**Universitt Freiburg, Abteilung für Mathematische Logik**
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# Blatt 6, Types

**Anwering 2 out of 5 suffices.**

**Aufgabe 1** (quantifier elimination and types)**.** Show that a theory has quantifier elimination if and only if every type $p$ is implied by the quantifier free formulas in $p$. Let us also express the 'if' condition this way: for every $M, N \models T$ and $\bar{a} \in M$ and $\bar{b} \in N$

$$tp_{qf}^M(\bar{a}) = \mathrm{tp}_{qf}^N(\bar{b}) \text{ implies } \mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(\bar{b})$$

where $\mathrm{tp}_{qf}^M(\bar{a})$ is the the set of quantifier free formulas satisfied by $\bar{a}$ in $M$.

**Aufgabe 2** (describing types in RCF)**.**

1. Describe 1-types in models of RCF: let $R$ be a real closed field. Show that 1-types over $R$ (=types in $S^1(R)$) correspond to cuts in the ordering $(R, <)$. (This means, supposing that $R$ is a model of RCF and $R \subseteq A \models$RCF and $A$ is $|R|^+$-saturated and $x \in A - R$, then $\mathrm{tp}^A(x/R)$ is determined by the cut of $x$ in $R$; that is if $x$ and $y$ in $A$ are such that for all $a_1, a_2$ in $R$, $a_1 < x < a_2$ if and only if $a_1 < y < a_2$ then $\mathrm{tp}^A(x/R) = \mathrm{tp}^A(y/R)$. By $A$ being $|R|^+$-saturated we mean that all types in $S^1(R)$ are indeed types of elements in $A$: $p \in S(R)$ then $p = \mathrm{tp}^A(x/R)$ for some $x \in A$).

2. Show that RCF has no countable saturated models: $T$ has a countable saturated model if and only if $|S_n(T)| \leq \aleph_0$ for all $n$. You need to characterise types in $S_1(\mathrm{RCF})$ and show that $|S_1(\mathrm{RCF})| = 2^{\aleph_0}$.

   (In case it is not yet covered in the lecture, a model $M$ of $T$ is called saturated if every consistent set of formulas in variables $\bar{x}$ is realised in $M$ by some $\bar{a}$).

**Aufgabe 3.**

1. Show that a type $p$ in $S_n(T)$ is isolated if $\{p\} = [\phi]$ for some $\phi$; this means that $p$ is an *isolated point* in the Stone topology.

2. Suppose that $M$ is an $L$-structure, $A$ is a subset of $M$ and $b \in M$ is algebraic over $A$ (= it is in $\mathrm{acl}^M(A)$, see Blatt 5 for the definition). Show that $\mathrm{tp}^M(b/A)$ is isolated.

**Aufgabe 4.**

1. In $L$ the language of DLO, prove that if $a, b \in \mathbb{Q}$, then $\mathrm{tp}^{\mathbb{Q}}(a/\mathbb{N}) = \mathrm{tp}^{\mathbb{Q}}(b/\mathbb{N})$ if and only if there is an automorphism $\sigma$ of $\mathbb{Q}$ that fixes $\mathbb{N}$ pointwise and sends $a$ to $b$.

2. Let $A = \{1 - \frac{1}{n} : n = 1, \ldots\} \cup \{2 + \frac{1}{n} | n = 1, \ldots\}$. Show that 1 and 2 realise the same types over $A$, but there is no automorphism of $\mathbb{Q}$ fixing $A$ pointwise sending 1 to 2.

3. Is the previous item contradictory with what you expect from elements of the same type? Perhaps, you can formulate a theorem that relates 'realising the same type' to 'one being sent to the other by an automorphism'.

**Aufgabe 5** (describing types in ACF)**.** Suppose that $K \models \mathrm{ACF}$ and $k \subseteq K$ is a field. Show that $n$-types over $k$ are determined by prime ideals in $k[X_1, \ldots, X_n]$: for every type $p$ find a prime ideal $I_p$ such that the map $p \mapsto I_p$ is a bijection between $S_n^K(k)$ and $\operatorname{Spec} k[X_1, \ldots, X_n] = \{\text{prime ideals of } k[X_1, \ldots, X_n]\}$. (if interested, prove that this map is continuous.)

**Hinweis** (Note that this hint spoils the exercise!)**.** To avoid confusion, let me first mention that the letter $p$ is for a type and the letter $P$ is for a prime ideal.

First, prove that given a type $p$ in $S_n^K(k)$, the set $I_p = \{f | \text{the formula } f(\bar{X}) = 0 \text{ is in } p\}$ is a prime ideal of $K[\bar{X}]$.

For converse, Consider the prime ideal $P$ in $k[\bar{X}]$. It is of the form

$$Q \cap k[\bar{X}]$$

for a prime ideal $Q$ in $K[\bar{X}]$ (we take this for granted, but if you are interested in proving it, one of the main ingredients you may need is noetherianity of $K[\bar{X}]$).

Now, since $Q$ is prime, $K[\bar{X}]/Q$ is an integral domain. Let $F_1$ be the fraction field of $K[\bar{X}]/Q$. I remind you quickly that $D$ is called an integral domain if $ab \neq 0$ for all non-zero $a$ and $b$. If $D$ is an integral domain, then $F := \{\frac{a}{b} | a, b \in D\}$ with addition and multiplication of fractions is a field and is called the filed of fractions of $D$.

Let $F_2$ be the algebraic closure of $F_1$; luckily you have proved in Blatt 5 that 'algebraic closure' is the same thing in both model theoretic and algebraic senses.

Since ACF is model-complete, $F_2$ is an elementary extension of $K$:

$$k \subseteq K \preceq F_2 = \operatorname{acl}(\operatorname{Frac}(K[X_1, \ldots X_n]/Q)).$$

Consider elements $X_1/Q, \ldots, X_n/Q$ in $F_2$. Let $p$ be the type of the tuple $(X_1/Q, \ldots, X_n/Q)$ over $k$; that is

$$p := \operatorname{tp}^{F_2}((X_1/Q, \ldots, X_n/Q)/k).$$

Show that $I_p = P$.

Now, using the information above and quantifier elimination of ACF, show that $p \mapsto I_p$ is a bijection.