mathbb{N}, <, 0, s)$ where $s(x) = x+1$ has quantifier elimination and every definable subset of $\mathbb{N}$ is either finite or cofinite (=its complement is finite).

Aufgabe 2. Consider the theory of $(\mathbb{Z}, +, 0, 1)$ in the language where we add predicates p_n for the elements divisible by n. First axiomatise this theory and then prove that it has quantifier elimination. We call this the theory of $\mathbb{Z}$-groups.

Aufgabe 3. Show that in $(\mathbb{Z}, +, 0, 1)$ we cannot define the ordering (also discuss how we can, if we have a symbol for multiplication in the language; see the first Aufgabe, Blatt 1, after Definierbarkeit).

Hinweis. Remember that a definable set is preserved by automorphisms.

Aufgabe 4. Show that modulo the theory T of the structure $(\mathbb{Z}, +, 0, 1, <)$, the formula $\exists y 2y = x$ is not equivalent to a quantifier free formula (, or the set of even numbers is not defined by a quantifier free formula).

Aufgabe 5. We call $\mathcal{M} \models T$ existentially closed if for all quantifier free $\phi(\bar{x}, \bar{y})$, whenever $\mathcal{N} \models T$, $\mathcal{M} \subseteq \mathcal{N}$, $\bar{a} \in M$ and $\mathcal{N} \models \exists \bar{x} \phi(\bar{x}, \bar{a})$, we have $\mathcal{M} \models \exists \bar{x} \phi(\bar{x}, \bar{a})$.

a) Show that if T is $\forall\exists$-axiomatisable then is has existentially closed models. Indeed if $M \models T$ then there is $\mathcal{N} \supseteq \mathcal{M}$ existentially closed with $|N| = |M| + |L| + \aleph_0$.

b) Suppose that T has an infinite non-existentially closed model. Prove that T has non-existentially closed models of cardinality κ for any infinite $k \geq |L|$.

   Hinweis. Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are models of T and $\mathcal{N}$ satisfies an existential formula not satisfied in $\mathcal{M}$. Consider models of the theory of $\mathcal{N}$ where we add a unary predicate for M.

c) Show that if $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M}$ is existentially closed, then there is $\mathcal{M}_1 \models T$ such that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}_1$ with $\mathcal{M} \preceq \mathcal{M}_1$.

d) Show that T is model complete if and only if every model of T is existentially closed (we call T model complete if for all $\mathcal{M} \subseteq \mathcal{N}$, models of T, we have $\mathcal{M} \preceq \mathcal{N}$).

   Hinweis. ($\Leftarrow$) Suppose that $\mathcal{M}_0 \subseteq \mathcal{N}_0$ are models of T. Use c) to build $\mathcal{M}_0 \subseteq \mathcal{N}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_2 \ldots$, a chain of models of T such that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ and $\mathcal{N}_i \preceq \mathcal{N}_{i+1}$.

References

[1] D. Marker. *Model Theory : An Introduction*. Graduate Texts in Mathematics. Springer, 2002.

[2] K. Tent and M. Ziegler. *A Course in Model Theory*. Lecture Notes in Logic. Cambridge University Press, 2012.

[3] Lou van den Dries. The field of reals with a predicate for the powers of two. *Manuscripta Math.*, 54(1-2):187–195, 1985.