Model Theory 2, Session 9

Definition 1. $p \in S_n(B)$ is definable/C if for each $\phi(\bar{x}, \bar{y}) \in L$ there exists a formula $\psi_\phi(\bar{y})$ in $L(C)$ such that

$$\forall \bar{b} \quad \left[\phi(\bar{x}, \bar{b}) \in p \Leftrightarrow \models \psi_\phi(\bar{b})\right]$$

Example 1. If T is strongly minimal, then each $p \in S(A)$ is definable.

Let us establish what we require for working this example out. Our T is arbitrary but complete.

Lemma 1. If $b \in \mathrm{acl}(aA)$ then

$$\mathrm{RM}(b/A) \leq \mathrm{RM}(a/A).$$

So

$$\mathrm{acl}(a) = \mathrm{acl}(b) \Rightarrow \mathrm{RM}(a) = \mathrm{RM}(b).$$

This was proved last session, modulo a small error which is to be settled in our tutorial session.

Lemma 2. Assuming that T is strongly minimal, we have

$$\mathrm{RM}(a_1, \ldots a_n/B) = \dim(a_1, \ldots, a_n/B).$$

Proof. By the above lemma, we assume that $a_1, \ldots, a_n$ are algebraically independent over B. Let $\psi \in \mathrm{tp}(a_1, \ldots, a_n/B)$. We first show that $\mathrm{RM}(\psi) \geq n$, and then show that $\mathrm{RM}(tp(a_1, \ldots, a_n/B)$ is to n.
By induction hypothesis, we have

$$\mathrm{RM}(a_1, \ldots, a_n/Ba_1) = n - 1.$$

This means that $\mathrm{rank}(\chi) \geq n - 1$, where χ is the following formula:

$$(x_1 = a_1) \wedge \psi(x_1, \ldots, x_{n-1}).$$

At the same time, all conjugates of χ over B are disjoint with rank $\geq n - 1$. So $\mathrm{RM}(\psi) \geq n$.
Let $B' \supset B$.

Reminder 1. If T is strongly minimal and $M \models T$ and A, B are independent subsets of M and $f : A \to B$ is a bijection, then f is an elementary map:

$$\forall a_1 \ldots a_n \in A \quad \mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(f(a_1), \ldots, f(a_n)).$$

In particular if $a_1, \ldots, a_n$ are independent and $b_1, \ldots, b_n$ are independent, then

$$\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n).$$

There is only one type $p \in S_n(B')$ realised by a B'-independent sequence of elements. This implies that $\mathrm{rank}(\mathfrak{C}^n) \leq n$. □

Corollary 1. Assuming that T is strongly minimal and ψ a formula over B, we have

$$\mathrm{RM}(\psi) = \max\{\dim(\bar{a}/B) | \mathfrak{C} \models \psi(\bar{a})\}.$$

In strongly minimal theories, Morley rank is definable.

Corollary 2. Suppose that T is strongly minimal. For each k, The following is a definable class:

$$\{\bar{b} | \operatorname{RM} \psi(x_1, \ldots, x_n, \bar{b}) = k\}.$$

Proof. By induction on n we will show that "$\mathrm{RM}(\psi(\bar{x}, \bar{b}) \geq k$" for $|\bar{x}| = n$ is expressed in $\mathrm{tp}(\bar{b})$ (it is an elementary property of $\bar{b}$). A property P being an elementary property of b means that there is formula ϕ such that $b \models \phi$ and for every a if $a \models \phi$ then a also has this property; in other words $\phi(\mathfrak{C})$ is the set of elements with that property.
For $n = 1$ we have

$$\operatorname{RM} \psi(x_1, \bar{b}) \geq 1 \Leftrightarrow \exists_{\text{infinitely many}}\, x_1 \quad \psi(x_1, \bar{b}).$$

The above is an elementary property of $\bar{b}$ since T is strongly minimal.
(Induction step) By the previous corollary,

$$\begin{aligned} \mathrm{RM}(\psi(\bar{x}, \bar{b})) \geq k &\text{ if and only if} \\ \exists a_1 \quad \mathrm{RM}(\psi(a_1, x_2, \ldots, x_n, \bar{b}) &\geq k \\ &\text{or} \\ \exists a_1 \notin \mathrm{acl}(\bar{b}) \quad \mathrm{RM}(\psi(a_1, x_2, \ldots, x_n, \bar{b}) &\geq k - 1. \end{aligned}$$

verify that the above is expressible by $\mathrm{tp}(\bar{b})$ (the first one by the induction hypothesis, the second one by the fact that $\mathrm{RM}(\psi(a_1, x_2, \ldots, x_n, \bar{b}) \geq k-1$ can be expressed by a formula $\theta(a_1, \bar{b})$ and the condition is equivalent to there are infinitely many x_1 such that $\theta(x_1, b)$.

□

Let $p \in S(A)$ have $\mathrm{RM}(p) = \alpha$ and $\deg(p) = d$. Then there is $\phi \in p$ such that $\operatorname{RM} \phi = \alpha$ and $\deg \phi = d$. we have

$$\forall \psi \quad \psi \in p \Leftrightarrow \mathrm{RM}(\phi \wedge \neg\psi) < \alpha.$$

So

$$p = \{\psi | \psi \in L(A), \mathrm{RM}(\phi \wedge \neg\psi) < \alpha\}.$$

Back to example 1 let $\phi_0 \in p$ be such that $(\operatorname{RM} \phi, \deg \phi)$ is minimal.

$$\psi(x, a) \in p \Leftrightarrow \mathrm{RM}(\phi_0(x) \wedge \neg\psi(x, a)) < k$$

By corollary 2 this is a property expressed in $\mathrm{tp}(\bar{a})$.
We proved in the previous session that:

Reminder 2. Let $p \in S(M)$ be a definable type. Then for every $B \supseteq M$, p has a unique extension $q \in S(B)$ definable over M. q is the unique heir of p and

$$q = \{\phi(x, \bar{b}) | \phi(x, \bar{y}) \in L, \bar{b} \in B, \mathfrak{C} \models d_p x \phi(x, \bar{b})\}.$$

Proposition 1. Let T be strongly minimal, $M \models T$ and assume that B includes M. Then

$$[\mathrm{tp}(a/B) \text{ is an heir of } \mathrm{tp}(a/M)] \Leftrightarrow [\mathrm{RM}(a/B) = \mathrm{RM}(a/M)].$$

Remark 1. We have seen that if T is strongly minimal, then

$$\mathrm{RM}(a/B) = \mathrm{RM}(a/M) \Leftrightarrow a \underset{M}{\overset{acl}{\downarrow}} B$$

Since this notion is symmetric, if T is strongly minimal then (heir=coheir). This is the case also in stable theories (and we will see this).
The mentioned fact about strongly minimal theories implies that

$$T \text{ strongly minimal} \quad M \models T \quad M \subseteq B \quad p \in S(M)$$
$$\Rightarrow$$
$$\exists_{unique} q \supseteq p \quad \mathrm{RM}(q) = \mathrm{RM}(p).$$

The above is the case for all totally transcendental theories.

proof of Proposition 1. assume that $\mathrm{RM}(\mathrm{tp}(a/M)) = k$. Let $\phi \in \mathrm{tp}(a/M)$ be such tha $(\mathrm{RM}(\phi), \deg \phi) = (\mathrm{RM}\, p, \deg p)$. By the previous lemma, the following is the unique heir of p on B:

$$\{\psi(x) | \psi \in L(B), \mathrm{RM}(\phi_0 \wedge \neg\psi(x, a)) < k\}.$$

The above set is contained in all q with $q \in S(B)$, $p \subseteq q$, $\mathrm{RM}(q) = k$. □

Stability

In our Model theory 1 course, we defined a theory to be stable if it is κ-stable for some κ. We defined a theory to be κ-stable if whenever $|A| = \kappa$ we have $|S(A)| = \kappa$. In this section we are going to provide an alternative formulation for stability based on a property satisfied by a formula.

Definition 2. For $\phi(x, y)$ a formula, we define

$S_\phi(B) :=$ maximal consistet set of formulae of the form $\phi(x, b)$ or $\neg\phi(x, b), b \in B$

Definition 3.

1. ϕ is **stable** if there exists a cardinal λ such that

$$|B| \leq \lambda \Rightarrow |S_\phi(B)| \leq \lambda.$$

2. T is **stable** if all its formulae are.

3. ϕ has the **order property** (OP) if there are $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ with

$$i < j \text{ if and only if } \models \phi(a_i, b_j);$$

equivalently (to be proven) if there are $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ with

$$i > j \text{ if and only if } \models \phi(a_i, b_j).$$

4. ϕ has the **binary tree property** (BTP) if there exists a binary tree $(b_s | s \in 2^{<\omega})$ of parameters b_s each of whose branches is consistent; accurately: for each $\sigma \in 2^\omega$ the following set is consistent.

$$\{\phi^{\sigma(n)}(x, b_{\sigma \upharpoonright n}) | n < \omega\}.$$

Theorem 1. The following are equivalent for a formula ϕ.

a. ϕ is stable.

b. B infinite $\Rightarrow |S_\phi(B)| \leq |B|$.

c. ϕ does not have OP.

d. ϕ does not have BTP.

Proof. $a \Rightarrow d$. Let μ be minimal with $2^\mu > \lambda$. The tree $I = 2^{<\mu}$ has cardinality $\leq \lambda$. If ϕ has BTP, by compactness there is a tree of parameters indexed by I to witness this. By completing each branch to a ϕ-type, we will see that over the parameter set $B = \{b_s | s \in I\}$, we have $|B| \leq \lambda < 2^\mu$, and $|S_\phi(B)| \geq 2^\mu$.
$d \Rightarrow c$. Let $I = 2^{<\omega}$. There is a linear order on I such that

$$\forall \sigma \in I \quad \forall n \in \omega \quad \sigma < \sigma \upharpoonright n \Leftrightarrow \sigma(n) = 1.$$

By standard lemma, we can find $(a_i b_i)_{i \in I}$ such that

$$i < j \Leftrightarrow \phi(a_i, b_j).$$

Now $(\phi(x, b_s) | s \in 2^{<\omega})$ witnesses BTP.
$c \Rightarrow b$ (next session) ☐