Universitt Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 1, ws2014-2015
Prof. Dr. Heike Mildenberger
Dr. Mohsen Khani

Blatt 4, Model Companion, Positive Quantifier Elimination

Aufgabe 1. Suppose that T and T' are L-theories. We say that T' is a **model companion** of T if

1. T' is model-complete (as defined in Blatt 3, Aufgabe 5d),
2. every model of T has an extension that is a model of T', and
3. every model of T' has an extension that is a model of T (2 and 3 together mean: $T_\forall = T'_\forall$).

a) Show that each theory has at most one model companion.

b) Show that DLO (dense linear order without endpoints) is the model companion of the theory of discrete linear orders.

c) Suppose that T is $\forall\exists$ axiomatisable. Show that if T' is a model companion of T, then T' is the theory of existentially closed models of T.

Definition. We say that an L-formula $\phi(\bar{x})$ is **positive** if it is in the smallest collection of L-formulas containing the atomic formulas and closed under $\wedge$, $\vee$, $\exists$ and $\forall$.

Definition. We say that $\eta : \mathcal{M} \to \mathcal{N}$ is an L-homomorphism if:

1. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constants;
2. $\eta(f^{\mathcal{M}}(\bar{x})) = f^{\mathcal{N}}(\eta(\bar{x}))$ for all $\bar{x} \in M$ and function symbols f;
3. if $\bar{x} \in R^{\mathcal{M}}$, then $\eta(\bar{x}) \in R^{\mathcal{N}}$ for all $\bar{x} \in M$ and relation symbols R.

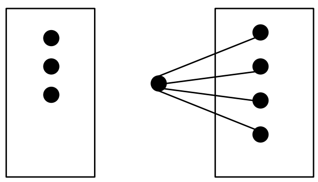
Aufgabe 2. Let T be a complete L-theory and $\phi(\bar{x})$ be an L-formula such that $T \models \exists\bar{x}\phi(\bar{x})$. Show that the following are equivalent:

1. There is a positive quantifier-free formula $\psi(\bar{x})$ such that $T \models \forall\bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.
2. For all $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{A} \subseteq \mathcal{M}$, if $f : \mathcal{A} \to \mathcal{N}$ is an L-homomorphism, $\bar{a} \in A$ and $\mathcal{M} \models \phi(\bar{a})$, then $\mathcal{N} \models \phi(f(\bar{a}))$.

Hinweis. For 2$\to$ 1, put $\Gamma = \{\psi(\bar{x}) : \psi \text{ is positive quantifie free and } T \models \psi(\bar{x}) \to \phi(\bar{x})\}$. Let $\Sigma = T \cup \{\neg\psi(\bar{c}) : \psi \in \Gamma\} \cup \{\phi(\bar{c})\}$. Show that Σ is unsatisfiable.

Definition. A random graph is a graph in which, given any sets $X = \{x_0, \ldots, x_m\}$ and $Y = \{y_0, \ldots, y_n\}$ of vertices with $X \cap Y = \emptyset$, there is a vertex z (with $z \notin Y$) such that there is an edge between z and all elements of X and there no edge between z and any element of Y. So the theory of random graphs is the union of the theory of the graphs with the following axiom scheme:

$$\forall x_1 \ldots x_m \forall y_1, \ldots y_n [\bigwedge_{i,j} \neg x_i = y_j \rightarrow$$
$$\exists z (\bigwedge_{i=1,\ldots,m} zRx_i) \wedge (\bigwedge_{j=1,\ldots,n} \neg zRy_j) \wedge \bigwedge_{j=1,\ldots,n} \neg (z = y_j)].$$



Aufgabe 3.

1. Show that the theory of random graphs has quantifier elimination and is complete;
2. show that it is the model companion of the theory of graphs.

Aufgabe 4. Let K be an algebraically closed field and $D \subseteq K^n$ be definable. Show that every injective polynomial map from D to D is surjective.