Extension of a Row in Mutually Indiscernible Arrays in Strong Theories

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Abstract

Given a mutually indiscernible array $A = (a_{ij})_{i,j<\omega}$ and a tuple b, it is possible to adhere a copy of b to the first row of A and obtain a mutually indiscernible array $A^{0b} = ((a_{0j}b_{0j})_{j\in\omega}(a_{ij})_{i\in\omega,i\neq 0,j\in\omega})$ in which $a_{00}b_{00} \equiv^{\sigma} a_{00}b$. That is, if we wish to add b itself to the array, then we need to apply σ to A^{0b} and hence possibly change the first column. Nevertheless, if T is strong, and $A = (a_{ij})_{i<\kappa,j\in\omega}$ is mutually indiscernible and κ is large enough, then we can adhere b itself to a_{00} and obtain a mutually indiscernible array $(a'_{0j}b_{0j})(a'_{kj})_{k\in I,j\in\omega}$ for some countable $I = \{i_1, i_2, \ldots\} \subseteq \kappa$, where $a'_{00}b'_{00} = a_{00}b$ and $(a'_{kj})_{j\in\omega} \equiv^{EM} (a_{kj})_{j\in\omega}$ and $a'_{k0} = a_{k0}$ for each $k \in I$.
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1 Introduction

Extension of a mutually indiscernible array may be required in various situations, especially those in which a witness to a property is to be replaced by a witness with additional desired properties. The array extension lemma presented here, was obtained during an unsuccessful attempt by the author to suggest some strongness theorem for the ac-valued difference fields, following the proof of the NTP_2 theorem by Chernikov and Hils in [3].

In [3] they have proved that given a theory T of an ac-valued difference field $\mathcal{K} = (K, k, \Gamma, \sigma, \mathrm{ac}, v)$ of residue characteristic 0 that eliminates quantifiers from K in the natural three sorted language, if k (as a difference field) and Γ (as a difference ordered abelian group) are NTP2, then so is K. To prove this by contradiction, they begin with a *good* witness $(A = (a_{ij})_{i,j\in\omega}, \phi)$ for TP2 (in which A is in particular, a strongly indiscernible array). They then construct a sequence of *good* array extensions $A = A^0 \subseteq A^1 \subseteq \ldots \subseteq A^\omega$ with the following feature: if a^0_{00} is the enumeration of a substructure of $\mathbb{M}$ (the monster model), then in each step l countably many elements b_l are added to it and each entry a^l_{00} is the enumeration an elementary substructure of a^{l+1}_{00}. The elements b_l are in some of the steps added from sorts $k(\mathbb{M})$ and $\Gamma(\mathbb{M})$ and in others generally from the main sort. When they are from $k(\mathbb{M})$ or $\Gamma(\mathbb{M})$, in adding them safely, a rather complicated array extension lemma is used to *respect* the type of a realization a of the first column that remains in all these extensions in place, and to guarantee that each of the A^l witnesses TP_2 with ϕ. When b_l come form the main filed sort, they are added by a rather trivial array extension lemma. At the end (by compactness) an array A^ω is obtained in which the a^ω_{00} is an enumeration of a structure K^ω such that $K^\omega\langle a\rangle/K^\omega$ is an immediate extension of valued difference fields. Hence (implied by the fact the quantifiers from the main sort are eliminated) the formula $\phi(x, a^\omega_{00})$ is implied by a quantifier-free formula ψ and A^ω witnesses TP_2 with ψ. Since quantifier-free formulae in T are NIP (and hence NTP_2), this is a contradiction.

The author tried the same idea for resilience and strongness, of course with mutually indiscernible arrays instead of strongly indiscernible arrays, non of which was successful. He then decided to extract an extension lemma which he conceives as useful on its own. This lemma is to correspond to Lemma 3.8 in [3].

In the rest of the paper, we do not distinguish notationally between (infinite) tuples and singletons. We have fixed a complete theory T and its monster model $\mathbb{M}$.

Theorem 1. Assume that D is a stably embedded definable set the theory of whose induced structure is strong, $b \in D$ is countable, $A = (a_{ij})_{i\in\kappa,j\in\omega}$ is a mutually indiscernible array and $\kappa \geq \aleph_0^+.\omega$. Then there is a set $I = \{i_1, i_2, \ldots\} \subseteq \kappa$ and an array $A^{0b} = ((a'_{0j}b'_{0j})_{j\in\omega}(a'_{kj})_{k\in I,j\in\omega})$ such that

- A^{0b} is mutually indiscernible,
- $a'_{00}b'_{00} = a_{00}b$,
- $(a'_{kj})_{j\in\omega} \equiv_{a_{k0}} (a_{kj})_{j\in\omega}$ and in particular, $a'_{k0} = a_{k0}$, for each $k \in I$.

2 Setting and Definitions

We begin by defining theories with NTP_2 and strong theories. In the next definition, x and y are finite tuples of variables.

Definition 2. Let T be any theory.

1. We say that T has/is *TP_2*, or has *the tree property of the second kind*, if there is a formula $\phi(x, y)$, an ordinal $k < \omega$ and an array $A = (a_{ij})_{i,j<\omega}$ such that for each i the family $\{\phi(x, a_{ij})|j \in \omega\}$ of formulae is k-inconsistent, and for each path $f : \omega \to \omega$ the family $\{\phi(x, a_{if(i)})|i \in \omega\}$ of formulae is consistent. We call T *NTP_2* if it does not have TP_2.
2. We say that T is *not-strong*, if there is an array $A = (a_{ij})_{i,j\in\omega}$ and a family $\Phi = \{\phi_i(x, y)\}_{i\in\omega}$ of formulae and a family $\{k_i\}_{i\in\omega}$ of ordinals $k_i \in \omega$, such that for each i the family $\{\phi_i(x, a_{ij})|j \in \omega\}$ of formulae is k_i-inconsistent and for each path $f : \omega \to \omega$ the family $\{\phi_i(x, a_{if(i)})|i \in \omega\}$ of formulae is consistent. In this case we also say that (A, Φ) *witnesses not-strong.* If there are no (A, Φ) to witness not-strong, then T is called *strong.*

Strong theories form a subclass of NTP_2 theories and are introduced in [1] and further studied in [4]. For more on NTP_2 we refer the reader to [2].

We continue by gathering facts and lemmas most of which come from [3].

Definition 3 ([2])**.** An array $A = (a_{ij})_{i,j\in\omega}$ is called *mutually indiscernible* if for each $i < \omega$, the i'th row $(a_{ij})_{j<\omega}$ is an indiscernible sequence over the rest of the array $(a_{kj})_{k\neq i,k\in\omega,j\in\omega}$.

Equivalently, the array $A = (a_{ij})_{i,j<\omega}$ is mutually indiscernible if fixing any finite number of rows, the type of each finite subarray of A on those rows, depends only on the $\{=, <\}$-types of the indices for the columns. The sequence of columns of a mutually indiscernible array is an indiscernible sequence (of sequences). Mutually indiscernible arrays have a so-called 'Ramsey' structure. That is, as in the next fact, it is possible to apply Ramsey's lemma to 'extract' a mutually indiscernible array from a given array. The following fact is indeed Lemma 3.5(2) in [3] written in a different format.[1].

Fact 4. Let $A = (a_{ij})_{i,j<\omega}$ be any array. We can find a mutually indiscernible array $A' = (a'_{ij})_{i,j<\omega}$ such that for each choice $I = i_1 < \ldots < i_n$ for rows and corresponding choices of columns $J_{i_1} = j^1_{i_1} < \ldots < j^m_{i_1}, \ldots, J_{i_n} = j^1_{i_n} < \ldots < j^m_{i_n}$, for the finite subarray $a' = (a'_{ij}|i \in I, j \in J_i)$ of A' associated to these choices, and each formula ϕ with $\models \phi(a')$, there is a finite subarray of A of the form $a = (a_{ij})_{i\in I, j\in J'_i}$ such that $\models \phi(a)$, for some choices $J'_{i_1} = j'^1_{i_1} < \ldots < j'^m_{i_1}, \ldots, J'_{i_n} = j'^1_{i_n} < \ldots < j'^m_{i_n}$ of columns for A.

We say that the array A' in the fact above *realises the Ehrenfeucht-Mostowski type of* A.

Note that the array A is called *strongly indiscernible* if it is mutually indiscernible and the sequence of its rows is an indiscernible sequence of sequences. We refer the reader to Lemmas 3.6 and 3.8 in [3], for the extensions of strongly indiscernible arrays (the latter in NTP_2 theories). By fact 4, for strongness we need only to deal with mutually indiscernible arrays. The next corollary is not directly relevant to the rest of the paper, but it justifies our attempt of dealing with mutually indiscernible arrays.

Corollary 5. If T is not-strong, then there is a pair (A, Φ) to witness it, in which A is a mutually indiscernible array. If T is NTP_2, then there is a witness (A, ϕ) to it, in which A is strongly indiscernible and can be considered as having as great number of rows $k \geq \omega$ as we wish.

In the definition of strong and NTP_2, it is enough to deal only with formulae with a single free variable x. For strong this fact is due to the property of sub-multiplicity of *burden*.

The burden of a type $p(x)$, denoted by $\text{bdn}(p)$, is the supremum of the cardinals κ for which there is an inp-pattern for p of depth κ. By an inp-pattern of depth κ for p we mean an array $A = (a_{ij})_{i<\kappa, j\in\omega}$ and a family $\{\phi_i(x, y)\}_{i<\kappa}$ of formulae and a family $\{k_i\}_{i<\kappa}$ of ordinals smaller than ω, such that for each i, the family $\{\phi_i(x, a_{ij})|j \in \omega\}$ is k_i-inconsistent and for each path $f : \kappa \to \omega$, the family $\{\phi_i(x, a_{if(i)})|i < \kappa\}$ is consistent with p (see [1]). Burden is sub-multiplicative; that is, if $\text{bdn}(a_i) := \text{bdn}(\text{tp}(a_i)) < \kappa_i$ for each $i < n$, then $\text{bdn}(a_0 \ldots a_{n-1}) < \prod_{i<n} k_i$ (see [2]).

[1] We have borrowed this equivalent presentation from Casanovas' notes on NTP_2, for a model theory seminar in University of Freiburg.

Notation 6. For a given array A, by $\mathrm{EM}(A)$, we mean the Ehrenfeucht-Mostowski type of A regarded as in Fact 4. We write $A' \models \mathrm{EM}(A)$ if A' is an array as in Fact 4 that realises the Ehrenfeucht-Mostowski of A. By $A' \simeq A$ (A' is equivalent to A) we mean that A' is an array which, considering the whole arrays as tuples, is equivalent to (=has the same types as the type of) A. We mean similarly by $I' \simeq I$ for I and I' sequences of elements. Whenever a and b are tuples with equivalent types over c, by $a \equiv_c^{\sigma'} b$ we indicate that a and b have the same type over c and σ' is an automorphism that sends a to b and fixes c. We mean similarly by $A' \simeq^{\sigma'} A$. In Lemma 7 we have used the notation A^{0b} for an array $\big((a_{0j}b_{0j})_{j\in\omega}(a_{ij})_{i>0,i\in\omega,j\in\omega}\big)$ whose 0'th row is obtained by adding the type of b to a copy of the 0'th row of A, and the rest of it equal to the rest of A. For a definable set D, by D_{ind} we have denoted the structure induced on D.

The following is a rather trivial row extension lemma for a mutually indiscernible array.

Lemma 7 (trivial extension lemma)**.** Let $A = (a_{ij})_{i,j\in\omega}$ be a mutually indiscernible array and b a given (possibly infinite) tuple. Then there is a sequence $(b_{0j})_{j\in\omega}$ such that $A^{0b} = \big((a_{0j}b_{0j})_{j\in\omega}(a_{ij})_{i>0,i,j\in\omega}\big)$ is an array with the following properties:

- it is mutually indiscernible,
- $a_{00}b_{00} \equiv a_{00}b$.

Proof. Let (b'_{0j}) be a sequence such that for each j we have $a_{0j}b_{0j} \equiv_{(a_{ij})_{i>0,i,j\in\omega}} a_{00}b$. Let $B = \big((a_{0j}b'_{0j})_{j\in\omega}(a_{ij})_{i>0,i,j\in\omega}\big)$ and let $B' = \big((a''_{0j}b''_{0j})_{j\in\omega}(a''_{ij})_{i>0,i,j\in\omega}\big) \models \mathrm{EM}(B)$. Then $A'' = (a''_{ij})_{i,j\in\omega} \simeq^{\sigma} A$, and $\sigma(B')$ is the array we are looking for. ☐

Remark 8. As mentioned earlier, the problem with the trivial extension lemma is that to have $a_{00}b$ itself in the first entry, we need to apply the automorphism σ^* in $a_{00}b_{00} \equiv^{\sigma^*} a_{00}b$ to the array, and thereby change the rest of the first column, and (as is implied by the review of the proof of NTP_2 in the introduction) this is not desired in the proofs involving extension of arrays. Having $a_{00}b$ in the first entry without needing to change the first column, requires some indiscerniblity of the rest of the array over b. We will see that this is possible if b comes from a strong part of the structure.

3 Proof of Theorem 1.

We will give the proof of Theorem 1 after two auxiliary lemmas.

Lemma 9.

1. Let I and J be indiscernible sequences over some small set C such that I is indiscernible over JC. If J' is an indiscernible sequence over IC and $J' \models EM(J/IC)$, then I and J' are mutually indiscernible over C.

2. In above I and J may be arrays.

3. Let $(a_i)_{i\in\omega}$ be an indiscernible sequence and b a (possibly infinite) tuple and C a small set. Let $p(x, a_0) = \mathrm{tp}(b/a_0C)$. If $\bigcup_{i\in\omega} p(x, a_i) = \{\psi(x, a_i c) | \models \psi(b, a_0 c), c \in C\}$ is consistent, then there is a sequence $(a'_i)_{i\in\omega}$ indiscernible over Cb such that $(a'_i)_{i\in\omega} \equiv_C (a_i)_{i\in\omega}$ and $a'_0 = a_0$. (From [3]).

Proof. 1 and 2 are easy to check, and 3 is proved in the reference. □

The following lemma is indeed no more than a revisiting of lemma 2.4 in [2].

Lemma 10 (row replacement lemma)**.**

1. Assume that C is a small set and D is a stably embedded definable set the theory of whose induced structure is strong, and let $b \in D$ be countable. Let $A = (a_{ij})_{i<\kappa, j\in\omega}$ be mutually indiscernible over C and $\kappa \geq (\aleph_0)^+$. Then there is some $i < \kappa$ and a sequence $(a'_{ij})_{j\in\omega}$ indiscernible over Cb such that $(a'_{ij})_{j\in\omega} \simeq_C (a_{ij})_{j\in\omega}$ and $a'_{i0} = a_{i0}$.

2. More generally, assume that $\mathrm{bdn}(D_{ind}) < \kappa'$ and $b \in D$ and $|b| < \lambda$. Let $A = (a_{ij})_{i\in\kappa, j\in\omega}$ be mutually indiscernible over C and $\kappa \geq (\kappa' + \lambda)^+$. Then there is $i < \kappa$ and a sequence $(a'_{ij})_{j\in\omega}$ indiscernible over Cb such that $(a'_{ij})_{j\in\omega} \simeq_C (a_{ij})_{j\in\omega}$ and $a'_{i0} = a_{i0}$.

Proof. We prove the first item and the second can be proved similarly. By item 3 in lemma 9 we need to find $i < \kappa$ for which $\bigcup_{j\in\omega} p_{ij}(x, a_{ij})$ is consistent, where $p_{i0}(x, a_{i0}) = \mathrm{tp}(b/a_{i0}C)$. To get a contradiction, and regarding the fact that each row of A is indiscernible over C, we assume otherwise: for each $i < \kappa$ there is $m_i \in \omega$, $c_i \in C$ and a formula $\chi_i \in \bigcup_{j\in\omega} p_{ij}(x, a_{ij})$ and a tuple

y_i of variables in $x = \{x_1, x_2, \ldots\}$ such that $\{\chi_i(y_i, a_{i0}c_i), \ldots, \chi_i(y_i, a_{im_i}c_i)\}$ is inconsistent (note that b is of size ω and x is a variable of size $|b|$). Since $\kappa \geq (\aleph_0)^+$ and $|b| = \omega$ there are at least ω rows in which the same variable y and the same parameter c occurs and we let $I = \{i_1, i_2, \ldots\}$ be a choice of such rows. Since D is stably embedded, we have

$$\chi_i(y, a_{i0}c) \cap D = \xi_i(y, e_{i0}), \text{ for each } i \in \{i_1, i_2, \ldots\},$$

for some $e_{i0} \in D$ and $\xi_i \in L$. By mutual indiscernibility over C there are e_{ik} for each $k \in \omega$ such that

$$\chi_i(y, a_{ik}c) \cap D = \xi_i(y, e_{ik}), \text{ for each } i \in \{i_1, i_2, \ldots\}.$$

Hence, the pair (B, Ξ) with $B = (e_{ij})_{i \in I, j \in \omega}$ and $\Xi = \{\xi_i | i \in I\}$ witnesses that the theory of D_{ind} is not-strong, contradicting our assumptions. $\square$

Finally, in the following theorem we extract an array with an extended first row from an array with enough number of rows. Unlike the trivial extension lemma, if b comes from a definable D as in the assumptions of the previous lemma, then we can add b itself to the first row, but in the expense of reducing the number of rows of the array.

Theorem 11 (strong row extension lemma)**.**

1. Let D be definable and stably embedded with induced structure D_{ind} whose theory is strong. Let $b \in D$ and $|b| = 1$. Assume that $A = (a_{ij})_{i \in \omega^2, j \in \omega}$ is a mutually indiscernible array. Then we can find $I = \{i_1, i_2, \ldots\} \subseteq \omega^2$ and an array $A^{0b} = ((a'_{0j}b'_{0j})_{j \in \omega}(a'_{kj})_{k \in I, j \in \omega})$ such that
   - A^{0b} is mutually indiscernible.
   - $a'_{00}b'_{00} = a_{00}b$.
   - $(a'_{kj})_{j \in \omega} \simeq_{a_{k0}} (a_{kj})_{j \in \omega}$, and in particular, $a'_{k0} = a_{k0}$, for each $k \in I$.
2. Assume that D is as in above, $b \in D$ is countable, $A = (a_{ij})_{i \in \kappa, j \in \omega}$ is a mutually indiscernible array with $\kappa \geq \aleph_0^+.\omega$. Then there is a countable set $I = \{i_1, i_2, \ldots\} \subseteq \kappa$ and an array $A^{0b} = ((a'_{0j}b'_{0j})_{j \in \omega}(a'_{kj})_{k \in I, j \in \omega})$ with the following properties:
   - it is mutually indiscernible,
   - $a'_{00}b'_{00} = a_{00}b$,

- $(a'_{kj})_{j\in\omega} \simeq_{a_{k0}} (a_{kj})_{j\in\omega}$, and in particular, $a'_{k0} = a_{k0}$, for each $k \in I$.

Proof. We prove only the second item and the proof of the first item is similar. Let $(b_{0j})_{j\in\omega}$ be a sequence such that $a_{0j}b_{0j} \equiv_{(a_{ij})_{i>0,i\in\kappa,j\in\omega}} a_{00}b_{00}$ and $a_{00}b_{00} = a_{00}b$. Use Ramsey and the fact that $a_{0j}b_{0j} \equiv_{(a_{ij})_{i>0,i\in\kappa,j\in\omega}} a_{00}b_{00}$ to find a sequence $(a''_{0j}b''_{0j})_{j\in\omega}$ such that $(a''_{0j}b''_{0j})_{j\in\omega}$ is indiscernible over $(a_{ij})_{0<i<\kappa,j\in\omega}$ and $a''_{00}b''_{00} = a_{00}b$. To simplify the notation, in the rest we assume that $a''_{0j} = a_{0j}$ for each $j \in \omega$ and $(a_{0j}b_{0j})_{j\in\omega}$ is a sequence indiscernible over $(a_{ij})_{0<i<\kappa,j\in\omega}$ with $a_{00}b_{00} = a_{00}b$.

Apply Lemma 10 to the array $(a_{ij})_{0<i<\aleph_0^+,j\in\omega}$ with $b = (b_{0j})_{j\in\omega}$ and $C = (a_{0j})_{j\in\omega} \cup (a_{ij})_{\aleph_0^+<i<\kappa}$, to find $0 < i_1 < \aleph_0^+$ and a sequence $(a'_{i_1j})_{j\in\omega}$ such that $(a'_{i_1j})_{j\in\omega} \simeq_{(a_{0j})_{j\in\omega}\cup(a_{ij})_{\aleph_0^+<i<\kappa}} (a_{i_1j})_{j\in\omega}$ and $(a'_{i_1j})_{j\in\omega}$ is indiscernible over $(a_{0j}b_{0j})_{j\in\omega} \cup (a_{ij})_{\aleph_0^+<i<k}$. Let

$$(a'_{0j}b'_{0j}) \models \mathrm{EM}\big((a_{0j}b_{0j})_{j\in\omega}/(a'_{i_1j})_{j\in\omega} \cup (a_{ij})_{\aleph_0^+<i<\kappa}\big).$$

Then by Lemma 9 (item 1), the two sequences $(a'_{0j}b'_{0j})_{j\in\omega}$ and $(a'_{i_1j})_{j\in\omega}$ are mutually indiscernible over $(a_{ij})_{\aleph_0^+<i<\kappa}$ and in addition, we have $a'_{00}b'_{00} \equiv^{\sigma^0}_{a_{i_10}(a_{ij})_{\aleph_0^+<i<\kappa,j\in\omega}} a_{00}b_{00}$ (note that $a'_{i_10} = a_{i_10}$). Applying σ^0 to both these sequences we obtain two mutually indiscernible rows $(a^0_{0j}b^0_{0j})_{j\in\omega}$ and $(a^0_{1j})_{j\in\omega}$ such that

- $a^0_{0j}b^0_{0j} = a_{00}b$,
- $a^0_{10} = a_{i_10}$,
- $(a^0_{0j})_{j\in\omega} \simeq (a_{0j})_{j\in\omega}$,
- $(a^0_{1j})_{j\in\omega} \simeq (a_{i_1j})_{j\in\omega}$,
- $((a^0_{0j}b^0_{0j})_{j\in\omega}(a^0_{1j})_{j\in\omega})$ is mutually indiscernible over $(a_{ij})_{\aleph_0^+<i<\kappa}$.

We denote the array $((a^0_{0j}b^0_{0j})_{j\in\omega}(a^0_{1j})_{j\in\omega})$ by A^b_1 and the array $(a^0_{ij})_{i=0,1,j\in\omega}$ by A_1.

Assume that the mutually indiscernible array $A^b_n = \big((a^n_{0j}b^n_{0j})_{j\in\omega}(a^n_{ij})_{0<i<n+1,j\in\omega}\big)$ is constructed in which

- $a^n_{00}b^n_{00} = a_{00}b$,

- $a_{k0}^n = a_{i_k 0}$ for $k = 1, \ldots, n$ and elements $0 < i_1 < \ldots < i_n < \kappa$ (kappa), with $\aleph_0^+.(k-1) < i_k < \aleph_0^+.k$,
- $(a_{0j}^n)_{j\in\omega} \simeq (a_{0j})_{j\in\omega}$,
- $(a_{kj}^n)_{j\in\omega} \simeq (a_{i_k j})_{j\in\omega}$, for $k = 1, \ldots, n$ and $i_1 < \ldots < i_n$ as in above,

and so that

- A_n^b is mutually indiscernible over $(a_{ij})_{\aleph_0^+.n<i<\kappa, j\in\omega}$.

Let $A_n = (a_{ij}^n)_{i<n+1, j\in\omega}$. Apply Lemma 10 to the the array $(a_{ij})_{\aleph_0^+.n<i<\aleph_0^+.(n+1), j\in\omega}$ with $C = A_n \cup (a_{ij})_{\aleph_0^+.(n+1)<i<\kappa, j\in\omega}$ and $b = (b_{0j}^n)_{j\in\omega}$, to find i_{n+1} with $\aleph_0^+.n < i_{n+1} < \aleph_0^+.(n+1)$ and a sequence $(a'_{i_{n+1}j})_{j\in\omega}$ to have

- $(a'_{i_{n+1}j})_{j\in\omega} \simeq_C (a_{i_{n+1}j})_{j\in\omega}$,
- $a'_{i_{n+1}0} = a_{i_{n+1}0}$, and
- $(a'_{i_{n+1}j})_{j\in\omega}$ is indiscernible over $A_n^b \cup (a_{ij})_{\aleph_0^+.(n+1)<i<\kappa, j\in\omega}$.

Let $D = \big((d_{0j}c_{0j})_{j\in\omega}(d_{ij})_{0<i<n+1, j\in\omega}\big)$ be such that $D \models \mathrm{EM}\left(A_n^b/(a'_{i_{n+1}j})_{j\in\omega} \cup (a_{ij})_{\aleph_0^+.(n+1)<i<\kappa, j\in\omega}\right)$. Then by Lemma 9 the array $\left(D, (a'_{i_{n+1}j})_{j\in\omega}\right)$ is mutually indiscernible over $(a_{ij})_{\aleph_0^+.(n+1)<i<\kappa, j\in\omega}$. Further

$$d_{00}c_{00} \equiv^{\sigma^n}_{a_{i_{n+1}0}(a_{i_k0})_{k=1,\ldots,n}(a_{ij})_{\aleph_0.(n+1)<i<\kappa, j\in\omega}} a_{00}b.$$

We now apply the automorphism above to the array $(D, (a'_{i_{n+1}j})_{j\in\omega})$ to obtain an array $A_{n+1}^b = \big((a_{0j}^{n+1}b_{0j}^{n+1})_{j\in\omega}(a_{ij}^{n+1})_{0<i<n+2, j\in\omega}\big)$, such that

- A_{n+1}^b is mutually indiscernible over $(a_{ij})_{\aleph_0^+.(n+1)<i<\kappa}$,
- $a_{00}^{n+1}b_{00}^{n+1} = a_{00}b$,
- $a_{k0}^{n+1} = a_{i_k 0}$ for $k = 1, 2, \ldots, n+1$,
- $(a_{kj}^{n+1})_{j\in\omega} \simeq (a_{i_k j}^{n+1})_{j\in\omega}$ for $k = 1, \ldots, n+1$.

Let $I = \{i_1, i_2, \ldots\}$ be obtained by induction as above. Now by compactness and regarding the induction above, there is an array $A^{0b} = \big((a'_{0j}b'_{0j})_{j\in\omega}(a'_{kj})_{0<k\in\omega, j\in\omega}\big)$ with the following properties:

- A^{0b} is mutually indiscernible,
- $a'_{00}b'_{00} = a_{00}b$,
- $(a'_{0j})_{j\in\omega} \simeq (a_{0j})_{j\in\omega}$,
- $a'_{k0} = a_{i_k 0}$ for each $k \in \omega$, and
- $(a'_{kj})_{j\in\omega} \simeq (a_{i_k j})_{j\in\omega}$ for each $k \in \omega$;

and this finishes the proof. □

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