The field of reals with a predicate for the real algebraic numbers and a predicate for the integer powers of two

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Abstract

Given a theory T of a polynomially bounded o-minimal expansion R of $\bar{\mathbb{R}} = \langle \mathbb{R}, +, ., 0, 1, < \rangle$ with field of exponents $\mathbb{Q}$, we introduce a theory $\mathbb{T}$ whose models are expansions of dense pairs of models of T by a discrete multiplicative group. We prove that $\mathbb{T}$ is complete and admits quantifier elimination when predicates are added for certain existential formulas. In particular, if $T = \mathrm{RCF}$ then $\mathbb{T}$ axiomatises $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$, where $\mathbb{R}_{alg}$ denotes the real algebraic numbers. We describe types and definable sets in our models and prove that $\mathbb{T}$ is dependent.
Keywords. O-minimality, dense pairs, integer powers of two.

1 Introduction

Throughout, $\bar{\mathbb{R}}$ is the structure $\langle \mathbb{R}, +, ., 0, 1, < \rangle$ and L_{or} its language. In [2] van den Dries proved quantifier elimination for the theory of the structure

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$\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ in a language L^* containing as well as usual L_{or}-symbols, a predicate symbol (in our notation) G, predicates $\{P_n\}_{n\in\mathbb{N}}$, and a function symbol λ. In $\mathbb{R}$ the predicate G is interpreted as the the set of the integer powers of two, $2^{\mathbb{Z}}$. Each P_n represents the set $2^{n\mathbb{Z}}$ and to each $x \in \mathbb{R}$, the function λ assigns the largest integer power of two less than or equal to x. In fact, $\langle 2^{\mathbb{Z}}, \{2^{n\mathbb{Z}}\}_{n\in\mathbb{N}}, ., < \rangle$ is a model of Presburger arithmetic and the quantifier elimination of this structure plays role in the proof. By quantifier elimination, every definable subsets of $\mathbb{R}$ is the union of an open set and a discrete set. Hence $\mathbb{Z}$ is not definable, and therefore this structure is not subject to the Gödel Phenomenon.

In [6] Günaydin extends the results of [2] to expansions of the field of reals with a multiplicative subgroup generated by 2 and 3 and a further predicate for the subgroup generated by 2, and notices in particular that in this structure, $3^{\mathbb{Z}}$ is not definable. In [9] Hieronymi proves that for every $(\alpha, \beta) \in \mathbb{R}^2$, if $\log_\alpha(\beta) \notin \mathbb{Q}$ then the structure $\langle \bar{\mathbb{R}}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}} \rangle$ defines $\mathbb{Z}$.

In [3] van den Dries proved that given a complete o-minimal theory (in our notation) T' which extends the theory of ordered abelian groups, the theory T'_d whose models $\langle M, N \rangle$ are dense pairs of models of T' is complete. He formulated T'_d in a language L'_d comprising of L', the language of the o-minimal theory T', and a predicate U for N and proved that every L'_d-formula is equivalent to a Boolean combination of formulas of the form $\exists \bar{y} \quad U(\bar{y}) \wedge \phi(\bar{x}, \bar{y})$ with $\phi(x, y)$ an L'-formula. A particular example of a dense pair of o-minimal structures is $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$. He further proved that every closed definable subset of $\mathbb{R}$ is a finite union of points and open intervals and therefore $\mathbb{Z}$ is not definable in this structure. He characterises types and definable sets in models of T'_d and proves that the open L'_d-definable sets are L'-definable.

Our aim is to bring together the quantifier elimination theorems of van den Dries referred to, to axiomatise and prove a similar elimination theorem for $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ in a joint language of L^* and L'_d and observe that $\mathbb{Z}$ is not definable in this structure. The idea of amalgamating these two proofs is suggested by Miller in [11] in order to prove that every open definable set in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ can be defined in $\langle \mathbb{R}, 2^{\mathbb{Z}} \rangle$. As the question of open definable sets is studied by Fornasiero in [4] and [5] we focus only on describing formulas, definable sets and types, and then using the methods developed by Hieronymi and Günaydin in [7] we will prove that the theory of this structure is dependent.

Let us fix an expansion R of $\bar{\mathbb{R}}$ with $T = \mathrm{Th}(R)$ and $L = L(T)$. We will assume that T is o-minimal and polynomially bounded with the field of exponents $\mathbb{Q}$. The latter notion will be explained in subsection 1.1. Let $\mathbb{L}$ be the language $L \cup \{N_{\mathbb{L}}, G_{\mathbb{L}}, \lambda_{\mathbb{L}}, \{P_{n_{\mathbb{L}}}\}_{n\in\mathbb{N}}\}$ where $N_{\mathbb{L}}$ and $G_{\mathbb{L}}$ are unary predicates respectively for the dense substructure and the discrete subgroup, $\lambda_{\mathbb{L}}$ is a unary function symbol and $\{P_{n_{\mathbb{L}}}\}_{n\in\mathbb{N}}$ are countably many unary predicates. We drop the index $\mathbb{L}$ for the elements of the language when it causes no confusion with the components of the structure $\mathbb{M} = \langle M, N, G, \lambda, \{P_n\}_{n\in\mathbb{N}}\rangle$.

Let $\mathbb{T}$ be a theory in the language $\mathbb{L}$ every model $\mathbb{M} = \langle M, N, G, \lambda, \{P_n\}_{n\in\mathbb{N}}\rangle$ of which satisfies the following axioms:

1. Axioms expressing that $M \models T$ and $N \prec M$ is dense in M and $N \neq M$.
2. $G \subseteq N$.
3. Axioms for G:
    - G is a multiplicative subgroup of positive elements of M.
    - $2 \in G \wedge \forall x(1 < x < 2 \rightarrow x \notin G)$.
    - $\forall x > 0 \quad \exists y \in G \quad y \leq x < 2y$.
4. Axioms for P_n ($n \in \mathbb{N}$):
    - $\forall x \quad P_n(x) \leftrightarrow \exists y \in G \quad x = y^n$.
    - Axioms expressing that for each $x \in G$ and each $n \in \mathbb{N}$, one and only one of $\{x, 2x, \ldots, 2^{n-1}x\}$ is in P_n.
5. Axioms for λ:
    - $\forall x \quad \lambda(x) \in G$.
    - $\forall x \quad \lambda(x) \leq x < 2\lambda(x)$.
    - $\forall x \quad x \leq 0 \rightarrow \lambda(x) = 0$.

Theorem 1.

1. The theory $\mathbb{T}$ is complete.
2. Every $\mathbb{L}$-formula (with free variables $\bar{z}$) has an equivalent which is a Boolean combination of formulas of the form

$$\exists \bar{x} = (x_1, \ldots, x_n)\exists \bar{y} = (y_1, \ldots, y_m)$$
$$\Big(\bigwedge_{i=1,\ldots,n} N_{\mathbb{L}}(x_i) \wedge \bigwedge_{i=1,\ldots,m} G_{\mathbb{L}}(y_i) \wedge \phi(\bar{x}, \bar{y}, \bar{z})\Big)$$

with $\phi(\bar{x}, \bar{y}, \bar{z})$ in L.

3. $\mathbb{T}$ is dependent.

Note that our structures in particular generalises the structure $\langle R, 2^{\mathbb{Z}} \rangle$ for R as above, considered by Miller in [11]. Also note that, if $R = \bar{\mathbb{R}}$, then $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}}, \lambda, \{2^{n\mathbb{Z}}\}_{n\in\mathbb{N}} \rangle$ is axiomatised by $\mathbb{T}$, where for each x, $\lambda(x)$ is the largest integer power of two less than or equal to x.

We will prove *1* and *2* in section 2, then in section 3 we use *2* to describe definable sets and types, and finally, in section 4, we will prove *3*. In the following subsection we explain the valuation inequality which is an ingredient of our proofs in the next section.

1.1 Valuation Inequality

If $\mathbb{M} = \langle M, N, G, \lambda, \{P_n\}_{n\in\mathbb{N}} \rangle$ is a model of $\mathbb{T}$, then M and N are real closed fields. Finite and infinitesimal elements of M are respectively the sets:

$$\begin{aligned} \mathrm{Fin}(M) &= \{x \in M : \quad \exists n \in \mathbb{N} \quad |x| < n\}, \\ \mu(M) &= \{x \in M : \quad \forall n \in \mathbb{N} \quad |x| < 1/n\}. \end{aligned}$$

$\mathrm{Fin}(M)$ is a valuation ring for M. We say x, y are in the same Archimedean class if x/y and y/x are both in $\mathrm{Fin}(M)$. We denote by $\Gamma(M)$ or Γ, the group of all Archimedean classes of M. The valuation ring $\mathrm{Fin}(M)$ is then obtained by the standard valuation on M which sends each nonzero element x of M to its Archimedean class $v(x)$ in Γ.

In Γ the order is defined by $v(x) > 0 \leftrightarrow |x| \in \mu(M)$. Since by our axioms, for each positive x we have $\lambda(x) \leq x < 2\lambda(x)$, each Archimedean class of M in $\mathbb{M}$ is represented by an element of G and $v(G) = \Gamma$ and $G/2^{\mathbb{Z}} \cong \Gamma$.

Clearly we can also define $\mathrm{Fin}(N)$ and $\mu(N)$ and as N is dense in M and $G \subseteq N$, the valuation groups of N and M coincide.

Since M is real closed, Γ is divisible and hence so is $G/2^{\mathbb{Z}}$, implying that $\langle G, ., <, \{P_n\} \rangle$ is a model of Presburger arithmetic (this can also be inferred from items 3 and 4 in the axioms).

The field of exponents of R is the following field, denoted by K:

$$K = \{r \in \mathbb{R} : \text{ the function } x \to x^r \text{ on } (0, \infty) \text{ is 0-definable in } R\}.$$

We call R polynomially bounded (Miller, [10]) if for every unary definable function f there exists some $N \in \mathbb{N}$ such that for all sufficiently large positive elements x of $\mathbb{R}$ we have $|f(x)| \leq x^N$.

In [10] it is proved that if $\mathcal{R}$ is an expansion of $\langle \mathbb{R}, < \rangle$ then either $\mathcal{R}$ defines e^x, or for every ultimately nonzero $\mathcal{R}$-definable $f : \mathbb{R} \to \mathbb{R}$ there exists a nonzero $c \in \mathbb{R}$ and a 0-definable real power function x^r such that $f(x) = cx^r + o(x^r)$.

For the rest of the paper we keep our assumptions that T is o-minimal, polynomially bounded and has field of exponents $\mathbb{Q}$ (i.e. $K = \mathbb{Q}$). With these assumptions in place, we have:

Proposition 2 (The valuation inequality, Wilkie, van den Dries, Speisseger, [12], [8])**.** Let $M_1, M_2 \models T$ and $M_1 \preceq M_2$. Then $\dim_{\mathbb{Q}}(\frac{\Gamma(M_2)}{\Gamma(M_1)}) \leq \mathrm{rank}_{M_1}(M_2)$.

In the above proposition, $\mathrm{rank}_{M_1}(M_2)$ is the cardinality of a basis for M_2 over M_1 regarding the fact that the definable closure in o-minimal theories is a pregeometry, and $\dim_{\mathbb{Q}}(\frac{\Gamma(M_2)}{\Gamma(M_1)})$ is the dimension as $\mathbb{Q}$-vector spaces. We will need a special case of this proposition for when $\mathrm{rank}_{M_1}(M_2) = 1$. In this case the above proposition implies that when we expand a model of T with one element, then we obtain at most one new Archimedean class (up to linear independence over $\mathbb{Q}$). More precisely, if $x \in M_2 - M_1$ is such that $v(x) \notin \Gamma(M_1)$ then for every element t in $M\langle x \rangle$, $v(t)$, the Archimedean class of t, has a representative in $\Gamma(M_1).v(x)^{\mathbb{Q}} \subseteq \Gamma(M_2)$.

1.2 Notation

Models of $\mathbb{T}$ are denoted by blackboard-bold letters, say $\mathbb{M}$, and it is always assumed that $\mathbb{M}$ is the structure $\langle M, N, G, \lambda, \{P_n\}_{n \in \mathbb{N}} \rangle$. We usually drop the index $n \in \mathbb{N}$. We often fix $\mathbb{M}$ and prove statements about its parts M, N, G, etcetera. Apart from $\mathbb{M}$, we have used the same notation for a structure and its universe. By $M\langle x \rangle$ we mean the L-definable closure of $M \cup \{x\}$. For a set Λ, by $M\langle \Lambda \rangle$ we mean the L-definable closure of $M \cup \{x : x \in \Lambda\}$. The notion of rank and independence throughout the paper and the notation $\mathrm{rank}_{M_1}(M_2)$ is as pointed out after Proposition 2.

As mentioned earlier, to prove the completeness of $\mathbb{T}$, we rely extensively on the techniques in [2] and [3].

2 Towards the proofs

For the back and forth argument that we will employ in the proof of Theorem 1, we need the following definitions and lemmas. The following definition is an adaption of a similar definition from [3] to our setting.

Definition 2.1. Let $\mathbb{M} = \langle M, N, G, \lambda, \{P_n\}\rangle \subseteq \mathbb{M}_1$, where $\langle M, N\rangle$ is an elementary pair of models of T. We say that $\mathbb{M}_1$ is a *free* extension of $\mathbb{M}$, or $\mathbb{M}$ is a free substructure of $\mathbb{M}_1$, if for every $Y \subseteq N_1$, in M_1, Y is independent over N if and only if it is independent over M.

Under the conditions of the above definition, if $Z \subseteq N_1$, then $M\langle Z\rangle \cap N_1 = N\langle Z\rangle$.

We will also need the following lemma whose statement and proof can be found in [3].

Lemma 3 ([3])**.** Let T be a complete o-minimal theory which extends the theory of ordered abelian groups. Then

i) if the dense pair $\langle M, N\rangle$ of models of T is κ-saturated for $\kappa > |T|$, then $\mathrm{rank}_N(M) \geq \kappa$.

ii) If $\langle M, N\rangle$ is a dense pair of models of T, then $M - N$ is dense in M.

In the following lemmas we identify the structure generated by elements over a given structure.

Lemma 4. Let $\mathbb{M}_1 = \langle M_1, N_1, G_1, \lambda_1, \{P_{1n}\}_{n\in\mathbb{N}}\rangle$ be a free extension of $\mathbb{M}$ and $x \in G_1 - M$. Then $M\langle x\rangle$ is closed under λ_1 and $\langle M\langle x\rangle, N\langle x\rangle, M\langle x\rangle \cap G_1, \lambda_1|_{M\langle x\rangle}, \{P_{1n}|_{M\langle x\rangle}\}_{n\in\mathbb{N}}\rangle$ is a free substructure of $\mathbb{M}_1$.

Proof. It follows from the freeness of $\mathbb{M}_1$ over $\mathbb{M}$ that $M\langle x\rangle \cap N_1 = N\langle x\rangle$. We need to prove is that $M\langle x\rangle$ is closed under λ_1. This simply implies that $M\langle x\rangle \cap G_1 = \lambda_1(M\langle x\rangle)$ and $\langle M\langle x\rangle \cap G_1, P_{1n}|_{M\langle x\rangle}, ., <\rangle$ satisfies axioms 3,4,5.

Let $t \in M\langle x\rangle$. By the valuation inequality, the Archimedean class of t has a representative in $\Gamma(M).v_1(x)^{\mathbb{Q}}$. Now as mentioned in Subsection 1.1, $\Gamma(M) \cong G/2^{\mathbb{Z}}$, hence there is $a \in G$ and $q \in \mathbb{Q}$ such that $\frac{t}{ax^q} \in \mathrm{Fin}\, M$. Since $\lambda_1(t)$ is in the same Archimedean class as t, $\frac{\lambda_1(t)}{ax^q} \in \mathrm{Fin}\, M$. Assuming $q = u/v$, and raising to the power of v we get $\lambda_1(t)^v/a^v x^u \in \mathrm{Fin}\, M$. But $\lambda_1(t)^v/a^v x^u$ is also in G_1 and this implies that there is $n \in \mathbb{N}$ such that $\lambda_1(t)^v = 2^n a^v x^u$, by which $\lambda_1(t) \in M\langle x\rangle$. $\square$

We denote the structure obtained in the above lemma by $\mathbb{M}\langle x\rangle$ and we denote $M\langle x\rangle \cap G_1$ and $P_{1n}|_{M\langle x\rangle}$ respectively by $G\langle x\rangle$ and $P_n\langle x\rangle$. Note that

$\langle G\langle x\rangle, \{P_n\langle x\rangle\}, ., <\rangle$ is a model of Presburger arithmetic generated by x over $\langle G, \{P_n\}, ., <\rangle$ in $\langle G_1, \{P_{1n}\}, ., <\rangle$.

In the following, for a sequence Λ of elements in G_1, we denote $M\langle\Lambda\rangle \cap G_1$ by $G\langle\Lambda\rangle$ and $P_{1n}|_{M\langle\Lambda\rangle}$ by $P_n\langle\Lambda\rangle$.

Lemma 5. Let $\mathbb{M}_1$ be a free extension of $\mathbb{M}$ and $x \in N_1 - M$. Then there is a countable sequence $\Lambda = (a_i)_{i\in\mathbb{N}}$ in G_1 such that $\mathbb{M}\langle x, \Lambda\rangle := \langle M\langle x, \Lambda\rangle, N\langle x, \Lambda\rangle, G\langle\Lambda\rangle, \lambda_1|_{M\langle x,\Lambda\rangle}, \{P_n\langle\Lambda\rangle\}\rangle$ is a free extension of $\mathbb{M}$ and a free substructure of $\mathbb{M}_1$. Also each a_i is $h_i(x)$ for $L \cup \{\lambda\}$-definable functions h_i with parameters in M.

Proof. Consider the L-structure $M\langle x\rangle$ and two cases. First, when $M\langle x\rangle$ has no more Archimedean classes than M does. In this case, $\langle M\langle x\rangle, N\langle x\rangle, G, \lambda, \{P_n\}\rangle$ is the $\mathbb{L}$-structure we are looking for, and we let $\Lambda = \emptyset$. The second case is when $M\langle x\rangle$ has Archimedean classes that are not represented in M. By the valuation inequality, we need only one element a_1 in G_1 so that $G\langle a_1\rangle$ contains representatives for all Archimedean classes in $M\langle x\rangle$. Furthermore a_1 is $\lambda_1(f_1(x))$ for some L-definable f with parameters in M. Now consider the structure $\mathbb{M}\langle a_1\rangle$ as described in the previous lemma. By induction, if there is an Archimedean class in $M\langle x, a_1, \ldots, a_n\rangle$ which is not represented in $G\langle a_1, \ldots, a_n\rangle$, then let $a_{n+1} \in G_1 - M\langle a_1, \ldots, a_n\rangle$ be the representative of this class in M_1; otherwise let $a_{n+1} = a_n$. Note that a_{n+1} is $\lambda_1(f_{n+1}(x, a_1, \ldots, a_n))$ for some L-definable f_{n+1} with parameters in M. Let $\Lambda = (a_i)_{i\in\mathbb{N}}$. We claim that $M\langle x, \Lambda\rangle$ is closed under λ_1, as a result of which $M\langle x, \Lambda\rangle \cap G_1 = M\langle\Lambda\rangle \cap G_1 = G\langle\Lambda\rangle$. So see this note that if $t \in M\langle x, \Lambda\rangle$ then $t \in M\langle x, a_{i_1}, \ldots, a_{i_m}\rangle$ for some $a_{i_1}, \ldots, a_{i_m} \in \Lambda$. By induction hypothesis, $\lambda_1(t)$ is in $M\langle x, a_1, \ldots, a_k\rangle$ for a $k > i_m$, and hence in $M\langle x, \Lambda\rangle$. By freeness of $\mathbb{M}_1$ over $\mathbb{M}$, $M\langle x, \Lambda\rangle \cap N_1 = N\langle x, \Lambda\rangle$, and the structure $\mathbb{M}\langle x, \Lambda\rangle$ as in the statement of the theorem is an $\mathbb{L}$-structure extending $\mathbb{M}$ and free over it. $\square$

For the next lemma, we suppose that $\mathbb{M}_1$ is a free extension of $\mathbb{M}$ and $x \in M_1 - M\langle N_1\rangle$. In this case $M\langle x\rangle \cap N_1 = N$ and we have the following:

Lemma 6. For $\mathbb{M}_1$, $\mathbb{M}$ and x as in the above, there is a countable sequence $\Lambda = (a_i)_{i\in\mathbb{N}}$ in G_1 such that $M\langle x, \Lambda\rangle$ is closed under λ_1, and $\mathbb{M}\langle x, \Lambda\rangle := \langle M\langle x, \Lambda\rangle, N\langle\Lambda\rangle, G\langle\Lambda\rangle, \lambda_1|_{M\langle x,\Lambda\rangle}, \{P_n\langle\Lambda\rangle\}\rangle$ is a free extension of $\mathbb{M}$ and a free substructure of $\mathbb{M}_1$.

We omit the proof of the above lemma as it is by slight modifications in the proof of Lemma 5.

Back and forth argument, the proof of Theorem 1(1,2).
Suppose that $\mathbb{M}_1$ and $\mathbb{M}_2$ are fixed κ-saturated models of $\mathbb{T}$ for $\kappa > |T| + |\mathbb{L}| + \aleph_0$. Denote by Σ the collection of isomorphisms $f : \mathbb{M} \cong \mathbb{M}'$ (as in the following diagram) where $\mathbb{M}$ is a free substructure of $\mathbb{M}_1$, $\mathbb{M}'$ is a free substructure of $\mathbb{M}_2$ and $|M|, |M'| \leq \kappa$. We will show that Σ has the back and forth property. This implies that $\mathbb{M}_1$ and $\mathbb{M}_2$ are elementarily equivalent. Since this holds for all such $\mathbb{M}_1$ and $\mathbb{M}_2$, it also implies that $\mathbb{T}$ is complete.

$$\begin{array}{c} \mathbb{M}_1 = \langle M_1, N_1, G_1, \lambda_1, \{P_{1n}\}_{n\in\mathbb{N}} \rangle \\ \uparrow \\ \mathbb{M} = \langle M, N, G, \lambda, \{P_n\}_{n\in\mathbb{N}} \rangle \\ f \,\|\!\wr \\ \mathbb{M}' = \langle M', N', G', \lambda', \{P'_n\}_{n\in\mathbb{N}} \rangle \\ \downarrow \\ \mathbb{M}_2 = \langle M_2, N_2, G_2, , \lambda_2, \{P_{2n}\}_{n\in\mathbb{N}} \rangle . \end{array} \tag{2.1}$$

Let $x \in M_1 - M$. What we need is a $y \in M_2 - M'$, and a structure containing x and extending $\mathbb{M}$ which is isomorphic to a structure containing y and extending $\mathbb{M}'$, with an isomorphism that extends f and sends x to y.

According to 'where' in $\mathbb{M}_1$ the element x comes from, we have the following cases:

Case one: $x \in G_1 - M$.
Consider the structure $\mathbb{M}\langle x\rangle$, generated over $\mathbb{M}$ by x as described in Lemma 4. Let $y \in G_2 - M'$ be an element which satisfies the same Presburger arithmetic type over $\langle G', \{P'_n\}, ., < \rangle$ as x does over $\langle G, \{P_n\}, ., < \rangle$. Such a y exists since as mentioned earlier $\langle G_1, \{P_{1n}\}, ., < \rangle$ and $\langle G_2, P_{2n}, ., < \rangle$ are models of Presburger arithmetic and this theory admits elimination of quantifiers. We claim that y realises the same cut over M' as x does over M (via the isomorphism f); that is, for each $t \in M$, if $x < t$ then $y < f(t)$. To see this, first note that as f is an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$, for each $t \in M$, $f(\lambda_1(t)) = \lambda_2(f(t))$. Now suppose that $x < t$ and $\lambda_1(t) = t$. Then $\lambda_2(f(t)) = f(t)$. So $f(t) \in G'$ and by the choice of y, $y < f(t)$. Also if $\lambda_1(t) < t < 2\lambda_1(t)$ and $x < \lambda_1(t) < t$ then $y < f(\lambda_1(t)) = \lambda_2(f(t)) < f(t)$. Note that there is no case $\lambda_1(t) < x < t$; as then, since $x \in G$, by the axioms we have $\lambda_1(t) = x$. One can simply verify that the two structures $\mathbb{M}\langle x\rangle$ and $\mathbb{M}\langle y\rangle$, both as in Lemma 4 are isomorphic.

Case two: $x \in N_1 - M$ and $x \notin G_1$.
In this case, consider the structure $\mathbb{M}\langle x, \Lambda\rangle$ as described in Lemma 5 with $\Lambda = (a_i)_{i\in\mathbb{N}}$ a sequence in G_1. Let $b_1 \in G_2 - G'$ be an element, as in case one, that

realises the same Presburger arithmetic type over $\langle G', \{P'_n\}, ., <\rangle$ as a_1 does over $\langle G, \{P_n\}, ., <\rangle$. Let b_{n+1} be an element that realises the same Presburger arithmetic type over $G'\langle b_1, \ldots, b_n\rangle$ as does a_{n+1} over $G'\langle a_1, \ldots, a_n\rangle$. Denote by Λ' the sequence $(b_i)_{i\in\mathbb{N}}$ obtained this way. Now let $y \in M_2$ be an element realising the same cut in $M'\langle\Lambda'\rangle$ as the cut of x in $M\langle\Lambda\rangle$. It is now easy to check that the two structures $\mathbb{M}\langle x, \Lambda\rangle$ and $\mathbb{M}'\langle y, \Lambda'\rangle$ are isomorphic.

Case three: $\boldsymbol{x \in M\langle N_1\rangle}$ and $\boldsymbol{x \notin N_1}$.
In this case, there are elements $x_1, \ldots, x_n$ in N_1 such that $x \in M\langle x_1, \ldots, x_n\rangle$. Now, one can combine the arguments for cases one and two to get the result.

Case four: $\boldsymbol{x \in M_1}$ and $\boldsymbol{x \notin M\langle N_1\rangle}$.
For this case we first construct the structure $\mathbb{M}\langle x, \Lambda\rangle$ as in Lemma 6. Now, as in case two, we can find a sequence $\Lambda' = (b_i)_{i\in\mathbb{N}}$ of elements in G_2 such that $\mathbb{M}\langle\Lambda\rangle$ and $\mathbb{M}\langle\Lambda'\rangle$ are isomorphic. By Lemma 3, $M'\langle N_2\rangle = N_2\langle M'\rangle \neq M_2$, and since $M_2 - N_2$ is dense in M_2 and M_2 is saturated, we can find $y \in M_2 - M'\langle N_2\rangle$ that realises the same cut in $M'\langle\Lambda'\rangle$ as the cut of x in $M\langle\Lambda\rangle$. In this case, the structures $\mathbb{M}\langle x, \Lambda\rangle$ and $\mathbb{M}'\langle y, \Lambda'\rangle$ as in Lemma 6 are isomorphic.

The above four cases exhaust all possibilities and as mentioned before, the completeness of $\mathbb{T}$ results from the fact that by our argument $\mathbb{M}_1$ is elementarily equivalent to $\mathbb{M}_2$. □

From here onwards when we say $\mathbb{M}_1$ is a 'sufficiently' saturated extension of $\mathbb{M}$, we mean saturated as in the setting of the proof of the above theorem. Considering the axioms for λ, in the following we have stated item 2 of Theorem 1 in a slightly different way.

Theorem 7. Every $\mathbb{L}$-formula (with free variables $\bar{y}$) has an equivalent which is a Boolean combination of formulas of the form

$$\exists \bar{x} = (x_1, \ldots, x_n) \quad (\bigwedge_{i=1,\ldots,n} (N_{\mathbb{L}}(x_i)) \wedge \phi(\bar{x}, \bar{y})) \tag{*}$$

with $\phi(\bar{x}, \bar{y})$ in $L \cup \{\lambda_{\mathbb{L}}\}$.

Proof. Consider the situation as in Diagram 2.1. Suppose that $\bar{a} = (a_1, \ldots, a_n) \in M_1^n$ and $\bar{b} = (b_1, \ldots, b_n) \in M_2^n$ realise the same formulas of the form (*). We will prove that then $\mathrm{tp}_{\mathbb{M}_1}(\bar{a}) = \mathrm{tp}_{\mathbb{M}_2}(\bar{b})$, and this is equivalent to the statement of the theorem. Suppose that $\mathrm{rank}_{N_1}(N_1\langle\bar{a}\rangle) = \mathrm{rank}_{N_2}(N_2\langle\bar{b}\rangle) = r \leq n$ and without loss of generality we assume that $a_1, \ldots, a_r$ are independent over N_1 and so are $b_1, \ldots, b_r$ over N_2 (the fact that $\mathrm{rank}_{N_1}(N_1\langle\bar{a}\rangle) = \mathrm{rank}_{N_2}(N_2\langle\bar{b}\rangle)$

follows from the assumption that $\bar{a}$ and $\bar{b}$ realise the same formulas of the form (*). To be precise, suppose that $a_r \in \mathrm{dcl}(a_1, \ldots, a_{r-1}, N_1)$. Then there is an L-definable function f such that $\bar{a} = (a_1, \ldots, a_n)$ satisfies the formula:

$$\exists \bar{n} = (n_1 \ldots, n_m) \quad \bigwedge_{i=1,\ldots,m} N_{\mathbb{L}}(n_i) \wedge x_r = f(\bar{n}, x_1, \ldots, x_{r-1}) \tag{2.2}$$

The above formula is satisfied by $\bar{b} = (b_1, \ldots, b_n)$ and this means that $b_r \in \mathrm{dcl}(b_1, \ldots, b_{r-1}, N_2)$).

Now $\mathrm{rank}_{N_1}(N_1\langle\bar{a}\rangle) = r$, implies that there is a tuple $\bar{c}$ of elements in N_1 such that $\mathrm{rank}_{\langle\bar{c}\rangle}(\langle\bar{a}, \bar{c}\rangle) = r$. Remember that $\langle\bar{a}, \bar{c}\rangle$ is the L-definable closure of $\{\bar{a}, \bar{c}\}$ in M_1.

Consider the following type $\Phi(\bar{y})$ in $\mathbb{M}_2$:

$$\Phi(\bar{y}) = \{\phi(\bar{b}, \bar{y}) \wedge N_{\mathbb{L}}(\bar{y}) : \phi(\bar{x}, \bar{y}) \in L \cup \{\lambda_{\mathbb{L}}\} \text{ and } \mathbb{M}_1 \models \phi(\bar{a}, \bar{c})\}.$$

As $\bar{a}$ and $\bar{b}$ realise the same formulas of the form (*) and $\mathbb{M}_2$ is saturated, this type is satisfied in M_2 by a tuple $\bar{d}$ of elements of N_2. With a similar argument to that in parentheses above, one can check that then $\mathrm{rank}_{\langle\bar{d}\rangle}(\langle\bar{b}, \bar{d}\rangle) = \mathrm{rank}_{\langle\bar{c}\rangle}(\langle\bar{a}, \bar{c}\rangle) = r$.

As elementary pairs, $\langle\langle\bar{a}, \bar{c}\rangle, \langle\bar{c}\rangle\rangle$ is isomorphic to $\langle\langle\bar{b}, \bar{d}\rangle, \langle\bar{d}\rangle\rangle$ via a map, say i. In the following we will find an $\mathbb{L}$-structure isomorphism between two free $\mathbb{L}$-substructures of $\mathbb{M}_1$ and $\mathbb{M}_2$ that extends this isomorphism.

Take an $x > 0$ in $\langle\bar{a}, \bar{c}\rangle$ with $\lambda_1(x) \notin \langle\bar{a}, \bar{c}\rangle$. Let $y = i(x) \in \langle\bar{b}, \bar{d}\rangle$. We want to show that $\lambda_2(y)$ realises the same cut in $\langle\bar{b}, \bar{d}\rangle$ as does $\lambda_1(x)$ in $\langle\bar{a}, \bar{c}\rangle$, via i, and hence we have the isomorphism: $\langle\langle\bar{a}, \bar{c}, \lambda_1(x)\rangle, \langle\bar{c}, \lambda_1(x)\rangle\rangle \cong \langle\langle\bar{b}, \bar{d}, \lambda_2(y)\rangle, \langle\bar{d}, \lambda_2(y)\rangle\rangle$ between two elementary pairs.

Suppose that $\lambda_1(x) < t$ for some $t \in \langle\bar{a}, \bar{c}\rangle$. We can consider t as $f(\bar{a}, \bar{c})$ for some L-definable function $f : M_1 \to M_1$ with no parameters. Let also $x = g(\bar{a}, \bar{c})$ for some definable function g. Then the formula $\psi(\bar{z}) := \exists u \quad N_{\mathbb{L}}(u) \wedge [u = \lambda_{\mathbb{L}}(g(\bar{z}, \bar{c}))] \wedge [u < f(\bar{z}, \bar{c})]$, is satisfied by $\bar{a}$. Hence, by the choice of $\bar{d}$, the corresponding formula $\exists u \quad N_{\mathbb{L}}(u) \wedge [u = \lambda_{\mathbb{L}}(g(\bar{z}, \bar{d}))] \wedge [u < f(\bar{z}, \bar{d})]$ is satisfied by $\bar{b}$, which means $\lambda_2(y) < i(t)$. Consequently $\lambda_1(x)$ and $\lambda_2(y)$ satisfy the same cuts respectively in $\langle\bar{a}, \bar{c}\rangle$ and $\langle\bar{b}, \bar{d}\rangle$ and we have two isomorphic elementary pairs $\langle\langle\bar{a}, \bar{c}, \lambda_1(x)\rangle, \langle\langle\bar{c}, \lambda_1(x)\rangle\rangle$ and $\langle\langle\bar{b}, \bar{d}, \lambda_2(y)\rangle, \langle\bar{d}, \lambda_2(y)\rangle\rangle$.

Now let $\Lambda_1 = \{\lambda_1(x) : x \in \langle\bar{a}, \bar{c}\rangle\}$ and $\Lambda'_1 = \{\lambda_2(x) : x \in \langle\bar{b}, \bar{d}\rangle\}$. Also let $\Lambda_{n+1} = \{\lambda_1(x) : x \in \langle\bar{a}, \bar{c}, \Lambda_1, \ldots, \Lambda_n\rangle$ and $\Lambda'_{n+1} = \{\lambda_2(x) : x \in \langle\bar{b}, \bar{d}, \Lambda'_1, \ldots, \Lambda'_n\rangle\}$. Then $\langle\bar{a}, \bar{c}, \bigcup_{i\in\mathbb{N}} \Lambda_i\rangle$ and $\langle\bar{b}, \bar{d}, \bigcup_{i\in\mathbb{N}} \Lambda'_i\rangle$ are respectively closed under λ_1 and λ_2 and L-isomorphic. By a similar argument to the above, it is easy to check that the two $\mathbb{L}$-structures

$$\mathbb{M}_3 = \langle\langle\bar{a}, \bar{c}, \bigcup_{i\in\mathbb{N}} \Lambda_i\rangle, \langle\bar{c}, \bigcup_{i\in\mathbb{N}} \Lambda_i\rangle, \langle\bar{a}, \bar{c}, \bigcup_{i\in\mathbb{N}} \Lambda_i\rangle \cap G_1, \lambda_1|_{\langle\bar{a},\bar{c},\bigcup_{i\in\mathbb{N}}\Lambda_i\rangle}, P_{1n}|_{\langle\bar{a},\bar{c},\bigcup_{i\in\mathbb{N}}\Lambda_i\rangle}\rangle$$

and

$$\mathbb{M}_4 = \langle\langle\bar{b}, \bar{d}, \bigcup_{i\in\mathbb{N}} \Lambda'_i\rangle, \langle\bar{b}, \bigcup_{i\in\mathbb{N}} \Lambda'_i\rangle, \langle\bar{b}, \bar{d}, \bigcup_{i\in\mathbb{N}} \Lambda'_i\rangle \cap G_2, \lambda_2|_{\langle\bar{b},\bar{d},\bigcup_{i\in\mathbb{N}}\Lambda'_i\rangle}, P_{2n}|_{\langle\bar{b},\bar{d},\bigcup_{i\in\mathbb{N}}\Lambda'_i\rangle}\rangle$$

are isomorphic and this isomorphism is in the back and forth system Σ. So, the isomorphism between these two implies that $\mathbb{M}_1$ and $\mathbb{M}_2$ are elementarily equivalent by which $\mathrm{tp}_{\mathbb{M}_1}(\bar{a}) = \mathrm{tp}_{\mathbb{M}_2}(\bar{b})$, and this finishes the proof. $\square$

3 Types and definable sets

We use the described quantifier elimination in the previous section for characterising definable sets and types in our structures.

Theorem 8. Let $\mathbb{M}$ be a common elementary substructure of $\mathbb{M}_1, \mathbb{M}_2 \models \mathbb{T}$. Then

1. if $\bar{g}_1 \in G_1^n$ and $\bar{g}_2 \in G_2^n$ realise the same Presburger arithmetic-types over $\langle G, \{P_n\}, ., <\rangle$ then they realise the same $\mathbb{L}$-types over M in $\mathbb{M}_1$ and $\mathbb{M}_2$.

2. If $\bar{n}_1 \in N_1^n$ and $\bar{n}_2 \in N_2^n$ realise the same $L \cup \{\lambda\}$-types over M (over N) in $\mathbb{M}_1$ and $\mathbb{M}_2$, then they realise the same $\mathbb{L}$-types over M (over N) in $\mathbb{M}_1$ and $\mathbb{M}_2$.

Proof of 1. Suppose that $\bar{g}_1 = (g_{1,1}, \ldots, g_{1,n})$ and $\bar{g}_2 = (g_{2,1}, \ldots, g_{2,n})$. Then by the assumption of the theorem, $g_{1,1}$ and $g_{2,1}$ realise the same Presburger arithmetic types over $\langle G, \{P_n\}, ., <\rangle$ and therefore (by similar argument to case one for x and y) the same cuts in M. Now the two structures $\mathbb{M}\langle g_{1,1}\rangle$ and $\mathbb{M}\langle g_{1,2}\rangle$ as obtained in Lemma 4 are isomorphic and this isomorphism (which sends $g_{1,1}$ to $g_{2,1}$) is in the back and forth system Σ defined in page 8, in the proof of Theorem 1. This implies that $\mathrm{tp}_{\mathbb{M}_1}(g_{1,1}/M) = \mathrm{tp}_{\mathbb{M}_2}(g_{2,1}/M)$. Now assume that we have two isomorphic structures $\mathbb{M}\langle g_{1,1}, \ldots, g_{1,i}\rangle$ and $\mathbb{M}\langle g_{2,1}, \ldots, g_{2,i}\rangle$ (for $i < n$). Then $g_{1,i+1}$ and

$g_{2,i+1}$ realise the same cuts in $G\langle g_{1,1},\ldots,g_{1,i}\rangle$ and $G\langle g_{2,1},\ldots,g_{2,i}\rangle$ and hence the same cuts in $M\langle g_{1,1}\ldots,g_{1,i}\rangle$ and $M\langle g_{2,1},\ldots,g_{2,i}\rangle$. Again by the arguments of the case one, we obtain two isomorphic structures $\mathbb{M}\langle g_{1,1},\ldots,g_{1,i+1}\rangle$ and $\mathbb{M}\langle g_{2,1},\ldots,g_{2,i+1}\rangle$ in our back and forth system Σ and this leads to the result we are looking for. $\square$

Proof of 2. Suppose that $\bar{n}_1=(n_{1,1},\ldots,n_{1,n})$ and $\bar{n}_2=(n_{2,1},\ldots,n_{2,n})$. We do the argument for n_{11} and n_{21} and as in *Proof of 1* above the result for the tuples follows. Suppose that n_{11} and n_{21} realise the same $L\cup\{\lambda\}$-types over N. There is a sequence $\Lambda=(a_i)_{i\in\mathbb{N}}$ of elements in G_1 as in Lemma 5 such that $M\langle\Lambda\rangle$ is closed under λ_1. There is also a sequence $\Lambda'=(b_i)_{i\in\mathbb{N}}$ of elements in G_2 such that $M\langle\Lambda\rangle$ is isomorphic to $M\langle\Lambda'\rangle$. As n_{11} and n_{21} realise the same $L\cup\{\lambda\}$-types over N and by the construction of the sequences Λ and Λ' as in Lemma 5 (in which each a_n, $n>1$, in Λ is the witness for the new Archimedean class obtained by adding n_{11} to $M\langle a_1,\ldots,a_{n-1}\rangle$ and equivalently to $N\langle a_1\ldots,a_{n-1}\rangle$), two structures $\mathbb{M}\langle n_{11},\Lambda\rangle$ and $\mathbb{M}\langle n_{21},\Lambda'\rangle$ are isomorphic. This isomorphism is in our back and forth system Σ and this is what we need. $\square$

The second part of the above theorem is equivalent to the statement of the following corollary for definable subsets of N^n. Note that definable here always means with parameters.

Corollary 9 (definable subsets of N^n)**.** Let $\mathbb{M}=\langle M,N,G,,\lambda,\{P_n\}_{n\in\mathbb{N}}\rangle$ be a model of $\mathbb{T}$ and $Y\subseteq N^n$ be definable in $\mathbb{M}$. Then $Y=Z\cap N^n$ for some $Z\subseteq M^n$ definable by an $L\cup\{\lambda\}$-formula.

We can now explain the main definability feature we were looking for: a subset of $\mathbb{R}_{alg}$ is definable in $\langle\bar{\mathbb{R}},\mathbb{R}_{alg},2^{\mathbb{Z}}\rangle$ if and only if it is of the form $Z\cap\mathbb{R}_{alg}$ for Z a definable set in $\langle\bar{\mathbb{R}},2^{\mathbb{Z}}\rangle$. The set Z is then the union of an open set and finitely many discrete sets (by the d-minimality of $\langle M,G\rangle$ (see for example [11]). So $Z\cap\mathbb{R}_{alg}$ is not equal to $\mathbb{Q}$ which implies that $\mathbb{Q}$ and hence $\mathbb{Z}$ are not definable in $\langle\bar{\mathbb{R}},\mathbb{R}_{alg},2^{\mathbb{Z}}\rangle$.

The following theorem and corollary show that each discrete set defined with parameters in N is a subset of N.

Theorem 10. Let $\mathbb{M}=\langle M,N,G,\lambda,\{P_n\}\rangle$ be a model of $\mathbb{T}$. Then N is definably closed in $\mathbb{M}$.

Proof. Let $b \in M - N$. We need to show that b is not definable with parameters of N. Let $\mathbb{M}_1 = \langle M_1, N_1, G_1, \lambda_1, \{P_{1n}\}\rangle$ be a saturated model of $\mathbb{T}$ which is a free extension of $\mathbb{M}$. Let $b_1 \in M_1 - N_1$ be an element which satisfies the same cut in N as does b and $b_1 \neq b$. We claim that b_1 satisfies the same type over N as does b. Clearly $\lambda(b_1) = \lambda(b) \in N$. Since for each $x \in M \supseteq N\langle b\rangle$, $\lambda(x) \in N$, the structure $\langle N\langle b\rangle, N, \lambda(N), \lambda|_{N\langle b\rangle}, \{P_n|_N\}\rangle$ is a model of $\mathbb{T}$ which, as we will prove, is isomorphic to $\langle N\langle b_1\rangle, N, \lambda_1(N\langle b_1\rangle), \lambda_1|_{N\langle b_1\rangle}, \{P_{1n}|N\langle b_1\rangle\}\rangle$. Note that since b_1 realises the same cut in N as b does, for each L-definable function f with parameters in N, $f(b_1)$ and $f(b)$ realise the same cuts in N, and $\lambda_1(f(b_1)) = \lambda(f(b))$, that is, $N\langle b_1\rangle$ is also closed under λ_1 and for each $x \in N\langle b_1\rangle$, $\lambda_1(x) \in N$. This isomorphism, as in the proof of quantifier elimination, implies that b_1 and b satisfy the same types over N. From $b_1 \neq b$, we get that b is not definable with parameters in N. □

Corollary 11. Let $\mathbb{M} \models \mathbb{T}$ and $D \subseteq M$ be a discrete set defined with parameters in N. Then $D \subseteq N$.

Proof. Let $x \in D$. Then since D is discrete and N is dense in M, there are $a, b \in N$ such that $x \in (a, b)$ and x is the only point in this interval; that is, the singleton $\{x\}$ is definable with parameters in N and as N is definably closed, $x \in N$. □

The above corollary can be strengthened to the following statement: If D is a definable set in $\mathbb{M}$ which is a finite union of discrete sets, then D is a subset of N. To see this, note that D is a finite union of discrete sets if and only if $D^{[n]} = \emptyset$, for some $n \in \mathbb{N}$; where $D^{[0]} = D$ and for $i \leq n$, $D^{[i]} = D^{[i-1]} -$ isolated points of $D^{[i-1]}$.

As in [3], we also call a definable subset S of M, small, if there is an L-definable function $f : M^n \to M$ such that $S \subseteq f(N^n)$. In the next theorem, we prove that every definable set in a model of $\mathbb{T}$ is, up to a small set, definable with an $L \cup \{\lambda\}$-formula.

Theorem 12. Let $\mathbb{M} \models \mathbb{T}$ and $S \subseteq M$ be definable. Then there is a small set $X \subseteq M$ and an $L \cup \{\lambda\}$ -definable set S' for which to have $S - X = S' - X$.

Proof. It suffices to prove that for each definable function $F : M \to M$, there is an $L \cup \{\lambda\}$-definable function g and a small set $X \subseteq M$ such that for each $x \in M - X$, $F(x) = g(x)$. If this is true, then to get the result we replace F with the characteristic function of S.

For the proof of the above statement, take a sufficiently saturated elementary extension $\mathbb{M}_1$ of $\mathbb{M}$. By the proof of case four, for each $x \in M_1 - M\langle N_1\rangle$, there is a sequence $\Lambda^{(x)} = (a_i^{(x)})_{i\in\mathbb{N}}$ of elements of G_1 such that $M\langle x, \Lambda^{(x)}\rangle$ is closed under λ_1 and the universe of a model. So we have the following implication in $\mathbb{M}_1$:

$$x \notin M\langle N_1\rangle \Rightarrow F(x) \in M\langle x, \Lambda^{(x)}\rangle. \tag{3.1}$$

By the construction of $\Lambda^{(x)}$, there are $a_1^{(x)}, \ldots, a_n^{(x)}$ in G_1 such that $a_i^{(x)} = g_i^{(x)}(x)$ for some $L \cup \{\lambda\}$-definable function $g_i^{(x)}$ with parameters in M and $F(x) \in M\langle x, a_1^{(x)}, \ldots, a_n^{(x)}\rangle$. So, applying compactness theorem to Equation 3.1, there are L-definable functions $h_1, \ldots, h_m$ defined on $M^{n_{h_i}}$ and $L \cup \{\lambda\}$-definable functions $f_1, \ldots, f_n$ such that

$$\bigwedge_{i=1,\ldots,m} (x \notin h_i(N^{n_{h_i}})) \Rightarrow \bigvee_{i=1,\ldots,n} (F(x) = f_i(x)).$$

The above implication together with the following two facts gives the result. First, finite unions of small sets are small. Second, by the case four of our proof of quantifier elimination the sets $\{x \notin M\langle N_1\rangle : F(x) = f_i(x)\}$ is the intersection of $M_1 - M\langle N_1\rangle$ with an $L \cup \{\lambda\}$-definable set. $\square$

4 $\mathbb{T}$ has NIP

In [5], it is proved that if $\langle \mathbb{B}, \mathbb{A}\rangle$ is a dense pair of d-minimal structures, then its open core is $\mathbb{B}$; that is, every open set definable in $\langle \mathbb{B}, \mathbb{A}\rangle$ is definable in $\mathbb{B}$. Applying this result to our models we get the following: every open set definable in $\mathbb{M} \models \mathbb{T}$ can be defined by an $L \cup \{\lambda\}$-formula.

This section proves the dependency of $\mathbb{T}$ relying on the above fact and heavy use of the techniques developed in [7] for the dense pairs of o-minimal structures. However, it is recently pointed out to the author that the NIP for $\mathbb{T}$ follows from our theorem 7 and Corollary 2.6 of Chernikov in [1]. That being so, the only reason for having kept this proof here is that it relies more on the structural properties of our models.

Let us first recall the following definition of dependence.

Definition 4.1. Let T_1 be a complete theory in the language L_1 and M_1 be a monster model of T_1. We call T_1 dependent if for every L_1-formula $\phi(\bar{x}, \bar{y})$,

$\bar{b} \in M_1^{|\bar{y}|}$ and $(\bar{a}_i)_{i\in\mathbb{N}}$ an indiscernible sequence of tuples in $M_1^{|\bar{x}|}$, there exists a natural number N such that

- either for all $i > N$, $M_1 \models \phi(\bar{a}_i, \bar{b})$, or
- for all $i > N$, $M_1 \models \neg\phi(\bar{a}_i, \bar{b})$.

It is well-known that T_1 is dependent also if the above holds for all formulas $\phi(x, \bar{y})$ and $\bar{b} \in M_1^{|\bar{y}|}$. It is also well-known that Boolean combinations of dependent formulas are dependent.

We will also need the following lemma.

Lemma 13 ([7])**.** Let M_1 be a monster model of a theory T_1 and $(\bar{a}_i)_{i\in\mathbb{N}}$ be an indiscernible sequence. Let $\phi(\bar{x}, \bar{y})$ be a formula such that $M \models \exists\bar{y} \quad \phi(\bar{a}_i, \bar{y})$ for some i. Then there is an indiscernible sequence $(\bar{b}_i)_{i\in\mathbb{N}}$ such that for each i, $M \models \phi(\bar{a}_i, \bar{b}_i)$.

Theorem 14. $\mathbb{T}$ is dependent.

Proof. Let $\mathbb{M} = \langle M, N, G, \lambda, \{P_n\}\rangle$ be a monster model of $\mathbb{T}$ and $(a_i)_{i\in\mathbb{N}}$ an indiscernible sequence. Let $\phi(a_i, \bar{b}) = \exists\bar{z} \quad (N_{\mathbb{L}}(\bar{z}) \wedge \psi(a_i, \bar{b}, \bar{z}))$ be a formula in $\mathbb{L}$ with parameters $\bar{b}$ where ψ is an $L \cup \{\lambda\}$-formula. We need to prove that the following set $J \subseteq \mathbb{N}$ is finite or co-finite:

$$J := \{i \in \mathbb{N} : \mathbb{M} \models \phi(a_i, \bar{b})\}.$$

We break the proof of this down to the following cases.
Case 1. Suppose that all a_i's are in N. Let X be the set $\{x \in N : \phi(x, \bar{b})\}$. Then, by Corollary 9, $X = Y \cap N$ for Y a definable subset (possibly with other parameters than $\bar{b}$) in the language $L \cup \{\lambda\}$. So we have:

$$\begin{aligned} \mathbb{M} \models \phi(a_i, \bar{b}) &\text{ iff } a_i \in X \\ &\text{ iff } a_i \in N \cap Y \\ &\text{ iff } a_i \in Y \end{aligned}$$

But, $a_i \in Y$ is clearly expressible by an $L \cup \{\lambda\}$ formula. By Theorem 7.4 in [7], the theory of $\langle M, G\rangle$ is dependent. So only finitely or cofinitely many a_i's are in Y.
Case 2. Suppose that a_i's all lie in $M - N$ and $\bar{b} \in N$. For $\bar{z} \in N$, define $A_{\bar{z}} = \{x : \mathbb{M} \models \psi(x, \bar{b}, \bar{z})\}$. For a fixed $\bar{z}$, By d-minimality of $\langle M, G\rangle$, $A_{\bar{z}}$ is the union of an open set and finitely many discrete sets:

$$A_{\bar{z}} = O \cup D_1 \cup \ldots \cup D_n.$$

Let $a \in (a_i)_{i\in\mathbb{N}}$. If $a \in D_1 \cup \ldots \cup D_n$ then by the lines after corollary 11, $a \in N$, which is contradictory with our assumption that $a_i \notin N$. So, for each $\bar{z} \in N$,

$$a \in A_{\bar{z}} \text{ iff } a \in \mathrm{Int}(A_{\bar{z}}).$$

We now have:

$$\mathbb{M} \models \exists \bar{z} \in N \quad \psi(a, \bar{b}, \bar{z}) \text{ iff}$$
$$a \in \bigcup_{\bar{z}\in N} A_{\bar{z}} \text{ iff } a \in \bigcup_{\bar{z}\in N} \mathrm{Int}(A_{\bar{z}}).$$

As $\bigcup_{\bar{z}\in N} \mathrm{Int}(A_{\bar{z}})$ is an open definable set, by the first paragraph of this section, it is defined by an $L \cup \{\lambda\}$-formula. Again as the theory of $\langle M, G\rangle$ is dependent there are only finitely or cofinitely many a_i's in this set and the statement of the theorem in this case is proved.

Case 3. Let $(a_i)_{i\in\mathbb{N}}$ be an indiscernible sequence of elements not in N where the set $\{a_i : i \in \mathbb{N}\}$ is dependent over N. Then for some i, there exists an $i_0 < i$ such that $a_i \in N\langle a_0, \ldots, a_{i_0}\rangle$. So there exists an L-definable function $f : M \to M$ such that

$$\exists \bar{c} \quad \bar{c} \in N \wedge a_i = f(\bar{c}, a_0, \ldots, a_{i_0}).$$

Since $(a_i)_{i\in\mathbb{N}}$ is indiscernible, the above holds for all $i \geq i_0$ (and the same set$\{a_0, \ldots, a_{i_0}\}$). Now by Lemma 13 there is an indiscernible sequence $(\bar{g}_i)_{i\in\mathbb{N}}$ of tuples in N such that for all i ($i \geq i_0$)

$$f(\bar{g}_i, a_0, \ldots, a_{i_0}) = a_i.$$

So we have

$$\mathbb{M} \models \phi(a_i, \bar{b}) \leftrightarrow \phi(f(\bar{g}_i, a_0, \ldots, a_{i_0}), \bar{b}).$$

Since $(\bar{g}_i)_{i\in\mathbb{N}}$ is an indiscernible sequence of tuples in N, and by a similar argument to that for the first case, there are finitely or cofinitely many $(\bar{g}_i)$'s for which $\mathbb{M} \models \phi(f(\bar{g}_i, a_0, \ldots, a_{i_0}), \bar{b})$. So there are finitely or cofinitely many a_i's for which $\mathbb{M} \models \phi(a_i, \bar{b})$.

Case 4. Now consider the case where $(a_i)_{i\in\mathbb{N}}$ are independent over N. First note that for each i, $a_i \notin N\langle\bar{b}\rangle$. This is because if for an i_0, $a_{i_0} \in N\langle\bar{b}\rangle$, then all a_i's are in $N\langle\bar{b}\rangle$ and therefore $\mathrm{rank}_N(\{a_i, i \in \mathbb{N}\}) \leq \mathrm{rank}_N(\bar{b})$ which is impossible because $\mathrm{rank}_N(\{a_i, i \in \mathbb{N}\})$ is infinite.

Since for each x in M, $\lambda(x) \in N$, $N\langle\bar{b}\rangle$ is closed under λ. Hence $\langle M, N\langle b\rangle, G, \lambda, \{P_n\}\rangle$ is a model of $\mathbb{T}$ and by Theorem 10, $N\langle\bar{b}\rangle$ is definably closed in M. The rest of the proof is as in case 2:

for a fixed $\bar{z} \in N$, let $A_{\bar{z}} = \{x : \mathbb{M} \models \psi(x, \bar{b}, \bar{z})\}$. Then $A_{\bar{z}} = O \cup D_1 \ldots \cup D_n$ with O open and $D_1 \cup \ldots \cup D_n$ a finite union of discrete sets definable with parameters in $N\langle\bar{b}\rangle$. If $a_i \in A_{\bar{z}}$ then $a_i \in O$ (since again be the lines after corollary 11, $D_1 \cup \ldots \cup D_n \subseteq N\langle\bar{b}\rangle$). So

$$\mathbb{M} \models \phi(a_i, \bar{b}) \text{ iff } a_i \in \bigcup_{\bar{z} \in N} \mathrm{Int}(A_{\bar{z}}).$$

The set $\bigcup_{\bar{z} \in N} \mathrm{Int}(A_{\bar{z}})$ is an open definable set and hence is definable with an $L \cup \{\lambda\}$-formula and again by dependency of the theory of $\langle M, G\rangle$ the result follows. □

Remark 4.1. This paper is a shortened version of the intended paper by the author on the same subject. Having learnt of the independent work in [5] and [4], the author omitted results (especially those related to the open core of the models) that could follow more easily from that work.

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