Universitt Freiburg, Abteilung für Mathematische Logik
Übung zur Vorlesung Modelltheorie 1, ws2014-2015
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Blatt 5, Real and Algebraically Closed Fields

Nur nummerierte Aufgaben sind abzugeben.

Definition. Let $M \models T$ and A be a subset of M. By $\mathrm{acl}(A)$, algebraic closure of A in M, we mean the set of all y's in M for which there is a formula $\phi(x, \bar{a})$ with parameters $\bar{a}$ in A such that $\phi(M, \bar{a}) = \{y \in M | M \models \phi(y, \bar{a})\}$ is finite and $y \in \phi(M, \bar{a})$.

One can think of $\phi(x, \bar{a})$ as a 'polynomial' with coefficients in A and of y as a root of it. By $\mathrm{dcl}(A)$, definable closure of A in M, we mean the set of y's such that there is an $\bar{a} \in A$ such that y satisfies a formula $\phi(y, \bar{a})$ and y is the only element to satisfy this formula.

Aufgabe 1. Show that

1. in a model of DAG (the theory of torsion-free divisible abelian groups), algebraic closure and definable closure agree (= are the same thing!) and $\mathrm{acl}(A)$ is the $\mathbb{Q}$-vector space span of A.
2. Let $K \models \mathrm{ACF}$ (the theory of algebraically closed fields) and A be a subset of K. Show that $a \in \mathrm{acl}(A)$ if and only if a is algebraic over the subfield of K generated by A. This means that the model theoretic 'algebraic closure' and the algebraic closure in the sense of Algebra coincide for models of ACF.
3. Let $R \models \mathrm{RCF}$ (the theory of real closed fields) and A be a subset of R. Show that $\mathrm{acl}(A) = \mathrm{dcl}(A)$ and $\mathrm{acl}(A)$ is, similar to the previous item, the algebraic closure of the field generated by A in R.

Aufgabe 2. Show that the order on $\mathbb{R}$ is not quantifier-free definable in the language of rings.

Hinweis. Let c_1, c_2 be two algebraically independent elements over $\mathbb{R}$. First show that $\mathbb{R}(c_1, c_2)$, the field generated over $\mathbb{R}$ by c_1 and c_2, is formally real (that means -1 is not a sum of squares). Then note that if F is formally real and $a \in F$ is such that $-a$ is not a sum of squares, then there is an order $<$ on F such that $a > 0$. So there are two orders $<_1$ and $<_2$ on $\mathbb{R}(c_1, c_2)$ both extending the order of $\mathbb{R}$ such that $c_1 <_1 c_2$ and $c_2 <_2 c_1$. Now explain how this means that the order on $\mathbb{R}$ is not quantifier-free definable in the language of rings.

Aufgabe 3. (Real version of Nullstellensatz). Let F be a real closed field and I an ideal in $F[\bar{X}]$. Show that then, $v_F(I)$ is non-empty if and only if whenever $p_1, \ldots, p_m \in F[\bar{X}]$ and $\sum p_i^2 \in I$, then all p_i's are in I. By $v_F(I)$ we mean $\{\bar{a} | \bar{a} \in F \text{ and for all } f \in I \quad f(\bar{a}) = 0\}$.

Definition. We call an ordered structure $(M, <, \ldots)$, o-minimal (order minimal) if every definable subset of M can be defined using only $<$ and $=$; that is every definable subset of M is a finite union of points and intervals in M.

Aufgabe 4.

1. Show that $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$ is o-minimal.
2. Show that every model of $\mathrm{Th}(\mathcal{R})$ is o-minimal.
3. Show that whenever $(F, +, 0, \cdot, <)$ is an o-minimal field, F is real closed (note that a field is real closed if and only if it satisfies the intermediate value property).
4. Suppose that $M = (G, +, <, \ldots)$ is o-minimal and $(G, +, <)$ is an ordered group. Show that G is abelian.
5. In above show that G is also divisible.

Definition. If T' is a model companion (see Aufgabe 1 Blatt 4) of T and $T' \cup \mathrm{Diag}(M)$ is complete for any $M \models T$, then T' is a **model completion** of T. ($\mathrm{Diag}(M)$ is the set of quantifier-free formulas in the language $L(M)$ that hold in M.)

Definition. We say that T has **amalgamation property** if whenever M_0, M_1 and M_2 are models of T and $f_i : M_0 \to M_i$ are embeddings, there is $N \models T$ and $g_i : M_i \to N$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Aufgabe (continued from Aufgabe 1 on Blatt 4)**.**

1. Suppose that T' is a model companion of T. Show that T' is a model completion of T if and only if T has the amalgamation property.
2. Suppose that T has a universal axiomaisation and T' is a model completion of T. Show that T' has quantifier elimination.