NIP Seminar

Model Theory 2, University of Freiburg

(Reference: A guide to NIP theories, Simon)

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**Definition 1 (Independence Property)** A formula $\phi(x, y)$ has IP (the independence property) if there are sequences $(a_i)_{i \in \omega}$ and $(b_I)_{I \subseteq \omega}$ such that

$$\models \phi(a_i, b_I) \iff i \in I$$

A theory $T$ is called NIP if there is no formula in it with IP. If $\phi(x, y)$ is NIP, then there is a maximal integer $n$ such that there are no sequences $(a_i)_{i=1,...,n}$ and $(b_I)_{I \subseteq \{1,2,...,n\}}$ for which

$$\models \phi(a_i, b_I) \iff i \in I$$

This integer $n$ is called the VC-dimension of $\phi(x, y)$.

**Fact 2**

1. $\phi(x, y)$ is NIP if and only if $\phi(y, x)$ is NIP.

2. $\phi(x, y)$ has IP if and only if there is an indiscernible sequence $(a_i)_{i \in \omega}$ and a tuple $b$ such that

$$\models \phi(a_i, b) \iff i \text{ is even.}$$

3. $T$ is NIP if and only if all formulae $\phi(x, y)$ with $|y| = 1$ are NIP.

**Theorem 3 (Baldwin-Saxl)** Assume that $T$ is NIP, $G$ is a group definable in $T$, and $(H_a)_{a \in B}$ is a definable family of subgroups of $G$. Then there is an integer $N$ such that

$$\forall A_{\text{finite}} \exists A_0 \subseteq A \ (|A_0| \leq N \text{ and } \bigcap_{a \in A} H_a = \bigcap_{a \in A_0} H_a.)$$
**Definition 4 (Invariant types)** A global type $p \in S_\omega(M)$ is called $A$-invariant ($A$ is small) if

$$\forall \sigma \in \text{Aut}(M/A) \quad \sigma(p) = p.$$ 

That is, for each $a, b$ in the super-monster model

$$a \equiv_A b \Rightarrow \text{tp}(a/M) = \text{tp}(b/M).$$

**Fact 5**

- If $p$ is a definable type, then it is invariant. Indeed if $p$ is definable, then it is definable over some $A$ with $|A| \leq |T|$ and $p$ is $A$-invariant.
- If $p$ is finitely satisfiable in $A$ then it is $A$-invariant.

**Fact 6** Two ways for obtaining finitely satisfiable types:

- Let $A$ be a parameter set and $D$ an ultra-filter on $A^n$. The type $p_D$ defined as in the following is finitely satisfiable in $A$:

  $$\phi(x, b) \in p_D \iff \phi(A, b) \in D.$$ 

  Every global type finitely satisfiable $A$ is of the form $p_D$ for some ultra-filter $D$.

- Let $I = (a_i | i \in J)$ be indiscernible and $T$ be NIP. The type $p := \lim_{i \in J} \text{tp}(a_i/M)$ is finitely satisfiable in $I$ (the limit makes sense in the Stone topology.)

**Definition 7 (defining scheme of a type)** Let $p$ be global and $A$-invariant. For each $\phi(x, y) \in L$, let $D_\phi := \{\text{tp}(b/A) | \phi(x, b) \in p\}$. The family $\{D_\phi\}_{\phi \in L}$ is called the defining schema of $p$, and it determines $p$. Using this schema we can extend $p$ from $M$ to $M' \supset M$ by defining

  $$\phi(x, b) \in p \iff \text{tp}(b/A) \in D_\phi \text{ for each } b \in M'.$$

**Lemma 8** Let $p$ be global and $A$-invariant.

- If $p$ is definable, then it is $A$-definable.
- If $p$ is finitely satisfiable in some small set, then it is finitely satisfiable in some model $M \supset A$. 

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Definition 9 (Product of Types) Let $p(x)$ and $q(y)$ be global types and $p$ be $A$-invariant. We define $p(x) \otimes q(y)$ (a type with variables $x$ and $y$) by

$$p(x) \otimes q(y) = \text{tp}(ab/M)$$

where $b \models q$ and $a \models p|_{Mb}$

- $\otimes$ is associative, but, in general, not commutative.

- $p \otimes q$ is $A$-invariant if and only if $q$ is.

Definition 10 (Morley Sequence) Let $p$ be global and $A$ invariant and $B \supseteq A$. Let

$$p(n)(x_0, \ldots, x_n) := p(x_n) \otimes p(x_{n-1}) \otimes \cdots \otimes p(x_0),$$

and

$$p(\omega)(x_0, x_1, \ldots) := \lim p(n)(x_0, \ldots, x_n).$$

Each realization $(a_i)_{i \in \omega}$ of $p(\omega)|_B$ is called a Morley sequence of $p$ over $B$. Such a sequence is indiscernible over $B$ and $\text{tp}(a_1, \ldots, a_n/B) = p(n)|_B$.

We defined Morley sequences in model theory 2 as follows: $I = (a_i)_{i \in \omega}$ is Morley in $p$ over $A$ if each $a_i$ is a realisation of $p$ and $a_i \downarrow_A a_1 \ldots a_{i-1}$. Read example 7.2.10 (and exercise 7.1.4) in Ziegler-Tent model theory book to see why these two definitions are equivalent.

Lemma 11 Assume that $p(x)$ and $q(y)$ are global types, $p$ is definable, and $q$ is finitely satisfiable in some small model. Then $p(x) \otimes q(y) = q(y) \otimes p(x)$ (in particular, in stable theories, $\otimes$ commutes).

Theorem 12 (generically stable types) Let $T$ be NIP and $p$ be a global $A$-invariant type. Then the following are equivalent (and the type $p$ that satisfies these, is said to be generically stable)

1. $p = \lim(I)$ for each $I \models p(\omega)|_A$ (that is $p$ is the limit of each Morley sequence over $A$ in $p$).

2. $p$ is definable and finitely satisfiable in some small model $M$.

3. $p_x \otimes p_y = p_y \otimes p_x$.

4. Each Morley sequence of $p$ is totally indiscernible.
Definition 13 (a sequence being based on a set) Let $A \subseteq \mathcal{M}$ and let $I$ be an indiscernible sequence. We say that $I$ is based on $A$ if it is indiscernible over $A$ and for all $I_1, I_2 \models EM(I)$ (where $EM$ is for Ehrenfeucht-Mostowski type), there is some $a$ such that $a + I_1$ ($a$ is added to the sequence) and $a + I_2$ are both indiscernible over $A$.

Proposition 14 (eventual types) Let $T$ be NIP. Let $I$ be based on $A$. There is a unique global type $p$ with the following property (and this type is called the eventual type of $I$ over $A$, and is denoted by $Ev(I/A)$): For each $J \models EM(I/A)$ and each $B \subseteq \mathcal{M}$, there is $a \models p|_B$ such that $J + a$ is $A$-indiscernible. Moreover $p$ is invariant over $A$.

Definition 15 (dense meet-trees) A tree is a partially ordered set $(M, \leq)$ such that for each $a \in M$, the set $\{x \in M | x \leq a\}$ is linearly ordered by $\leq$ and for each $a, b \in M$ there is $c$ such that $c \leq a$ and $c \leq b$. We call $(M, \leq)$ a meet tree if in addition, for each $a, b \in M$ the set $\{x \in M | x \leq a, x \leq b\}$ has a greatest element, which we denote by $a \land b$. The theory of dense meet-trees, $T_{dt}$, is the theory defined by the following axioms:

- $\leq$ defines a meet tree and $\land$ is the meet relation.
- for each $c$ the set $\{x | x \leq c\}$ is dense and has no first element.
- for each $c$ there are infinitely many open cones centered at $c$.

$T_{dt}$ is $\aleph_0$ categorical and its unique countable model is the Fraïssé limit of finite meet trees. $T_{dt}$ is also NIP.

Remark 16 (indiscernible sequences in $T_{dt}$) Let $I = (a_i)_{i \in \omega}$ be an indiscernible sequence of single elements in $T_{dt}$. There are 6 possibilities for $I$:

1. $I$ is constant.
2. $I$ is increasing.
3. $I$ is decreasing.
4. Elements of $I$ are pairwise incomparable and $a_i \land a_j$ is constant for all $i, j$.
5. Elements of $I$ are pairwise incomparable and $a_i \land a_j$ for $i < j$ depends only on $i$ and is increasing with $i$. 

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6. Elements of \( I \) are pairwise incomparable and \( a_i \wedge a_j \) depends only on \( j \) and is decreasing with \( j \).

**Definition 17** Let \( \pi(x) \) be a partial unary type over \( \emptyset \). We say that \( \pi(x) \) is stably embedded if for every formula \( \phi(x_1, \ldots, x_n, b) \) (\( b \in \mathbb{C} \)) there is a formula \( \psi(x_1, \ldots, x_n, z) \) and \( d \in \pi(\mathbb{C}) \) such that

\[
\forall \bar{a} \in \pi(\mathbb{C}) \quad \phi(\bar{a}, b) \iff \psi(\bar{a}, d).
\]

If \( \pi \) is stably embedded, then \( \psi \) can be chosen so that it depends only on \( \phi \) and not the parameter \( b \). A stably definable set is defined similarly.

**Definition 18** Let \( A \subseteq \mathbb{C} \) be any set and \( B \subseteq \mathbb{C} \) be a small set of parameters. By \( A_{\text{ind}(B)} \), the induced structure over \( A \) by formulae in \( L(B) \), we mean the structure in the language \( L_B = \{ R_{\phi(x)}(\bar{x}) | \phi(x) \in L(B) \} \) whose universe is \( A \) and

\[
A_{\text{ind}(B)} \models R_{\phi}(\bar{a}) \iff \mathbb{C} \models \phi(\bar{a}).
\]

If \( M \prec N \) and \( N \) is \( |M|^+ \)-saturated, then \( M_{\text{ind}(N)} \) is called the Shelah expansion of \( M \) and is denoted by \( M^{sh} \). \( M^{sh} \) has quantifier elimination and is NIP.

**Definition 19** An externally definable subset of a model \( M \) is a subset \( D \subseteq M^k \) where \( D = \phi(M, b) \) for some \( b \in \mathbb{C} \) (where \( b \) may not be in \( M \)).

**Theorem 20** (Honest Definition) Let \( M \models T \), \( A \subseteq M \), \( \phi(x, y) \in L \) and \( b \in M \) be a \( |y| \)-tuple. Assume that \( \phi(x, y) \) is NIP. Then there is an elementary extension \( (M, A) \prec (M', A') \) and a formula \( \psi(x, z) \in L \) and a tuple \( d \) in \( A' \) such that

\[
\phi(A, b) \subseteq \psi(A', d) \subseteq \phi(A', b).
\]

(\( \psi(x, d) \) is called the honest definition of \( \phi(x, b) \)).

**Corollary 21** Assume that \( T \) is NIP. Let \( M \models T \) and \( A \subseteq M \). Let \( b \in M \) be a finite tuple. Let \( (M, A) \prec (M', A') \) be an \( |M|^+ \)-saturated extension. Then there is \( A_0 \subseteq A' \) of size at most \( |T| \) such that for all \( a, a' \in A^k \) we have

\[
a \equiv_{A_0} a' \Rightarrow a \equiv_{b} a'.
\]

**Lemma 22** (Shrinking of an indiscernible sequence) Let \( T \) be NIP. Let \( I = (a_i)_{i \in J} \) be an indiscernible sequence. Let \( \phi(\bar{x}, b) \in L(\mathbb{C}) \). Then there is a finite convex equivalence relation \( \simeq \) on \( J \) such that for all \( \bar{i}, \bar{j} \in J^n \) we have

\[
\bar{i} \simeq \bar{j} \Rightarrow \phi(a_{\bar{i}}, b) \iff \phi(a_{\bar{j}}, b).
\]