Strongly minimal theories and Morley's categoricity theorem.

1 Motivation

Let us begin by reminding ourselves of the final aim of this chapter [1]

Theorem 2 (Morley)**.** If T is a countable theory, then T is $\aleph_1$-categorical if and only if it is κ-categorical for all **uncountable** κ's.

Idea of the proof. This may remind you of the vector spaces: in some suitable sense every model of T is generated by a base of cardinality κ. This base is in a suitable sense independent and every two models whose bases have the same cardinality are isomorphic. □

In the rest, we are going to see what these suitable senses are! note that the word uncountable is in bold for a reason. For example ACF_0 is $\aleph_1$-categorical and not $\aleph_0$-categorical (there are $\aleph_0$ many algebraically closed fields of cardinality $\aleph_0$ (find out why).

The last ingredient we need for the proof of Morley's theorem is strong-minimality. Note that there are two concepts (related to one another of course): strongly minimal sets, and strongly minimal theories.

Strongly minimal theories are similar to vector spaces in the sense that they are concerned with the concepts of dimension, independence, basis, isomorphism of structures with similar bases, etc. You may now say, so they are also similar to algebraic closed fields, where algebraic independence replaces linear independence and there is a concept of basis and algebraic independence and transcendental elements in this sense. Indeed there is famous trichotomy conjecture by Zilber that (very crudely speaking) strongly minimal theories are of one of the following three kinds: either they are essentially trivial, or they are essentially like a vector space, or they are essentially like an algebraic

[1] (irrelevant and only for your knowledge:) also for stability Shelah has proved the following:

Theorem 1. For T complete in a countable L, one of the following holds:

1. T is not k-stable for any cardinal κ.
2. T is κ-stable for all $\kappa \geq 2^{\aleph_0}$
3. T is κ-stable if and only if $\kappa^{\aleph_0} = \kappa$.

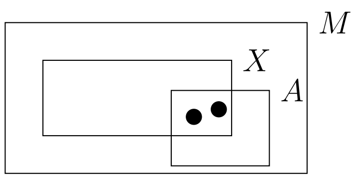
closed field (whatever essentially means!). This conjuncture is disproved by Hrushovski.
Strongly minimal theories are uncountably categorical and the reason is having an independent basis.

2 Strongly minimal sets

T is complete, countable, possesses infinite models. If $M \models T$, $\phi(x_1, \ldots, x_n, \bar{a})$ is a formula with $a \in M$, by $\phi(M, \bar{a})$ we mean $\{\bar{y} \in M^n : M \models \phi(\bar{y}, \bar{a})\}$
Sometimes it is easier to consider definable sets instead of the formulae defining them:

Definition 3. Suppose that $M \models T$, and $X \subseteq M^n$ is an infinite definable set (defined with parameters in M).

1. X is called minimal in M if for every definable (with parameters in M) set A either $X \cap A$ or $X \cap A^c$ is finite.

   X a definable set

   for every definable A either $A \cap X$ finite or $A^c \cap X$

2. A formula $\phi(\bar{x})$ (in $L(M)$) is strongly minimal if it defines a minimal set in all elementary extensions of M.

3. A non-algebraic type is strongly minimal if it contains a strongly minimal formula.

4. T is strongly minimal if every definable set in every model of T is either finite or cofinite (different wordings: T is strongly minimal if $x = x$ is a strongly minimal formula; T is strongly minimal if all sets are definable using only $\{=\}$)

Exercise 4. Show that the following theories are strongly minimal and in each case determine the closure of a given set X and make sense of the concepts of independence and bases:

1. The theory of ACF_0
2. The theory of K-vector spaces for a field K.

Exercise 5. if M is ω-saturated then every minimal formula is strongly minimal.

Theorem 6 (Exchange Principle for strongly minimal sets, direct proof)**.** Suppose that $M \models T$ and X is a strongly minimal subset of M defined without parameters. Let A be a subset of X and a, b be two elements of X. If $a \in \text{acl}(A \cup \{b\}) - \text{acl}(A)$, then $b \in \text{acl}(A \cup \{a\})$.

Proof. I do the proof for $A = \emptyset$ and the general case is no different. I thank Michael Lösch for having corrected and provided the following version of the proof: we want to prove that for all $a, b \in X$, if $a \in \text{acl}(b)$ then $b \in \text{acl}(a)$ provided that $a \notin \text{acl}(\emptyset)$.
We need a formula $\eta(x, a)$ such that $\eta(M, a)$ is finite and $b \in \eta(M, a)$. Let $\phi(x, b)$ be a formula witnessing the assumption $a \in \text{acl}(b)$, that is $a \in \phi(M, b)$ and $|\phi(M, b)| = n$ for some finite n, and assume that X is defined by the formula χ, that is $X = \{x \in M : M \models \chi(x)\}$. By our assumptions, the following hold for b:

- $b \in \phi(a, M)$,
- $b \in \psi(M)$, where $\psi(x)$ is the equivalent formula to $|\phi(M, x)| = n$,
- $b \in \chi(M)$.

Let us consider the formula $\eta(x, a) = \phi(a, x) \wedge \psi(x) \wedge \chi(x)$. By the items above, $b \in \eta(M, a)$ and it suffices to show that $\eta(M, a) = X \cap \psi(M) \cap \phi(a, M)$ is finite. To reach a contradiction, we assume that $\eta(M, a)$ is infinite. By strongly minimality of X, we get

$$|X - (\psi(M) \cap \phi(a, M))| = l$$

for some finite l. Let $\xi(x)$ be the formula expressing

$$|X - (\psi(M) \cap \phi(x, M))| = l.$$

We know that $a \in \xi(M)$. If $\xi(M)$ is finite then $a \in \text{acl}(\emptyset)$, and this is contradictory with our assumptions. Hence $\xi(M)$ is infinite and we can find

$n+1$-many elements, say $a_1, \dots a_{n+1}$, in $\xi(M)$. We define $B_i := X \cap \psi(M) \cap \phi(a_i, M)$. Since X is infinite and $|X \backslash B_i| = l$, we have

$$\bigcap_{i=1}^{n+1} B_i = X \backslash (\bigcup_{i=1}^{n+1} X \backslash B_i)$$

is infinite and in particular nonempty. So, let $\hat{b} \in \bigcap B_i$. Then for each i, $\phi(a_i, \hat{b})$ holds, that is $|\phi(M, \hat{b})| \geq n+1$. This contradicts the fact $\psi(\hat{b})$ holds. □

Definition 7. $A \subseteq X$ is independent if for every $a \in A$

$$a \notin \mathrm{acl}(A - \{a\}).$$

Let $C \subseteq X$, then A is independent over C means that for all $a \in A$

$$a \notin \mathrm{acl}(C \cup (A - \{a\}))$$

We say that A is a basis for $Y \subseteq X$ if $A \subseteq Y$, A is independent, and $\mathrm{acl}(A) = Y$.

As one expect from comparing the notion of basis with the notion of basis in vector spaces and in Algebraic closed fields, any two bases have the cardinality.

Lemma 8. If A and B are two bases for $Y \subseteq X$ then $|A| = |B|$.

Definition 9. If $Y \subseteq X$, then the dimension of Y is the cardinality of a basis of Y. Let $\dim(Y)$ denote it.

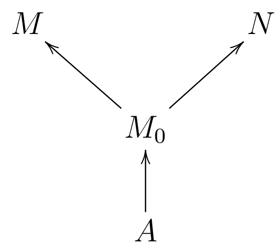
Every model of a strongly minimal theory is determined (up to isomorphism) by its dimension. This is what have we have stated in the theorem below:

Theorem 10. Suppose that T is strongly minimal and $M, N \models T$. Then $M \cong N$ if and only if $\dim(M) = \dim(N)$ (note that here $X = M$).

Idea of the proof. As we may expect, the bijection between the bases of M and N will extend to an isomorphism of M and N. □

Let us prove a more general theorem instead:

Theorem 11. Consider the following diagram:



$$M_0 \prec M, M_0 \prec N$$
$$A \text{ subset of } M_0$$

Let $\phi(x)$ be a strongly minimal formula with parameters from A. Suppose that $\dim(\phi(M)) = \dim(\phi(N))$. Then there is a partial elementary map $f : \phi(M) \to \phi(N)$.

The proof of this theorem will also remind you of linear algebra. Let us just remind ourselves of the following:

Observation 12. If a is algebraic over B then $\mathrm{tp}^M(a/B)$ is isolated.

proof of the observation. You have once proved this in your exercise sessions, but I still like to mention it again. Whenever we say $\phi(x)$ isolates $P(x)$ (P a type in variable x) it means that every for every x, if x satisfies ϕ then it satisfies all formulae in P, that is

$$\phi(M) = P(M)$$

Now let ϕ be the formula that witnesses that a is algebraic over B. Then $\phi \in P$ and let's say $\phi(M) = \{a_1, \ldots, a_n\}$. Then $P(M) \subseteq \{a_1, \ldots, a_n\}$. We will throw away those a_i's that do not realise all formulae in P. Let's say $a_{i1}, \ldots, a_{im}$ do not respectively satisfy $\psi_1, \ldots, \psi_m$. Then

$$(\phi \wedge \psi_1 \wedge \ldots \psi_m)(M) = P(M)$$

and this is what we need. $\square$

proof of the theorem. Let B be a transcendence basis for $\phi(M)$ and C be a transcendence basis for $\phi(N)$. Because $|B| = |C|$, there exists a bijection $f : B \to C$.

Claim 13. f is an elementary map.

hint on the proof of the claim. You have seen before (in the previous lecture), and you will prove again in the exercise sessions that in the diagram above, if $a_1, \ldots, a_n \in \phi(M)$ are independent over A and $b_1, \ldots, b_n \in \phi(N)$ are independent over A, then $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$. Indeed if $B \subseteq \phi(M)$ is infinite and and $C \subseteq \phi(N)$ is infinite, then B and C are sets of indiscernibles with the same type. □

Let

$$I := \{g : B' \to C' : B \subseteq B' \subseteq \phi(M), C \subseteq C' \subseteq \phi(N), f \subseteq g, g \text{ elementary}\}$$

By Zorn's lemma I has a maximal element $g : B' \to C'$. We now claim that indeed $B' = \phi(M)$ and $C' = \phi(N)$.

So suppose otherwise; let $b \in \phi(M) - B'$. Since $\mathrm{acl}(B) = \phi(M)$, we have b is algebraic over B and hence over B'. So $\mathrm{tp}(b/B')$ is isolated.
So there is $\psi(x, \bar{d})$ with $\bar{d} \in B'$ isolating $\mathrm{tp}(b/B')$. Because g is partial elementary, there is $c \in \phi(N)$ such that $\psi(c, g(\bar{d}))$ holds. So

$$\mathrm{tp}^M(b/B') = \mathrm{tp}^N(c/C')$$

and this means that we can extend g by sending b to c, and this is contradictory with the fact g is maximal. Thus $\phi(M) = B'$. Showing that $\phi(N) = C'$ is similar. □

Corollary 14. If a strongly minimal T is countable, then it is κ-categorical for all $\kappa > \aleph_0$.

Proof. We use Theorem 10. Let $M \models T$. Note that $M = \mathrm{acl}(B)$ for B its transcendence basis. Also

$$acl(B) = \bigcup_{\phi \in L, b \in B, \phi(M,b) \text{ finite}} (\phi(M, b))$$

so for every B the size of $\mathrm{acl}(B)$ is $\leq |B| + |T|$.
If M has size $\kappa \geq \aleph_1$ then $|B| = \kappa$. This means that two models of size κ have bases with the same dimension, so they are isomorphic. □

Corollary 15. Let T be a strongly minimal theory.

1. T is λ-stable for all $\lambda \geq |T|$.

2. T is totally transcendental.

3. T has no Vaughtian pairs.

Proof. 1. First note that if T is strongly minimal and A is a parameter set, then there can exactly one non-algebraic type be over A. Because if there are two types p_1 and p_2 then there are $\psi_1 \in p_1$ with $\neg\psi_1 \in p_2$. So both $\psi_1(M)$ and $\neg\psi(M)$ are infinite, which is not possible in strongly minima theories.

So, for every set A with $|A| = \lambda \geq |T|$ we have

$$|S(A)| \leq |\operatorname{acl}(A)| + 1(\text{the number of algebraic and non-algberaic types})$$

furthermore $|\operatorname{acl}(A)|$ is no greater than $|A|$ because it is a union of $\bigcup_{a\in A,\phi\in L} \phi(M, a)$ where each $\phi(M, a)$ is finite.

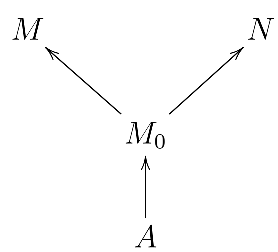
2. Suppose that there is a binary tree. Suppose that ϕ and $\neg\phi$ have appeared on top of the same node on the tree. One of $\phi(M)$ and $\neg\phi(M)$ is finite, let's say $\phi(M)$ is finite. The tree cannot contiune with infinite branches over ϕ, contradiction.

3. Suppose that (M, N, ϕ) form a Vaughtian pair with $N \prec M$ and $\phi(M) = \phi(N)$ infinite. So there are only finitely many elements in N not satisfying ϕ. Say there are n elements. Since the fact: 'there are n elements not satisfying ϕ' is expressible in the language, and $N \prec M$,

$$\{x \in M | M \models \neg\phi(x)\} = \{x \in N : N \models \neg\phi(x)\}$$

that is $M = N$, a contradiction. □

Exercise 16. Consider the following diagram:



$M_0 \prec M, M_0 \prec N$
A subset of M_0

1. let $\bar{a}$ be a tuple in A and $\phi(x, \bar{a})$ a formula. Then show that the fact that

$$\phi(x, \bar{a}) \text{ defines a strongly minimal set in } M$$

is an elementary property of $\bar{a}$ contained in the $\mathrm{tp}^M(\bar{a})$. It means that in the above diagram if $\phi(x, \bar{a})$ defines a strongly minimal set in M then it defines a strongly minimal set in N too.

2. Suppose that $a_1, \ldots, a_n$ in $\phi(M)$ are independent over A and $b_1, \ldots b_n \in \phi(N)$ are independent over A. Then show that $\mathrm{tp}^M(\bar{a}/A) = \mathrm{tp}^N(\bar{b}/A)$.

3. Let $B \subseteq \phi(M)$ be infinite and independent over A. Show that B is a set of indiscernibles over A (note that being a set of indiscernibles is a stronger property than being a sequence of indiscernibles).

4. Let $C \subseteq \phi(N)$ be infinite and independent over A. Show that C is a set of indiscernibles over A with the same type as type of B.

Exercise 17. 1. Let T be ω-stable. Show that if $M \models T$ then there is a minimal formula in M.

2. If $M \models T$ is $\aleph_0$-saturated and $\phi(\bar{x}, \bar{a})$ is minimal in M, then it is strongly minimal.

3 Baldwin-Lachlan theorem and Morley's theorem

Theorem 18. Let T be a complete theory with infinite models in a countable language. Let κ be an uncountable cardinal. Then

$$\begin{gathered} T \text{ is } \kappa\text{-categorical} \\ \text{if and only if} \\ T \text{ is } \omega\text{-stable and has no Vaughtian pairs.} \end{gathered}$$

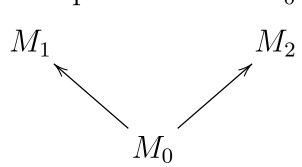
in particular (because the above characterisation does not depend on κ):

$$\begin{gathered} T \text{ is } \aleph_1\text{-categorical} \\ \text{if and only if} \\ T \text{ is } \kappa\text{-catergorical (for each uncountable } \kappa). \end{gathered}$$

Reminder: the following items are all proved in previous lectures and I have listed them only for our ease of reference.

1. if T is κ-categorical (uncountable κ) then T is ω-stable.
2. if T is κ-categorical (uncountable κ) then T has no Vaughtian pairs.
3. T has a prime model if and only if the isolated types are dense,
4. if T has no binary tree of consistent formulae then the isolated types are dense,
5. ω-stable theories have no binary tree of consistent formula (they are totally transcental)
6. (as a result of the facts above) if T is ω-stable, then it has a prime model (needed here).
7. if T is totally transcendental then there is a minimal formula in each $M \models T$,
8. if T eliminates $\exists^{\infty}$ then any minimal formula is strongly minimal,
9. a theory without Vaughtian pairs eliminates $\exists^{\infty}$,
10. (as a result of the facts above) if T is ω-stable with no Vaughtian pair, then there is a strongly-minimal formula (needed),
11. if T has no Vaughtian pairs and $M \models T$ and $A \subseteq M$ and ϕ a formula with parameters in A, then M is a minimal extension of $\phi(M) \cup A$. (needed)

proof of the theorem. One direction is clear by items 1 and 2. Let us do the more difficult direction: suppose that T is ω-stable and it has no Vauhtian pairs. then by item 10 above, there exists a strongly minimal formula $\phi(x)$. Now let M_1 and M_2 be two models of cardinality κ. By item 6 above, there exists a prime model M_0:



$M_0 \prec M_1, M_0 \prec M_2,$ all three models of T

We now want to prove that $\dim(M_1/M) = \dim(M_2/M) = \kappa$ and then this implies that M_1 and M_2 are isomorphic (by theorem 10).
Since T has no Vaughtian pairs, M_1 is a minimal extension of $\phi(M_1) \cup M_0$ and M_2 is a prime extension of $\phi(M_2) \cap M_0$ (item 11). Therefore $\phi(M_1)$ has cardinality κ and hence $\dim(M_1/M) = \dim(M_2/M) = \kappa$. □