Universitt Freiburg, Abteilung für Mathematische Logik
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Solving 2 out of 5 exercises suffices.

Blatt 9, Fraïssé's Construction

Aufgabe 1 (from [1])**.** Let p be a prime number and let $\mathbf{K}$ be the class of all finite fields of characteristic p. Show that $\mathbf{K}$ has heredity property, joint embedding property and amalgamation property, and the Fraïssé's limit of $\mathbf{K}$ is the algebraic closure of the prime field of characteristic p. (We need this observation that finite integral domain are fields because in Fraïssé's limit we talk about finitely generated **structures** not models.)

General algebra preliminaries

You may need some algebra of fields with finite characteristic. Let F be a field. Then $\operatorname{Char} F$ is the smallest n such that $n.1 = 0$. $\operatorname{Char} F$ is either a prime number, or it does not exist in which case we say $\operatorname{Char} F = 0$. Every field F with $\operatorname{Char} F = 0$ contains a copy of $\mathbb{Q}$ and every field F with $\operatorname{Char} F = p$ contains a copy of $\mathbb{Z}_p$.

If the filed F is finite, then $\operatorname{Char} F = p$ for some prime p. Also since F contains $\mathbb{Z}_p$, it is a vector space over $\mathbb{Z}_p$ and hence **as a vector space**,

$$F \cong \overbrace{\mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p}^{n \text{ times}}$$

for some n, that is the size of a finite field is always p^n for some p and n. Note that I haven't claimed that $\mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$ is a field! More interestingly, a finite field with p^n elements is the smallest field containing $\mathbb{Z}_p$ that includes all solutions of the equation

$$x^{p^n} = x$$

where $x^{p^n} - x \in \mathbb{Z}_p[X]$. This is also called the **splitting field** of the polynomial $x^{p^n} - x = 0$ over $\mathbb{Z}_p$.

Aufgabe 2 (from [1])**.** Let $\mathbf{K}$ be the class of finitely generated torsion-free abelian groups. Show that $\mathbf{K}$ has heredity property, joint embedding property and amalgamation property, and that the Fraïssé's limit of K is the direct sum of countably many copies of the additive group of rationals. Also discuss why countably many copies of additive group of integers is not the limit (it has to do with saturatedness).

Aufgabe 3. Let $\mathbf{K}$ be the class of all finite graphs. Show that the Fraïssé's limit of $\mathbf{K}$ is the countable random graph. Note that proving this, you will have also shown that the theory of random graphs has quantifier elimination.

Aufgabe 4.

1. Let **K** be the skeleton of M and M be **K**-saturated and countable. Show that M is ultrahomogeneous, meaning that each automorphism between finitely generated substructures of M extends to an automorphism of M.

2. Show that any two **K**-saturated structures are partially isomorphic (=there is a set of isomorphism between their substructures with the back and forth property).

Aufgabe 5 (back to types and $\aleph_0$-categoricity!)**.**

1. Suppose that T is $\aleph_0$-categorical and $M \models T$ and A is a finite subset of M. Show that $\mathrm{acl}(A)$ is finite.

2. Show that the theory of $(\mathbb{R}, 0, +)$ has exactly two 1-types and $\aleph_0$-many 2-types.

References

[1] W. Hodges. *Model Theory.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2008.