

THE FIRST ORDER THEORY OF A
DENSE PAIR AND A DISCRETE
GROUP

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The first order theory of a dense pair and a discrete group

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Let \tilde{T} be the theory of an expansion of $\langle \bar{\mathbb{R}}, +, \cdot, 0, 1, < \rangle$ which is o-minimal, model complete and polynomially bounded with \mathbb{Q} -exponents. We introduce a theory \mathbb{T} whose models are of the form $\mathbb{M} = \langle \tilde{M}, G, A \rangle$, where $\tilde{M}, \tilde{G} \models \tilde{T}$, \tilde{G} is an elementary substructure of \tilde{M} , G is dense in M , and A is a discrete multiplicative subgroup of $\langle M, \cdot \rangle$.

We will prove that \mathbb{T} is complete and hence it axiomatises $\text{Th}(\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle)$ when \tilde{T} is $\text{Th}(\bar{\mathbb{R}})$. We will then prove that if $\mathbb{M} \models \mathbb{T}$ and $\psi(\bar{z})$ is a formula in $L(\mathbb{T})(M)$, then it has an equivalent which is a Boolean combination of the formulas of the form

$$\exists \bar{x} \in G \exists \bar{y} \in A \phi(\bar{x}, \bar{y}, \bar{z})$$

where $\phi(\bar{x}, \bar{y}, \bar{z})$ is a formula in $L(\tilde{T})(M)$. Using this, we will characterise the definable sets and the types of tuples in M , for a model \mathbb{M} of \mathbb{T} . This characterisation, says in particular that $\langle \mathbb{Z}, +, \cdot, < \rangle$ is not interpretable in a model of \mathbb{T} and in spite of having a discrete and a dense subset in our structures, they are tame regarding the fact they do not exhibit the Gödel phenomenon.

We will note that the open definable subsets of M in $\langle \tilde{M}, G, A \rangle$ can be defined in $\langle \tilde{M}, A \rangle$, and towards proving that every open definable subset of \mathbb{R}^n in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ is definable in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$, we will prove that open definable subsets of \mathbb{R}^n which are defined by *special* formulas with parameters in \mathbb{R}_{alg} can be defined in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.

In our last chapter, we will prove that \mathbb{T} has NIP (not the independence property).

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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I summarise the whole experience of the last three years of my life in being a student of Alex Wilkie. It was mathematics for which I left home and nothing in England was as good as the weekly meetings with him each of which was the opening of a new chapter for me.

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Finally, I have to say that, this thesis, although it is not as strong as I always wanted my thesis to be, is written by a tired maths lover, who once had to sleep still in bed and not move for forty days for his backache, who was always awake in the long beautiful desert nights of Torbate Jam twisting his hair as he was thinking on his maths, who waited for six months before getting to the UK, who spent thirty days in the scolding hotness of Dubai just to apply for the visa to the UK, who was stranded for forty days in Italy for his stolen passport, who was very recently rejected for a visa to France and could not attend the final Maloa conference, and who, at the very moment, is not sure if he can extend his visa for the viva of this thesis. I wish any

spiritual reward of this fatigueness, to be bestowed upon the great soul of a father whose pains were far greater than mines and whom I missed more and more as I grew up.

To my loneliness,
who never leaves me alone.

Chapter 1

Introduction

‘Tameness’ or ‘Wildness’ of a topological space, algebraic structure or a model theoretic structure, is an intuitively understandable, but not uniquely defined, notion for mathematicians. One can say that a mathematical theory is ‘tame’ when the ‘behaviour’ of its models is predictable. Another approach can be calling a particular behavior ‘wildness’ and calling our models tame if they do not show that behaviour. As far as the author knows, labeling a ‘topological’ space tame was first due to Grothendieck, in ‘Esquisse D’un Program’, ‘Sketch of a Program’, [28],[13], in the part ‘Denunciation of so-called general topology and heuristic reflections toward a so-called “tame topology”’. In his—translated—words:

My approach toward possible foundations for a topology has been an axiomatic one, rather than declaring (which would indeed be a perfectly sensible thing to do) that the desired “tame spaces” are no other than (say) Hironaka’s semi-analytic spaces, and developing in this context the toolbox of constructions and notions which are familiar from topology supplemented by those which had not been developed up to now. For that very reason I preferred to work on extracting which exactly, among the geometrical properties of the semi-analytic sets in \mathbb{R}^n , make it possible to use these as “local models” for the notion of “tame space”... and what (hopefully!) makes this notion flexible enough to use it effectively as a fundamental notion for a “tame topology” which would express with ease the topological intuition for shapes.

In a similar vein to the first paragraph, a model theoretic structure can be called tame if its definable sets are ‘topologically’ tame. We can define wildness as the occurrence of a particular phenomenon. We will explain these two in the following.

As a negative answer to Hilbert’s Second Problem —crudely stated as whether or not the whole of our mathematics can be axiomatised— Gödel proved that, in the author’s terms, any effectively generated theory containing basic arithmetic, can not prove its own consistency. More specifically, there is a logically valid arithmetical sentence which, despite being true, is not provable from the natural axioms of Peano arithmetic. So, Peano arithmetic is undecidable and model theoretically a very wild theory with some models that do not bear any resemblance to natural numbers. As a result of this, the theory of the seemingly simple structure $\langle \mathbb{Z}, +, \cdot \rangle$ is undecidable and untame, or let us say, wild, simply because it exhibits the Gödel phenomenon.

To avoid this, if we use the trick of replacing the binary multiplication function with countably many unary predicates for $n\mathbb{Z}$, $n \in \mathbb{N}$, we get a model of the so-called Presburger arithmetic, $\langle \mathbb{Z}, +, \{n\mathbb{Z}\}_{n \in \mathbb{N}}, < \rangle$, which is decidable and eliminates quantifiers (=shows tame behaviour) [20].

Now consider the structure $\langle \mathbb{R}, +, \cdot, < \rangle$ —denoted usually by $\bar{\mathbb{R}}$. Quantifier-free definable sets in this structure are semi-algebraic sets. Tarski proved that the projections of semi-algebraic sets are semi-algebraic. In model theoretic terms, the structure $\bar{\mathbb{R}}$, and more generally, the theory of real closed fields, eliminates quantifiers and is decidable. In fact, the definable sets in this structure are ‘topologically’ tame.

The behaviour of $\langle \mathbb{R}, +, \cdot, < \rangle$ led Pillay and Steinhorn to introducing the concept of o-minimality [25]: Let L be a language which contains a symbol for the strict order. An L -structure in which $<$ is interpreted as a strict order, is o-minimal if its one-variable definable sets (with parameters) can be defined only with $<$, and a theory is o-minimal, if all its models are. Definable sets in o-minimal structures have similar ‘patterns’. They are finite unions of the so-called cells. $\langle \mathbb{R}, +, \cdot, < \rangle$ is o-minimal and its definable sets are semi-algebraic sets. $\langle \mathbb{R}, +, \cdot, <, \{f\}_{f \in F} \rangle$, where $F = \{\text{germs of analytic functions on a bounded closed box around zero}\}$, is o-minimal (Gabrielov [12], van den Dries [5]), and its definable sets are subanalytic sets. $\langle \bar{\mathbb{R}}, \exp \rangle$ is o-minimal, [32] its definable sets are ‘sub-exponential algebraic sets’. All o-minimal structures are topologically and—interestingly due to this— model theoretically tame; for example,

the Grothendieck ring of an o-minimal structure is $(\mathbb{Z}, +, \cdot, <)$ [6].

Now, let us combine the structure we described to be wild, $\langle \mathbb{Z}, +, \cdot \rangle$, with the one we described as tame, $\langle \mathbb{R}, +, \cdot, < \rangle$, to get the structure $\langle \mathbb{R}, \mathbb{Z}, +, \cdot, < \rangle$, in a language with symbols for $+$, \cdot , $<$, and a unary predicate for \mathbb{Z} . The first obvious observation is that as $\langle \mathbb{Z}, +, \cdot \rangle$ is interpretable in this structure, this structure is not tame. This is true, but not the whole truth! In fact, as the following proposition states, definable sets in this structure are sets which are of interest to descriptive set theorists. The definable sets in this structure build up the ‘Projective Hierarchy’.

Proposition. *A set $A \subseteq \mathbb{R}^n$ is projective if and only if it is definable in $\langle \mathbb{R}, \mathbb{Z}, +, \cdot, < \rangle$.*

Projective sets are sets obtained from Borel sets by the operations of projection (or continuous image) and complementation. For more on the Projective Hierarchy see [18]. The above theorem is Exercise 37-6 in the same reference.

When working on the Grothendieck rings of expansions of \mathbb{R} , the author noticed that $\langle \mathbb{R}, +, 0, 1, <, \text{floor function} \rangle$ eliminates quantifiers; obviously it is well-known and Miller has proved it in the appendix of [22]. In fact, as we will see, the floor function can be replaced by a similar function which takes values in a different discrete set and adding a floor function to a real closed field is a helpful tool when we want to add a discrete set to it.

So, expanding $\bar{\mathbb{R}}$ with a predicate for \mathbb{Z} has disappointing and encouraging sides. But what if we replace \mathbb{Z} with another (discrete) set? In [23], Miller also studies the more general question of which expansions of $\langle \mathbb{R}, +, \cdot \rangle$ can be called tame. We summarise some of his questions as follows:

Question.

1. *Which discrete sets can be added (with a predicate) to $\langle \mathbb{R}, +, \cdot \rangle$ where the Gödel phenomenon does not appear?*
2. *Which dense-codense sets can be added as a predicate to $\langle \mathbb{R}, +, \cdot \rangle$ with Gödel phenomenon not appearing?*
3. *Which Borel sets and cantor sets can be added to $\langle \mathbb{R}, +, \cdot \rangle$ where the Gödel phenomenon does not appear?*

In Chapter 2 and towards an answer to the first part of the above question, we will show how the following situation is dealt with. Let $2^{\mathbb{Z}}$ be the set $\{2^x : x \in \mathbb{Z}\}$. This set with multiplication (which it inherits from \mathbb{R}) forms an abelian group. One can easily check that the structure $\langle 2^{\mathbb{Z}}, 2^{n\mathbb{Z}}, \cdot, < \rangle$ is a model of Presburger arithmetic and hence subject to quantifier elimination. Van den Dries, proved in [8] that the structure $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}}, 2^{n\mathbb{Z}}, \lambda \rangle$, with the following definition for λ , admits quantifier elimination.

$$\lambda(x) = y \Leftrightarrow [y \in 2^{\mathbb{Z}} \text{ and } y \leq x < 2y].$$

The theory of the structure $\langle \mathbb{R}, 2^{\mathbb{Z}}, +, \cdot, < \rangle$ can be simply axiomatised by the most natural complete set of axioms. These axioms are those which say that in a model of our theory, there is a real closed field with a multiplicative subgroup which contains integer powers of two. In chapter 2 we will explain van den Dries's approach to this structure.

A definable set in $\langle \mathbb{R}, +, \cdot, <, 2^{\mathbb{Z}} \rangle$ is the union of a countable set and an open set, and hence \mathbb{Q} is not definable and as a result $\langle \mathbb{Z}, +, \cdot, < \rangle$ is not interpretable in this structure: we are safe from the Gödel phenomenon, and we have good definable sets. So, we can say that this structure shows tame behaviour.

Miller, in [23], extended this result to the following setting: let \mathcal{R} be a polynomially bounded expansion of $\langle \mathbb{R}, +, \cdot \rangle$ with field of exponents \mathbb{Q} , and $0 < \alpha \in \mathbb{R}$. Then $\langle \mathcal{R}, \alpha^{\mathbb{Z}} \rangle$ is model complete. Note that the cut of α in \mathbb{R} is part of the theory which axiomatises this structure.

In chapter 2, we address a slightly different proof for the model completeness of the theory of a model of RCF (=the theory of real closed fields) with a discrete set due to Wilkie (unpublished notes).

Miller called an expansion \mathcal{R} of $\langle \mathbb{R}, < \rangle$, d-minimal, short for discrete minimal, if for every $\mathcal{M} \equiv \mathcal{R}$, one-variable definable subsets of M are the union of an open set and finitely many discrete sets. In particular $\langle \mathbb{R}, +, \cdot, <, 2^{\mathbb{Z}} \rangle$ is d-minimal.

If \mathcal{R} is d-minimal, then for every finite collection \mathcal{A} of definable subsets of \mathbb{R}^n , there is a finite decomposition of \mathbb{R}^n into finitely many special C^0 -submanifolds compatible with \mathcal{A} . Note also that if \mathcal{R} is d-minimal, then for every m and definable $A \subseteq \mathbb{R}^{m+1}$ there exists $n \in \mathbb{N}$ such that for each $x \in \mathbb{R}^m$, A_x either has interior or is the union of n discrete definable sets.

Van den Dries asked about the structure $\langle \mathbb{R}, +, \cdot, <, 2^{\mathbb{Z}}, 3^{\mathbb{Z}} \rangle$ at the end of [7]. Hieronymi proved in [16], that this structure interprets $\langle \mathbb{Z}, +, \cdot \rangle$ and is, in this regard, not tame. The following dichotomy theorem is due to Hieronymi on the same matter:

Theorem (Hieronymi, [16]). *Let \mathcal{R} be an o-minimal expansion of $\langle \mathbb{R}, +, \cdot \rangle$ and let $D \subseteq \mathbb{R}$ be closed and discrete. Then either:*

- $\langle \mathcal{R}, D \rangle$ defines \mathbb{Z} or
- every subset of \mathbb{R} definable in $\langle \mathcal{R}, D \rangle$ has interior or is nowhere dense.

As a result of the above theorem, he also proved the following:

Theorem (Hieronymi, [16]). *Let $\alpha, \beta \in \mathbb{R}_{>0}$ with $\log_{\alpha} \beta \notin \mathbb{Q}$. Then $\langle \mathbb{R}, +, \cdot, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}} \rangle$ defines \mathbb{Z} .*

Getting back to the Question, we now describe the contents of Chapter 3. Let \mathbb{R}_{alg} denote the set of all algebraic elements of \mathbb{R} (=those which are roots of \mathbb{Q} -polynomials). Then \mathbb{R}_{alg} is countable and dense and codense in \mathbb{R} , and $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is a so-called dense pair of models of RCF (=the theory of real closed field). Van den Dries, [8], proved that, every formula in this structure is a boolean combination of the formulas of the form:

$$\exists \bar{x} \in \mathbb{R}_{alg} \quad \phi(\bar{x}, \bar{y})$$

for $\phi(\bar{x}, \bar{y})$ a formula in L_{or} . We will discuss this theorem and its consequences in chapter 3, but here we point out the simple result of this theorem that \mathbb{Z} is not definable in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$: this structure is tame in the sense of Gödel phenomenon. More intriguingly, this structure is proved tame, also in the second (topological) sense. This is the result of the following theorem and what follows in the paragraph after it:

Theorem (van den Dries, [8]). *Every open subset of \mathbb{R}^n , $n \in \mathbb{N}$, definable in $\langle \mathbb{R}, +, \cdot, <, \mathbb{R}_{alg} \rangle$ is definable in $\langle \mathbb{R}, +, \cdot, < \rangle$.*

The proof of the above theorem is a combination of works of Miller and Speiseger in [24], and van den Dries in [7]. We will sketch the proof in Chapter 5. We will also point out the following interesting topological consequence. Let A be a definable subset of \mathbb{R}^n in $\langle \mathbb{R}, +, \cdot, <, \mathbb{R}_{alg} \rangle$. Then there is a partition of \mathbb{R}^n into $\bar{\mathbb{R}}$ -cells with the following property. Each of the cells, say C , in this partition, is either distinct from

A , or contained in it, or has A dense and codense in it ($C \cap A$ dense and condense in C). The summary of this topological tameness property is the statement: ‘The open core of $\langle \mathbb{R}, +, \cdot, <, \mathbb{R}_{alg} \rangle$ is $\langle \mathbb{R}, +, \cdot \rangle$ ’.

The main chapter of this thesis is Chapter 4, where we have worked out a problem suggested by Hieronymi. In this chapter we try to establish the model theory of a structure comprised of all the the previous structures we referred to, i.e. the structure: $\langle \mathbb{R}, \mathbb{R}_{alg}, 2^{\mathbb{Z}}, +, \cdot, <, 0, 1 \rangle$.

Naturally, we first fix the most intuitive set of axioms for this structure. These axioms assert that in a structure $\langle \bar{M}, G, A \rangle$, M and G are real closed fields, G is an elementary substructure of M , G is dense in M , and $A \subseteq G$ is a multiplicative group that contains the integer powers of two. In the first draft of my work, I proved quantifier elimination and completeness for this theory, and described the types and definable sets in its models. In particular, the proofs easily suggested that \mathbb{Z} is not definable in a model of these axioms. Later, my supervisor noted that most of the proofs work in a more general setting, and I adapted all my notation and theorems, to a more general case, but all through, considered $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ as the main example.

So, Chapter 4 was generalised and is now the description of a theory axiomatised as follows. Let \tilde{T} be the theory of a polynomially bounded expansion of $\langle \mathbb{R}, +, \cdot \rangle$ which is model complete and has the field of exponents Q . Let \mathbb{T} be the set of axioms which assert that a structure $\mathbb{M} = \langle \tilde{M}, G, A \rangle$ is a model of \mathbb{T} , if $\tilde{M}, \tilde{G} \models \tilde{T}$, G is dense in M and A is a multiplicative group which contains the integer powers of a fixed element in G and is such that each element of M is between two successive elements of A . We will prove the following about \mathbb{T} in Chapter 4.

Theorem. *The following statements hold.*

- \mathbb{T} is complete.
- every formula in $L(\mathbb{T})$ is equivalent to a Boolean combination of formulas of the form:

$$\exists \bar{x} \in G \quad \exists \bar{y} \in A \quad \phi(\bar{x}, \bar{y}, \bar{z})$$

for $\phi(\bar{x}, \bar{y}, \bar{z})$ an $L(\tilde{T})$ -formula.

- if $\langle \tilde{M}, G, A \rangle \models \mathbb{T}$ then G is definably closed in this structure.

- if \mathbb{M}_1 and \mathbb{M}_2 are ‘free’ extensions (free extensions are defined in chapter 4) of \mathbb{M} , then due to the quantifier elimination, if $\bar{a} \in G_1$ and $\bar{b} \in G_2$ realise the same $L(\tilde{T} \cup \{A\})$ types over M in \mathbb{M}_1 and \mathbb{M}_2 , then they realise the same types; If $\bar{a} \in A_1$ and $\bar{b} \in A_2$, then the types of \bar{a} and \bar{b} in M are determined by their Presburger types in A_1 and A_2 respectively.

The above is the main result of Chapter 4. However in this chapter we also introduce the ‘small’ sets and study their properties. Small sets are the definable sets whose size and the pattern of distribution of whose elements, are similar to the field dense in the universe of our structure. The properties of small sets gives us insights on the properties of arbitrary definable sets. This is because we will prove:

Theorem. *If S is a definable set in $L(\mathbb{T})$, then up to an small set, S is equal to a set defined with an $L(\tilde{T}) \cup \{A\}$ -formula.*

In our proofs in chapter 4 we borrow many of the techniques employed by van den Dries in [8] and [7].

As mentioned above, \mathbb{Z} is not definable in our structure and we are safe from the wildness of Peano arithmetic. In chapter 5, we study some topological properties of models of \mathbb{T} . We first give the definition of the concept of ‘open core’ of an expansion of \mathbb{R} and then give the sketch of the proof of the theorem, referred to above, that the open core of $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is \mathbb{R} . Finally, we use this fact to conjecture that the open core of $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ is $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. In other words, if an open subset of \mathbb{R}^n is defined in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ it is defined in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. And in fact, this conjecture has been proven by Fornasiero in [10].

The open core of an expansion \mathcal{R} of $\bar{\mathbb{R}}$ was defined by Miller and Speissegger in [24], and is the reduct of \mathcal{R} obtained by adding predicates for all open sets defined in \mathcal{R} . Model theoretic properties of the open core of a structure justifies the topological properties of that structure; this is intuitively because of the fact that the closure and the interior of a definable set are definable.

In Chapter 6 we deal with a rather different model theoretic property of our theory. In this chapter we prove NIP (=not the independence property, and it is defined shortly) for \mathbb{T} . To prove that \mathbb{T} has NIP we use two facts. First that the discrete definable sets with parameters in G are subsets of G , and second that open definable

subsets of M in a model $\langle \tilde{M}, G, A \rangle$ of \mathbb{T} are defined in $\langle \tilde{M}, A \rangle$. Note that when a theory has either NIP or strict order property, then it is unstable.

Günayden and Hieronymi in [14] proved that the theories of $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ and $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ both have NIP and our proof in Chapter 6 relies on the techniques developed by them in this paper.

A theory T has NIP if all its formulas $\phi(\bar{x}, \bar{y})$ are ‘dependent’. A formula $\phi(\bar{x}, \bar{y})$ is said to be dependent for a theory T , if in every model $\mathcal{M} \models T$, for any finite sequence $\bar{b}_1, \dots, \bar{b}_n$ of elements in M the following happens: for each set Y of elements of this sequence, there is a tuple $a_Y \in M$ such that

$$M \models \phi(a_Y, \bar{b}) \Leftrightarrow \bar{b} \in Y.$$

Note that although we have in our language, which we will call \mathbb{L} in chapter 4, a predicate U for G , the dense substructure, we have sometimes for simplicity written $\bar{x} \in G$ instead of $\bigwedge_i U(x_i)$. The nature of the theory under study, always left me short of symbols and made me change them several times in each revision. I hope the symbols are now more reasonable and I need to say that the only misuse of the language left is using the symbol A for both the discrete set and the predicate representing it. The setting of each chapter is clearly stated at its beginning and the introductory chapters are made as short and yet detailed as possible.

Chapter 2

Adding a multiplicative group to a polynomially bounded structure

In [8], van den Dries axiomatised the theory of the structure $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ with a complete, decidable natural set of axioms and proved that this structure does not exhibit the Gödel phenomenon:

Despite $2^{\mathbb{Z}}$ being a discrete set, $\langle \mathbb{Z}, +, \cdot \rangle$ is not interpretable in the structure $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.

In his proof, he adds to L_{or} , the language of ordered rings, a predicate A , and sets the most natural axioms, T , in this language which simply assert that in a model $\langle \mathcal{M}, A \rangle \models T$, \mathcal{M} is a real closed field and A is a multiplicative group of the integer powers of two, and each element of \mathcal{M} falls between two successive elements of A .

To prove the completeness of T , he uses his modified version of classical quantifier-elimination tests and the already-known result of quantifier elimination of Presburger Arithmetic.

Let us first clarify what we meant in the previous paragraph by ‘his modified version of classical tests’. He first proves that T has algebraically prime models. That is, if $\mathcal{M} \models T_{\forall}$, then there is $\mathcal{N} \models T$ such that $\mathcal{M} \subseteq \mathcal{N}$ and for every $\mathcal{N}' \models T$, $\mathcal{N} \subseteq \mathcal{N}'$. This is part of a usual technique and does not, by itself, imply quantifier elimination. He then shows that if $\mathcal{C} \subseteq \mathcal{D}$ are both models of T , then there *exists* an element $d \in \mathcal{D} - \mathcal{C}$ such that $\mathcal{C}\langle d \rangle$, the structure generated by d over \mathcal{C} , can be embedded into an elementary extension of \mathcal{C} . What we called ‘modification of the previous tests’ is this *freeness* in the choice of b which, as explained in the next paragraph, van den Dries has exploited in his proof; while in the usual quantifier elimination proofs, back

and forth argument say, we need to ‘find’ an element in some model, corresponding to ‘any’ given element. A short sketch of his proof is as follows.

He adds to the language, predicates P_n for $2^{n\mathbb{Z}}$, $n \in \mathbb{N}$, which make $\langle 2^{\mathbb{Z}}, \{2^{n\mathbb{Z}}\}_{n \in \mathbb{N}}, \cdot, < \rangle$ a model of Presburger arithmetic. Now for the choice of d , in the previous paragraph, he picks an element given due to the quantifier elimination of Presburger arithmetic. To be more precise, suppose that $\langle \bar{M}, A, \{P_n\}_{n \in \mathbb{N}} \rangle \models T$ is a substructure of $\langle \bar{N}, B, \{Q_n\}_{n \in \mathbb{N}} \rangle \models T$. Now Let $\langle \bar{\mathbf{M}}, \mathbf{A}, \{\mathbf{P}_n\}_{n \in \mathbb{N}} \rangle \models T$ be a sufficiently saturated model of T extending $\langle \bar{M}, A, \{P_n\}_{n \in \mathbb{N}} \rangle$. Then as $\langle A, \{P_n\}_{n \in \mathbb{N}}, \cdot, < \rangle \subseteq \langle \mathbf{A}, \{\mathbf{P}_n\}_{n \in \mathbb{N}}, \cdot, < \rangle \models \text{Pr}$, by quantifier elimination of Pr , there exists a $b \in \mathbf{A}$ to make $\langle A\langle b \rangle, P_n\langle b \rangle \rangle$ a model of Pr , and this is the element he uses as *the* d we need. Note that $A\langle b \rangle$ and $P_n\langle b \rangle$ are the abelian groups generated over A and P_n by b .

He briefly states that a definable set in a model of T which is an expansion of \mathbb{R} , is the union of an open set and a countable set but he does not prove any cell-decomposition result for definable sets in higher dimensions.

Miller proved in [23] more generally that if \mathcal{R} is a polynomially bounded o-minimal expansion of \mathbb{R} with field of exponents \mathbb{Q} , and $\alpha > 0$ is an element of \mathbb{R} , then $\langle \mathcal{R}, \alpha^{\mathbb{Z}} \rangle$ has a complete axiomatisation and does not define \mathbb{Z} . His proof is essentially similar to van den Dries’s, but he gives a more general description of definable sets and, based on the pattern of them, he introduces a new notion of minimality, that is, d-minimality. we will expand on this notion in Chapter 4, but as a quick note, we suffice here to mention that a one dimensional definable set in a d-minimal structure is the union of an open set and finitely many discrete sets. If the d-minimal structure expands \mathbb{R} , then there is a countable decomposition of its definable sets into cells. More interestingly if it is of the form $\langle \mathcal{R}, \alpha^{\mathbb{Z}} \rangle$, for \mathcal{R} a polynomially bounded expansion of $\langle \mathbb{R}, < \rangle$ with the field of exponents \mathbb{Q} , then the definable sets have a countable decomposition into \mathcal{R} -cells.

Van den Dries’s proof is the first one that I learnt which I would have naturally put in my thesis. But Wilkie, noted to me his proof of the more general case, the case whose published proof, as I mentioned in the previous paragraph, is due to Miller in [23]. I decided to put his proof instead of van den Dries’ or Miller’s proof in my thesis. So the content of this chapter is merely Wilkie’s notes on this subject. Borrowing some of their techniques, one can also give a back and forth proof for the same theorem.

2.1 Setting

Before going through the details, we set up the notation we will be using throughout this chapter.

- $\bar{\mathbb{R}} := \langle \mathbb{R}, +, \cdot, 0, 1, \dots, < \rangle$ the ordered field of real numbers containing at least one constant $\omega > 1$.
- $\tilde{\mathbb{R}} := \langle \bar{\mathbb{R}}, \dots \rangle$, some expansion of $\bar{\mathbb{R}}$.
- $\tilde{T} := \text{Th}(\tilde{\mathbb{R}})$, the theory of $\tilde{\mathbb{R}}$.

Assume that

1. \tilde{T} is o-minimal.
2. \tilde{T} is polynomially bounded with field of exponents \mathbb{Q} .
3. \tilde{T} is model complete.

The terms ‘o-minimal’ and ‘polynomially bounded’ are explained in the sequel after the statement of the theorem and exploring some of its first consequences. The following is an example of \tilde{T} .

2.1.1 Main Example

For each $n \geq 0$, open $U \subseteq \mathbb{R}^n$, analytic function $f : U \rightarrow \mathbb{R}$ and closed, bounded box $B \subseteq U$, define $f^1 : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$f^1 := \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in B, \\ 0 & \text{otherwise} \end{cases}$$

Denote by F the collection

$$\{f^1 : f : U \rightarrow \mathbb{R} \text{ is analytic on } U \subseteq \mathbb{R}^n, n \in \mathbb{N}, \text{ and } B \text{ is an open box } \subseteq U\}.$$

Then $\mathbb{R}_{an} := \langle \mathbb{R}, \{f^1\}_{f \in F} \rangle$ satisfies 1,2 and 3 (Gabrielov [12], Denef-van den Dries [5]).

Obviously any reduct of \mathbb{R}_{an} also satisfies 1 and 2. Gabrielov in [12] showed that if we take any subcollection $\mathcal{S} \subseteq \{f^1\}_{f \in F}$ which is closed under differentiation, then

$$\mathbb{R}_{\mathcal{S}} := \langle \mathbb{R}, \mathcal{S} \rangle. \tag{2.1}$$

also satisfies 3.

2.1.2 Adding a multiplicative group

Let $G(\cdot)$ be a new unary predicate symbol. Let ω be a constant symbol of $L(\tilde{T})$ such that (in $\tilde{\mathbb{R}}$) $\omega > 1$.

Consider the following axioms DMG in the language $L(\tilde{T}) \cup \{G\}$:

$$\mathbf{DMG(1)} \quad \forall x \quad (G(x) \rightarrow x > 0);$$

$$\mathbf{DMG(2)} \quad \forall x, y \quad ((G(x) \wedge G(y)) \rightarrow G(x.y));$$

$$\mathbf{DMG(3)} \quad \forall x, y \quad ((G(x) \wedge x.y = 1) \rightarrow G(y));$$

$$\mathbf{DMG(4)} \quad \forall x > 1 \quad (G(x) \rightarrow x \geq \omega);$$

$$\mathbf{DMG(5)} \quad \forall x > 0 \quad \exists y \quad (G(y) \wedge y \leq x < \omega.y).$$

The y in **[DMG(5)]** is necessarily unique.

2.2 Main Theorem and Some Consequences

The following is the main theorem we will prove in this chapter.

The Main Theorem. *$\tilde{T} \cup \text{DMG}$ is complete and model complete.*

Before proving the Main Theorem, we state some of its consequences and explain the terms involved in its axiomatisation, like o-minimality and being polynomially bounded with field of exponent \mathbb{Q} .

Notice also that in the following way, model completeness of $T \cup \text{DMG}$ implies its completeness. Clearly the only expansion of $\tilde{\mathbb{R}}$ to a model of $\tilde{T} \cup \text{DMG}$ is

$$\langle \tilde{\mathbb{R}}, \omega^{\mathbb{Z}} \rangle.$$

Further, if $\tilde{M} \preceq \tilde{\mathbb{R}}$ is the minimal model of \tilde{T} then $\omega^{\mathbb{Z}} \subseteq M$ and $\langle \tilde{M}, \omega^{\mathbb{Z}} \rangle (\models \tilde{T} \cup \text{DMG})$ is embeddable in every model of $\tilde{T} \cup \text{DMG}$. So completeness follows from model-completeness.

2.2.1 Some consequences of the Main Theorem

A. Some model theory for real analytic periodic functions

Let $\tilde{\mathbb{R}} = \langle \mathbb{R}, +, \cdot, -, 0, 1, e^{2\pi}, < \rangle$. Let \mathcal{F} be any collection of analytic 2π -periodic functions (from \mathbb{R} to \mathbb{R}) which is closed under differentiation. Assume that the functions \sin and \cos are in \mathcal{F} . For $f \in \mathcal{F}$, define $f^* : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f^*(x) = \begin{cases} f(\log(x)) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

Let $G = e^{2\pi\mathbb{Z}}$, and note that

$$\forall x \in \mathbb{R} \quad \forall g \in G \quad f^*(gx) = f^*(x). \quad (2.2)$$

Corollary 2.1. $\text{Th}(\langle \tilde{\mathbb{R}}, \{f^* : f \in \mathcal{F}\} \rangle)$ is model complete.

Proof. Let \mathcal{S}' be the collection of all functions of the form

$$(a, \infty) \rightarrow \mathbb{R} : x \mapsto P(x^s, f_1^*, \dots, f_r^*)$$

for P a polynomial over \mathbb{Z} , $s \in \mathbb{Z}$ and $f_1, \dots, f_r \in \mathcal{F}$. Let $\mathcal{S} = \{F^1|_{[\frac{1}{n}, n]} : n \geq 1, F \in \mathcal{S}'\}$. Then \mathcal{S} is closed under differentiation, so $\langle \tilde{\mathbb{R}}, \mathcal{S} \rangle$ is a structure of type described by equation 2.1, and hence the main theorem applies to it. So $\text{Th}(\langle \tilde{\mathbb{R}}, \mathcal{S}, G \rangle)$ is model complete. But clearly, using equation 2.2, the structures $\langle \mathbb{R}, \mathcal{S}, G \rangle$ and $\langle \tilde{\mathbb{R}}, \{f^* : f \in \mathcal{F}\} \rangle$ are $\exists \cap \forall$ -bi-interpretable (note that $G = \{x > 0 : \sin \log x = 0, \cos \log x = 1\}$). \square

B. The structure of complex numbers with the function $z \mapsto z^i$

Let $\tilde{\mathbb{C}}$ be the expansion of the complex field by the (many valued) function $z \rightarrow z^i$, i.e. by the relation

$$\mathbf{g} := \{\langle e^u, e^{iu} \rangle \in \mathbb{C}^2 : u \in \mathbb{C}\}.$$

Note that

$$\forall z, w \in \mathbb{C} \quad \forall g, h \in e^{2\pi\mathbb{Z}} \quad [\langle z, w \rangle \in \mathbf{g} \Rightarrow \langle gz, hw \rangle \in \mathbf{g}].$$

Using this, it is easy to find a structure $\tilde{\mathbb{R}}$ of type described by equation 2.1 such that $\tilde{\mathbb{C}}$ is $\forall \cap \exists$ -interpretable in $\langle \tilde{\mathbb{R}}, e^{2\pi\mathbb{Z}} \rangle$, (via $\mathbb{C} \sim \mathbb{R}^2$).

The following theorem is proved in informal notes by Wilkie based on the Main Theorem of this chapter.

Theorem 2.2. $\tilde{\mathbb{C}}$ is quasi-minimal (i.e. every definable subset of \mathbb{C} is either countable or co-countable).

Wilkie's proof of this involves the following steps. First defining the G-derivation d , as a function from \mathbb{C} to \mathbb{C} which satisfies the chain rule for the multi-variable polynomials with exponents in $\mathbb{Z}[i]$. Then obtaining a pregeometry ζ on C by the following equation:

$$z \in \zeta(a_1, \dots, a_n) \Leftrightarrow dz \in \mathbb{C}da_1 \oplus \dots \oplus \mathbb{C}da_n$$

It follows from the implicit function theorem that if a point w_1 is in $\zeta(a_1, \dots, a_n)$ then there is an open neighborhood U of $\langle a_1, \dots, a_n \rangle$ and a holomorphic function $h_1 : U \rightarrow \mathbb{C}$ with $h_1(a_1, \dots, a_n) = w_1$ and $h_1(u_1, \dots, u_n) \in \zeta(u_1, \dots, u_n)$ for all $\langle u_1, \dots, u_n \rangle \in U$. The core of the proof is a lemma which says that if $\langle a_1, \dots, a_n \rangle$ is a generic point of \mathbb{C}^n and $\theta : [0, 1] \rightarrow \mathbb{C}^n$ is real analytic generic path with $\theta(0) = \langle a_1, \dots, a_n \rangle$, then the above germ h_1 has an analytic continuation along θ . In the proof of this lemma a complex version of valuation inequality (see 2.5) is used for $\tilde{\mathbb{C}}$ and the interpretability of $\tilde{\mathbb{C}}$ in $\langle \tilde{\mathbb{R}}, e^{2\pi\mathbb{Z}} \rangle$ is employed. Finally by a back and forth argument he proves that any two generic complex numbers satisfy the same $L_{\infty\omega}$ types.

2.3 O-minimal Structures

Full details about O-minimal structures and their properties can be found in [6]. In this section we just scratch the surface of the topic, and this is because the universe of our structure, with the symbols of our language apart obviously from its unary predicate for the discrete set, is o-minimal.

Let $\bar{M} = \langle M, +, \cdot, -, 0, 1, < \rangle$ be a model of RCF(=the theory of real closed fields). Let $\tilde{M} = \langle \bar{M}, \dots \rangle$ be an expansion of \bar{M} .

\tilde{M} is *o-minimal* if every definable subset of M is a finite union of open intervals and points. Let $\tilde{T} = \text{Th}(\tilde{M})$.

Facts

1. O-minimality implies many topological and geometrical finiteness conditions on arbitrary definable $S \subseteq M^n$. (Pillay, Steinhorn, Knight [25],[19], van den Dries's

book [6]).

2. If $\tilde{N} \equiv \tilde{M}$ then \tilde{N} is also o-minimal.

3. \tilde{T} is a Skolem theory and Skolem closure is a pregeometry:

Let $\tilde{M}_1, \tilde{M}_2 \models \tilde{T}, \tilde{M}_1 \preceq \tilde{M}_2$. Let $S \subseteq M_2$. Then the closure of $M_1 \cup S$ under the 0-definable functions (of \tilde{M}_2), denoted $M_1 \langle S \rangle$, is the domain of a (unique) elementary substructure, $\tilde{M}_1 \langle S \rangle$ of \tilde{M}_2 :

$$\tilde{M}_1 \preceq \tilde{M}_1 \langle S \rangle \preceq \tilde{M}_2.$$

Further, there exists $S_0 \subseteq S$ such that $M_1 \langle S_0 \rangle = M_1 \langle S \rangle$ (i.e. S_0 generates $M_1 \langle S \rangle$ over M_1) and for all $s \in S_0, s \notin M_1(S_0 - s)$ (i.e. S_0 is independent over M_1 .)

All such bases have the same cardinality, denoted $\dim_{\tilde{M}_1}(\tilde{M}_1 \langle S \rangle)$.

2.4 Polynomially bounded structures

For this chapter, we call an o-minimal structure \tilde{M} , *polynomially bounded (with \mathbb{Q} -exponents)* if for all definable $f : M \rightarrow M$, there is a $q \in \mathbb{Q}$ and a $c \in M$ such that $\frac{f(x)}{c \cdot x^q} \rightarrow 1$ as $x \rightarrow +\infty$ (in \tilde{M}). This property holds for the Main Example (see 2.1.1) and not for say $\langle \bar{\mathbb{R}}, \exp \rangle$ or $\langle \bar{\mathbb{R}}, x \mapsto x^{\sqrt{2}} \rangle$. The general definition is in chapter 4.

Now assume $\tilde{T} = \text{Th}(\bar{\mathbb{R}})$ with $\bar{\mathbb{R}}$ as in section 1. (Only 1 and 2 are needed here).

For $\tilde{M} \models \tilde{T}$ let $\Gamma(\tilde{M})$ be the multiplicative group of *skies* of $\langle M^{>0}, \cdot \rangle$. A *sky* is an equivalence class of the relation (for $a, b \in M^{>0}$)

$$a \sim b \Leftrightarrow \exists N \in \mathbb{N} - \{0\} \quad \frac{1}{N} \leq \frac{a}{b} \leq N.$$

$\Gamma(\tilde{M})$ inherits multiplication from \tilde{M} and is a divisible abelian group (since $\langle M^{>0}, \cdot \rangle$ admits n 'th root for all n), and hence it is a \mathbb{Q} -vector space (written multiplicatively!).

2.5 The Valuation Inequality

The following **A** and **B** are called ‘Valuation Inequality’ and are due to Wilkie, van den Dries and Speissegger (in [32] and [15]).

A Let $\tilde{M}_1, \tilde{M}_2 \models \tilde{T}, \tilde{M}_1 \preceq \tilde{M}_2$ and suppose that $\dim_{\tilde{M}_1}(\tilde{M}_2) = d (< \infty)$. Then $\dim_{\mathbb{Q}} \Gamma(\tilde{M}_2) / \Gamma(\tilde{M}_1) \leq d$.

B In particular (for $d = 1$), if $\tilde{M}_2 = \tilde{M}_1\langle a \rangle$ for some $a \in M_2$, and if there is a new sky, g/ \sim say, then

$$\forall \beta \in M_2 - \{0\} \quad \exists \alpha \in M_1 - \{0\} \quad \exists q \in \mathbb{Q} \quad \frac{1}{N} \leq \left| \frac{\beta}{\alpha.g^q} \right| \leq N$$

for some $N \in \mathbb{N} - \{0\}$.

2.6 Proof of the Main Theorem

Let $\tilde{T} = \text{Th}(\tilde{\mathbb{R}})$ satisfying assumptions 1,2,3 (Section 2.1). Our (and van den Dries'a proof) is based on the following:

Lemma 2.3. *Suppose that $\langle \tilde{M}_1, G_1 \rangle, \langle \tilde{M}_2, G_2 \rangle$ are both models of $\tilde{T} \cup \text{DMG}$ with $\langle \tilde{M}_1, G_1 \rangle \subseteq \langle \tilde{M}_2, G_2 \rangle$ (so that $\tilde{M}_1 \preceq_{L(\tilde{T})} \tilde{M}_2$ by 3). Let S be any subset of G_2 . Then $\langle \tilde{M}_1\langle S \rangle, M_1\langle S \rangle \cap G_2 \rangle \models \tilde{T} \cup \text{DMG}$.*

Proof. By induction, we may suppose that $S = \{g\}$ for some $g \in G_2$. Clearly we only have to show that $\langle \tilde{M}_1\langle S \rangle, M_1\langle S \rangle \cap G_2 \rangle \models \mathbf{DMG(5)}$ as all the other axioms are inherited from \tilde{M}_2 .

So let $a \in M_1\langle g \rangle$, $a > 0$. We have

$$\begin{aligned} \tilde{M}_1 &\preceq \tilde{M}_1\langle g \rangle \preceq M_2, \\ \omega \in G_1 &\subseteq G_2 \cap M_1\langle g \rangle \subseteq G_2 \end{aligned}$$

Since $\langle \tilde{M}_2, G_2 \rangle \models \mathbf{DMG(5)}$, there is some $h \in G_2$ such that $h \leq a < \omega.h$. We shall show that $h \in M_1\langle g \rangle$. Obviously we may assume that $g \notin M_1$ (otherwise $M_1 = M_1\langle g \rangle$ and the lemma is trivial). It follows that g/ \sim is a new sky (i.e. a new Archimedean class not represented in M_1), because otherwise There would be a b in M_1 and N in $\mathbb{N} - \{0\}$ such that

$$\frac{1}{N} \leq \frac{g}{b} \leq N$$

and by $\mathbf{DMG(5)}$ in \tilde{M}_1 we may assume that $b \in G_1$. But then:

$$\frac{g}{b} = \omega^m \text{ (for some } m \in \mathbb{Z}\text{)}.$$

So $g = b.\omega^m \in M_1$, a contradiction. Hence by the valuation inequality (2.5-B), there is some $e \in M_1$ and $q \in \mathbb{Q}$ and $N \in \mathbb{N} - \{0\}$ such that

$$\frac{1}{N} \leq \frac{a}{e.g^q} \leq N.$$

As above, we may suppose $e \in G_1$ and we may also replace a by h (note that $N' \geq \omega \geq \frac{1}{N'}$ for some $N' \in \mathbb{N} - \{0\}$). Also if $q = \frac{s}{t}$, ($s, t \in \mathbb{Z}, t \neq 0$), then

$$\frac{1}{N^t} \leq \frac{h^t}{e^t g^s} \leq N^s$$

note that the term in the middle is in G_2 and so

$$\frac{h^t}{e^t \cdot g^s} = \omega^m \quad (\text{for some } m \in \mathbb{Z}).$$

Thus $h^t = \omega^m \cdot e^t \cdot g^s \in M_1 \langle g \rangle$. But $\tilde{M}_1 \langle g \rangle$ is a real closed field, and hence $h \in M_1 \langle g \rangle$ as required. \square

2.7 Proof of the Main Theorem

By Robinson's test (see for example [17], page 375), and using the condition 3 in Section 2.1 that \tilde{T} is model complete, we must show the following:

Suppose $\langle \tilde{M}_1, G_1 \rangle, \langle \tilde{M}_2, G_2 \rangle \models \tilde{T} \cup \text{DMG}$, $\tilde{M}_1 \preceq \tilde{M}_2$, $G_1 = G_2 \cap M_1$ and $g_1, \dots, g_n \in G_2$. Then if $\phi(x_1, \dots, x_n)$ is a formula of $L(\tilde{T})$ with parameters in M_1 , and if $\tilde{M}_2 \models \phi(g_1, \dots, g_n)$, then for some $h_1, \dots, h_n \in G_1$, $\tilde{M}_1 \models \phi(h_1, \dots, h_n)$.

Now if $n \geq 2$, we may argue by induction, by replacing \tilde{M}_1 with $\tilde{M}_1 \langle g_1, \dots, g_{n-1} \rangle$, and using Lemma 2.3, provided we can do the case $n = 1$.

But since $\phi(x_1)$ defines a finite union of open intervals and points—with all endpoints in M_1 —this follows easily from the axioms **DMG**. \square (Main Theorem)

2.8 d-minimality

Most of the definitions in this section are borrowed from [23].

We say that \mathcal{R} (an expansion of $\langle \mathbb{R}, < \rangle$) is d-minimal if for every $\mathfrak{M} \equiv \mathcal{R}$, every subset of M definable in \mathfrak{M} is the union of an open set and finitely many discrete sets. By a compactness argument, \mathcal{R} is d-minimal if for every m and definable $A \subseteq \mathbb{R}^{m+1}$ there exists $n \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$, A_x has interior or is a union of N discrete sets. Note that if \mathcal{R} is d-minimal then every reduct of \mathcal{R} over $(\mathbb{R}, <)$ is d-minimal.

A d -dimensional C^p -submanifold M of \mathbb{R}^n is called special if there exists $\mu \in \Pi(n, d)$ such that for each $y \in \mu(M)$ there is an open box B about y such that each connected

component X of $M \cap \mu^{-1}(B)$ projects C^p -diffeomorphically onto B , i.e. $\mu|_M : M \rightarrow \mu M$ is a C^p -smooth covering map. A collection \mathcal{A} of subsets of \mathbb{R}^n is compatible with a collection \mathcal{B} of subsets of \mathbb{R}^n if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, either A is contained in B or A is disjoint from B .

Assume \mathcal{R} is d-minimal. Let \mathcal{A} be a finite collection of definable subsets of \mathbb{R}^n . Then there is a finite partition of \mathbb{R}^n into special C^0 -submanifolds, each of which is definable and compatible with \mathcal{A} . If \mathbb{R} expands $\bar{\mathbb{R}}$, then this holds with C^p instead of C^0 .

Every d-minimal expansion of $\langle \mathbb{R}, <, + \rangle$ admits countable cell decomposition. Every d-minimal expansion of $\bar{\mathbb{R}}$ admits countable C^p -decomposition.

If \mathcal{R} expands $\langle \mathbb{R}, <, + \rangle$ and has the uniform finiteness property, then every discrete definable subset of \mathbb{R} is finite. (so, obviously $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ does not have this property).

The following are equivalent:

1. \mathcal{R} is d-minimal.
2. For every m and definable $A \subseteq \mathbb{R}^{m+1}$ there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{R}^m$, A_x either has interior or is a union of N discrete sets.
3. For every m, n and definable $A \subseteq \mathbb{R}^{m+n}$ there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{R}^m$, either $\dim A_x > 0$ or A_x is the union of N discrete sets.

Proposition 2.4. *Every set definable in $(\mathbb{R}, +, \cdot, \alpha^{\mathbb{Z}})$ is a finite union of locally closed definable sets [24]. The theory of $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ has definable Skolem functions ([7]).*

In the following theorem we confirm that a similar property to o-minimal structures holds for d-minimal structures. The proof of this theorem relies on the fact that as in the above proposition, the theory of $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ has definable Skolem functions.

Theorem 2.5. *Let \mathcal{R} be a model of $\text{Th}(\bar{\mathbb{R}}, 2^{\mathbb{Z}})$. Let $\hat{\mathcal{R}}$ be a saturated expansion of \mathcal{R} such that all functions $f : R \rightarrow R$ that are definable in $\hat{\mathcal{R}}$ are definable in \mathcal{R} . Then all the functions $f : R^n \rightarrow R$ definable in $\hat{\mathcal{R}}$ are definable in \mathcal{R} .*

Proof. The case is clear for $n = 1$. Let $f : R^n \rightarrow R$ be a definable function. Let $a \in R$. Then by induction hypothesis the function $f_a : R^{n-1} \rightarrow R$ defined as $f_a(x) = f(a, x)$

is definable in \mathcal{R} so there are parameters \bar{c} and a definable function F in \mathcal{R} such that $f(a, x) = F(\bar{c}, x)$. So:

$$\forall a \quad \exists F \exists \bar{c} \quad [\forall x \quad f(a, x) = F(\bar{c}, x)].$$

Since $\hat{\mathcal{R}}$ is saturated there are finitely many F_i 's such that

$$\forall a \quad \exists \bar{c} \quad f(a, x) = \vee F_i(\bar{c}, x).$$

So there is one definable function F such that

$$\forall a \exists \bar{c} \quad [\forall x \quad f(a, x) = F(\bar{c}, x)].$$

By definability of skolem functions there is a map \bar{c} such that

$$\forall x \quad f(a, x) = F(\bar{c}(a), x).$$

Since each c_i is by hypothesis definable in \mathcal{R} the statement is proved. □

Chapter 3

Dense pairs of o-minimal structures

In the paper: ‘Solution of a problem of Tarski’, [27], Robinson proved the completeness of a theory axiomatising a real closed field with a predicate for a proper dense real closed subfield. This work was generalised in [7] by van den Dries to dense pairs of o-minimal expansions of ordered abelian groups.

In this chapter we will briefly discuss parts of the paper ‘dense pairs of o-minimal structures’, of van den Dries, [7], and most of the citations will be to this paper. These parts include: completeness of the theory of an elementary dense pair, its elimination of quantifiers up to ‘special’ formulas, definable sets in one dimension, definable functions and finally, the open core of a dense pair of o-minimal structures in the sense of [24].

We will see in this chapter that how a controlled back and forth argument helps in proving quantifier elimination and characterising types of elements. In the study of definable sets, we will see the special role that ‘small sets’ play. We will also see that if our structure expands $\langle \mathbb{R}, < \rangle$ then its open definable subsets are defined as if there were no ‘pair’ of structures involved. This means that if a set is open and is defined in a dense elementary pair of structures, then it can be defined in the bigger structure without resorting to the predicate for the smaller structure dense in it. This can be summarised in this more technical statement: ‘the open core of a dense pair of o-minimal structures is o-minimal’. To understand this last statement, we will bring a summary of a work of Miller and Speissegger [24] on the open definable subsets of a first order structure. But open core is introduced in more details in chapter 5. Finally we state that the theory of a dense pair of o-minimal structures has NIP and this is expanded on in chapter 6 where we prove NIP for our structure.

These notions will be made clearer when the setting of the chapter and the notations are introduced and definitions and theorems are precisely stated.

More general records and properties of dense pairs of first order structures can be found in more recent works like [11]. In this work, even dense tuples of topological structures are studied. In [10] Fonaciero considers dense pairs of d-minimal structures and proves similar results to [7]. Dense pairs (and lovely pairs) are studied in [3]. In the latter, lovely pairs of o-minimal structures are proved to be super-rosy of rank $\leq \omega$. In [4], a dense pair of o-minimal structures are proved to have thorn-rank one, hence a dense pair of o-minimal structures is in particular a lovely pair.

3.1 Setting

The setting of this chapter is as itemized below.

- Throughout, T denotes a complete o-minimal theory that expands the theory of ordered abelian groups.
- L is the language of T which extends $\{<, 0, 1, +, -\}$.
- T has definable Skolem functions.
- Models of T are denoted by $\mathcal{A}, \mathcal{B}, \dots$ and their universes by A, B, \dots
- $L^2 = L \cup \{U\}$ is a language obtained by adding a unary predicate U to L .
- T^2 denotes the theory in L^2 whose models are the elementary pairs $\langle \mathcal{B}, A \rangle$ of models of T (that is, if $M \models T^2$ then there are two models $\mathcal{B}, \mathcal{A} \models T$ such that $\mathcal{A} \preceq \mathcal{B}$ and $M = \langle \mathcal{B}, A \rangle$). Note that we are not obviously dealing with a two-sorted structure, as A is represented by the predicate U ; So axioms of T^2 first say that $\mathcal{B} \models T$, then using U and the other L -formulas, they assert that every formula with parameters in A which is true in \mathcal{B} is true in \mathcal{A} .
- T^d is the theory in L^2 whose models $\langle \mathcal{B}, A \rangle$ are such that $\mathcal{A}, \mathcal{B} \models T$, $A \neq B$, $\mathcal{A} \preceq \mathcal{B}$ and A is dense in B .

3.2 Quantifier elimination, completeness of T^d and description of a dense pair

Theorem 3.1 below, is the main theorem which sheds light on the way of proving all the other theorems concerning a dense pair. To get a better insight on what this theorem is to say, consider the following argument. Let \mathbb{R}_{alg} be the set of algebraic elements of \mathbb{R} . This set is obviously dense in \mathbb{R} and $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is a dense pair of models of $T := RCF$, the theory of real closed fields.

The following observations about this structure are key to the understanding of a dense pair. Let $\psi(\bar{y}) := \exists \bar{x} \ \phi(\bar{x}, \bar{y})$ be an L_{or} -formula and $\bar{a} \in \mathbb{R}$. Then

1. If $\bar{a} \in \mathbb{R}_{alg}$ then as $\bar{\mathbb{R}}_{alg} \preceq \bar{\mathbb{R}}$, if $\mathbb{R} \models \exists \bar{x} \ \phi(\bar{x}, \bar{a})$, then $\bar{\mathbb{R}}_{alg} \models \exists \bar{x} \ \phi(\bar{x}, \bar{a})$. That is, in this case, $[\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle \models \exists \bar{x} \in \mathbb{R}_{alg} \ \phi(\bar{x}, \bar{a})] \Leftrightarrow [\mathbb{R} \models \exists \bar{x} \ \phi(\bar{x}, \bar{a})]$.
2. Let $\bar{a} \notin \mathbb{R}_{alg}$. Then $\phi(\bar{\mathbb{R}}, \bar{a})$ is a collection of cells in \mathbb{R}^n (n the length of the variable \bar{x}). If any of these cells is of dimension n , then they are open, and as \mathbb{R}_{alg} is dense in \mathbb{R} , they contain elements of \mathbb{R}_{alg} . So, in this case $\bar{\mathbb{R}} \models \exists \bar{x} \ \phi(\bar{x}, \bar{a})$ implies $[\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle \models \exists \bar{x} \in \mathbb{R}_{alg} \ \phi(\bar{x}, \bar{a})]$.
3. Let $\bar{a} \notin \mathbb{R}_{alg}$ and the formula $\phi(\bar{x}, \bar{a})$ define only cells with dimensions less than n . In this case deciding whether or not there exist elements in \mathbb{R}_{alg} for which $\bar{\mathbb{R}} \models \phi(\bar{x}, \bar{a})$ is not as easy. This is because being an algebraic element, is not expressible with a first order formula.

Items 1,2,3 above suggest that in determining formulas of L^2 , existence of elements in the dense substructure plays a substantial role. This is precisely stated in the following theorem (of [7]).

Theorem 3.1 (van den Dries). *Let $\psi(\bar{y})$ be a formula in L^2 . Then in T^d this formula is equivalent to a boolean combination of formulas of the form*

$$\exists x_1, \dots, x_m \ [(\bigwedge_{i=1, \dots, m} U(x_i)) \wedge \phi(\bar{x}, \bar{y})],$$

for $\phi(\bar{x}, \bar{y})$ an L -formula.

In the next theorem, we will see that in a model $\langle \mathcal{B}, A \rangle$ of T^d , the subsets of A^n which are definable (with parameters) in L^2 , can be obtained by intersecting A^n with

L -definable subsets of B^n . This can be justified by similar reasonings to 1,2,3 just before Theorem 3.1 and is the initiative for labeling a class of definable sets as ‘small’. We will expand on small sets in the next very short section. In the next chapter, we will resort to some notion of smallness but in a slightly different way.

Theorem 3.2. *Let $\langle \mathcal{B}, A \rangle \models T^d$. Then for a set $Y \subseteq A^n$ the following are equivalent:*

1. *Y is definable in L^2 with parameters in B .*
2. *Y is of the form $Z \cap A^n$ where $Z \subseteq B^n$ is definable in L with the parameters in B .*
3. *Y is definable in the structure $\langle \mathcal{A}, (A \cap (0, b))_{0 < b \in B} \rangle$, which is an expansion of \mathcal{A} by the traces in A of intervals in B .*

The following theorem is along the same lines as the above theorem, but it is worth being mentioned separately:

Theorem 3.3. *Let $\langle \mathcal{B}, A \rangle \models T^d$ and $Y \subseteq B^n$ be A_0 -definable in L^2 for $A_0 \subseteq A$. Then $Y \cap A^n$ is A_0 -definable in \mathcal{A} .*

3.3 Theorems regarding small sets

Small sets are, roughly speaking, those L^2 -definable sets which are no bigger than the dense substructure with L -definable functions applied to it. Each definable subset of B in a model of $\langle \mathcal{B}, A \rangle \models T^d$ is proved to have a small part, and a part which is defined in B with an L -formula.

Definition 3.3.1. *Let $\langle \mathcal{B}, A \rangle \models T^d$ and $X \subseteq B$ be definable in L^2 with parameters. Then X is called A -small if there is L -definable function $f : B^n \rightarrow B$ with parameters in B such that $X \subseteq f(A^n)$.*

The following theorem says that a definable function in a model of T^2 is ‘almost’ definable in L .

Theorem 3.4. *Let $\langle \mathcal{B}, A \rangle \models T^d$. Let $F : B \rightarrow B$ be an L^2 -definable function with parameters. Then there is an A -small subset of B , say X , and an L -definable function $F' : B \rightarrow B$ such that F is equal to F' outside X (put in other words, every L^2 -definable function is, nearly and off a small set, equal to an L -definable function).*

If we replace the F above with the characteristic function of a definable set, we get the first part of the following theorem. The proof of the second part involves compactness theorem and results from different arguments.

Corollary 3.5. *Let $\langle \mathcal{B}, A \rangle \models T^d$.*

1. *Let $S \subseteq B$ be L^2 -definable (with parameters). Then there is an A -small set X and an L -definable (with parameters) set S' such that $S - X = S' - X$.*
2. *If $f : A^n \rightarrow A$ is L^2 -definable (with parameters), then there are functions f_1, \dots, f_n , all L -definable with parameters in A , such that for each $\bar{x} \in A^n$, $f(\bar{x}) = f_1(\bar{x})$, or ..., or $f(\bar{x}) = f_n(\bar{x})$.*

The possibility of decomposition of an o-minimal structure into cells makes it possible to understated the small sets. It is proved in [7](and is rather clear) that small sets can not contain an interval. But obviously they can be dense and codense in an interval, that is, if $\langle \mathcal{B}, A \rangle \models T^d$, and $a, b \in B$, then $A \cap (a, b)$ is an A -small set. The following theorem says that A -small sets are those densely and condensly accumulated in finitely many distinct areas of the universe and with no intersection with the rest of it. Let us see the precise statement of this:

Theorem 3.6. *Let $\langle \mathcal{B}, A \rangle \models T^d$ and $X \subseteq B$ be A -small. Then the universe B can be partitioned into finitely many intervals in the following form:*

$$B = (-\infty, b_0] \cup [b_0, b_1) \cup \dots \cup [b_n, +\infty).$$

such that for each i one the following happens:

- *either X has no intersection with (b_i, b_{i+1}) , or*
- *both $X \cap (b_i, b_{i+1})$ and $(b_i, b_{i+1}) - X$ are dense in (b_i, b_{i+1}) .*

The following corollary is the result of the above theorem and Corollary 3.5. Knowing that the points of a small set are dense in finitely many different areas of the universe, and that as Corollary 3.5 says every definable set is L -definable apart from a small part of it, we can expect the points of definable sets to gather densely in or to form cells.

Corollary 3.7. *Let $\langle \mathcal{B}, A \rangle \models T^d$ and $S \subseteq B$ be L^2 -definable. Then the universe B can be partitioned into finitely many intervals in the following form:*

$$B = (-\infty, b_0] \cup [b_0, b_1) \cup \dots \cup [b_n, +\infty).$$

such that for each i one the following happens:

- *either S has no intersection with (b_i, b_{i+1}) , or*
- *$(b_i, b_{i+1}) \subseteq S$, or*
- *both $S \cap (b_i, b_{i+1})$ and $(b_i, b_{i+1}) - S$ are dense in (b_i, b_{i+1}) .*

Combining theorems and corollaries of this section yields the following corollary.

Corollary 3.8. *Let $\langle \mathcal{B}, A \rangle \models T^d$. Let $f : B \rightarrow B$, definable in L^2 with parameters and continuous at all but finitely many points of B . Then f is definable in L .*

3.4 Free extensions

Let $\langle \mathcal{B}, A \rangle, \langle \mathcal{D}, C \rangle \models T^2$ and $\langle \mathcal{B}, A \rangle \subseteq \langle \mathcal{D}, C \rangle$. Then clearly $B \cap C = A$. Let x be an element in D . Then one needs to deal with the question of ‘what the structure generated by x over $\langle \mathcal{B}, A \rangle$ is’. We can guess that it is either of the form $\langle \mathcal{B}\langle x \rangle, A\langle x \rangle \rangle$ or $\langle \mathcal{B}\langle x \rangle, A \rangle$. Note that by $\mathcal{B}\langle x \rangle$ we mean the structure generated over \mathcal{B} by x , in the sense of the pregeometry in models of T . Certainly for $\langle \mathcal{B}\langle x \rangle, A\langle x \rangle \rangle$ to be a substructure of $\langle \mathcal{D}, C \rangle$, we need to have $B\langle x \rangle \cap C = A\langle x \rangle$. As we will see in chapter 4, this does not happen for all x ’s.

Under the condition of the above paragraph, we say that $\langle \mathcal{D}, C \rangle$ is a free extension of $\langle \mathcal{B}, A \rangle$, if for every set $Y \subseteq B$, the following happens:

Y is independent in D over A if and only if Y is independent over C in D .

‘Independent’ above is also with reference to the pregeometry arising from T .

Equivalently, we say that $\langle \mathcal{D}, C \rangle$ is a free extension of $\langle \mathcal{B}, A \rangle$, if for every set $Y \subseteq C$, the following happens:

Y is independent in D over A if and only if Y is independent over B in D .

If $\langle \mathcal{D}, C \rangle$ is a free extension of $\langle \mathcal{B}, A \rangle$ and $Z \subseteq C$, then $\langle \mathcal{B}, A \rangle \subseteq \langle \mathcal{B}\langle Z \rangle, A\langle Z \rangle \rangle \subseteq \langle \mathcal{D}, C \rangle$, and $\langle \mathcal{B}\langle Z \rangle, A\langle Z \rangle \rangle$ is also a free substructure of $\langle \mathcal{D}, C \rangle$.

Freeness plays an important role in the proof of quantifier elimination (Theorem 3.1). Let $\langle \mathcal{B}, A \rangle$ and $\langle \mathcal{D}, C \rangle$ be κ -saturated models of T^d . Then the following collection Γ has the back and forth property and this results in the completeness of T^d and its elimination of quantifiers:

$$\Gamma = \{i : i \text{ is an isomorphism between a free substructure of } \langle \mathcal{B}, A \rangle \\ \text{and a free substructure of } \langle \mathcal{D}, C \rangle\}.$$

3.5 Types

The proof of completeness and the quantifier elimination which follows from it, leads to a good description of types of elements. The following theorems explain more.

Theorem 3.9. *Let $\langle \mathcal{B}, A \rangle \models T^2$. Suppose that $\langle \mathcal{B}_1, A_1 \rangle$ and $\langle \mathcal{B}_2, A_2 \rangle$ are two free extensions of $\langle \mathcal{B}, A \rangle$. Then if $\bar{a}_1 \in (A_1)^n$ and $\bar{a}_2 \in (A_2)^n$ realise the same L -types over B in \mathcal{B}_1 and \mathcal{B}_2 , then they realise the same L^2 -types over B in $\langle \mathcal{B}_1, A_1 \rangle$ and $\langle \mathcal{B}_2, A_2 \rangle$.*

The next theorem is even more interesting in the sense that it provides us with conditions under which the whole type of an element over a model of T is determined by the cut it makes in the universe of that model.

Theorem 3.10. *Let $\langle \mathcal{B}_1, A_1 \rangle, \langle \mathcal{B}_2, A_2 \rangle \models T^2$. Let \mathcal{A} be a common elementary substructure of \mathcal{A}_1 and \mathcal{A}_2 . Suppose that $b_1 \in B_1 - A_1$ and $b_2 \in B_2 - A_2$ realise the same cut in A . Then they realise the same L^2 -types over A in $\langle \mathcal{B}_1, A_1 \rangle$ and $\langle \mathcal{B}_2, A_2 \rangle$ respectively.*

3.6 Definable Closure

The universe of the dense substructure is definably closed in a dense pair.

Theorem 3.11. *The following statements hold.*

1. *Let $\langle \mathcal{B}, A \rangle \models T^d$. Then A is definably closed in L^2 . That is if an element a is L^2 -definable with parameters in A , then it is in A .*
2. *Let $\langle \mathcal{B}, A \rangle \models T^d$ and $\mathcal{A}_0 \preceq \mathcal{A}$. Then A_0 is definably closed in $\langle \mathcal{B}, A \rangle$.*

3. Let $\langle \mathcal{B}_1, A_1 \rangle \models T^d$. Let also $\langle \mathcal{B}, A \rangle$ be a ‘free substructure of $\langle \mathcal{B}_1, A_1 \rangle$ ’. Then B is definably closed in $\langle \mathcal{B}_1, A_1 \rangle$.

3.7 Open Core

Chapter 5 covers the notion of open core, but in order to have all properties of a dense pair in the same chapter, we need to briefly introduce it here. Let \mathcal{R} be an expansion of $\langle \mathbb{R}, < \rangle$. Let Γ be the set

$$\{U : U \subseteq \mathbb{R}^n \text{ for some } n \text{ and } U \text{ is open and definable in } \mathcal{R}\}$$

Then the structure

$$\langle \mathbb{R}, <, (U)_{U \in \Gamma} \rangle$$

is called the ‘open core’ of \mathcal{R} is denoted by \mathcal{R}° . There are structures which are not o-minimal but their open core is. Such structures are, in the terms of [24], ‘topologically’ close to being o-minimal. By a theorem in [24], if every definable subset of \mathbb{R} is finite or uncountable, then \mathcal{R}° is o-minimal. As a result of this theorem, the open core of a dense pair of o-minimal structures whose universe expands \mathbb{R} is o-minimal. This fact and discussions in the last section of [7] lead to the following theorem.

Theorem 3.12 ([7], Theorem 5). *Let \mathcal{B} be an expansion of the ordered field of real numbers and $\langle \mathcal{B}, A \rangle \models T^d$. Then if $S \subseteq \mathbb{R}^n$ is open and definable with parameters in B in the language L^2 , then it is definable in the language L possibly with different parameters.*

3.8 Elimination of the quantifier “there exist infinitely many”

The property described in the following theorem is sometimes called ‘elimination of \exists^∞ ’.

Theorem 3.13. *Let $\langle \mathcal{B}, A \rangle \models T^d$, $m, n \in \mathbb{N}$ and $S \subseteq B^{m+n}$ be L^2 -definable with parameters. Then there exists a positive integer M such that whenever $\bar{x} \in B^m$ and $S_{\bar{x}}$ is finite, then $|S_{\bar{x}}| < M$.*

3.9 T^d is NIP

The theory T^d of a dense pair of models of T , has Not the Independence Property. That is if $\langle \mathcal{B}, A \rangle \models T^d$, $\phi(\bar{x}, \bar{y})$ is an L^2 -formula, (\bar{a}_i) is an indiscernible sequence, and \bar{b} an element, then one of the following sets is cofinite in \mathbb{N} :

- $X := \{i : \langle \mathcal{B}, A \rangle \models \phi(\bar{a}_i, \bar{b})\}$
- $Y := \{i : \langle \mathcal{B}, A \rangle \models \neg\phi(\bar{a}_i, \bar{b})\}$

This is a result of a theorem in [14], which we will discuss in Chapter 6. We will also explain more about NIP in that chapter.

Chapter 4

The first order theory of a dense pair and a discrete group

In chapter 2, we described the first order theory of a real closed field with a discrete group, we set a complete axiomatisation for it, and emphasised on the fact that, in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ as a typical model of such a theory, \mathbb{Z} is not definable and Gödel phenomenon does not happen. In Chapter 3, we treated the dense pairs of o-minimal structures, defined small sets and described the definable sets. We also pointed out that \mathbb{Z} is not definable there as well.

In this chapter, we will study the model theory of a more complicated structure comprised of a real closed field, a dense substructure and a discrete group: a combination of the two structures described above. We will present a set of axioms, which, we will prove, form a complete theory whose models are structures of the same format as we need. Completeness of this theory will be proved by a back and forth argument on a certain collection of substructures of two saturated models. Note that the structure in which we take particular interest is $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$.

We will prove that our theory, which we will call \mathbb{T} , eliminates quantifiers up to predicates we add for certain existential formulas. We will then describe the types of elements in a given model of our theory. We will show that depending on being in the discrete group or the dense substructure, the type of a tuple is determined by its type of simpler formulas. In other words, if $\mathbb{M} = \langle M, G, A, \dots \rangle$ is a model of our theory, where G is dense in M and A is the discrete subgroup, then the type a tuple of elements in A makes in \mathbb{M} is determined by its type in $\langle A, \dots \rangle$. Also if our tuple is

in G then its type in $\langle G, A, \dots \rangle$ determines its type in $\langle M, G, A, \dots \rangle$.

We will finally characterise the definable subsets of a model of \mathbb{T} and will leave the topological descriptions of our structure to chapter 5.

4.1 Setting

We write RCF for the theory of real closed fields. Throughout $\bar{\mathbb{R}} = \langle \mathbb{R}; +, \cdot, -, 0, 1, < \rangle$ is the ordered field of reals with L_{or} its language. Let $\tilde{\mathbb{R}} = \langle \bar{\mathbb{R}}; \omega, \dots \rangle$ be some expansion of $\bar{\mathbb{R}}$ with a distinguished constant $\omega > 1$ with $\tilde{T} = \text{Th}(\tilde{\mathbb{R}})$. Denote by \tilde{L} the language $L(\tilde{T})$. For the rest of this chapter we assume that \tilde{T} is fixed and satisfies the following assumptions $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$.

\tilde{T}_1) **o-minimality.**

\tilde{T}_2) **polynomially boundedness with \mathbb{Q} -exponents.** To remind the reader, an o-minimal structure \mathcal{M} is called *polynomially bounded (with \mathbb{Q} -exponents)* if for all definable functions $f : M \rightarrow M$, there is a $q \in \mathbb{Q}$ and a $c \in M$ such that $\frac{f(x)}{c \cdot x^q} \rightarrow 1$ as $x \rightarrow +\infty$ (in \mathcal{M}). This property certainly does not hold for $\langle \bar{\mathbb{R}}, \exp \rangle$.

\tilde{T}_3) **model completeness.**

As mentioned in Chapter 2, under these assumptions \tilde{T} is a *Skolem theory* and the Skolem closure is a pregeometry for models of this theory:

Let $\tilde{M}_1, \tilde{M}_2 \models \tilde{T}, \tilde{M}_1 \preceq \tilde{M}_2$. Let $S \subseteq M_2$. Then the closure of $M_1 \cup S$ under the 0-definable functions (of \tilde{M}_2), denoted $M_1 \langle S \rangle$, is the domain of a (unique) elementary substructure $\tilde{M}_1 \langle S \rangle$ of \tilde{M}_2 :

$$\tilde{M}_1 \preceq \tilde{M}_1 \langle S \rangle \preceq \tilde{M}_2.$$

Further, there exists $S_0 \subseteq S$ such that $M_1 \langle S_0 \rangle = M_1 \langle S \rangle$ (i.e. S_0 generates $M_1 \langle S \rangle$ over M_1) and for all $s \in S_0, s \notin M_1 \langle S_0 - s \rangle$ (i.e. S_0 is independent over M_1). All such bases have the same cardinality, denoted $\dim_{\tilde{M}_1}(\tilde{M}_1 \langle S \rangle)$ or $\text{rank}(\tilde{M}_1 \langle S \rangle | \tilde{M}_1)$. All the reference to structures generated over models of \tilde{T} are with regard to this pregeometry.

Standard valuation is defined on the universe of each model \tilde{M} of \tilde{T} as follows. Let $x, y \in M$, We say x and y are in the same Archimedean class if

$$\frac{x}{y}, \frac{y}{x} \in \text{Fin } M := \{x \in M : \exists N \in \mathbb{N} \quad |x| < N\}.$$

The set of all Archimedean classes of M with the multiplication inherited from M forms a group $\Gamma(M)$. As M is a real closed field $\Gamma(M)$ is a divisible abelian group and hence a \mathbb{Q} -vector space. With the following ordering, $\Gamma(M)$ is also an ordered group:

$$x > 0 \text{ if } x \in \mu(M) := \{x \in M : \forall n \in \mathbb{N}, |x| < \frac{1}{n}\}.$$

The mapping $v : M \rightarrow \Gamma(M)$, which sends each element of M to its Archimedean class $v(x)$, is called the ‘standard valuation’. In some of our proofs we invoke the ‘valuation inequality’, described in section 2.5 Chapter 2.

A predicate for a dense subset

Add a unary predicate U to \tilde{L} to get the language $\tilde{L}(U)$. Let \tilde{T}_{dense} in $\tilde{L}(U)$ be the theory with the following (informally written) axioms.

$\langle \tilde{M}, G \rangle \models \tilde{T}_{dense}$ if

$\mathbb{T}_1)$ $\tilde{M}, \tilde{G} \models \tilde{T}$.

$\mathbb{T}_2)$ $\tilde{G} \preceq \tilde{M}$ and $G \neq M$.

$\mathbb{T}_3)$ G is dense in M .

By [7], and as we saw in Chapter 3, $\tilde{T}_{dense} = \{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3\}$, as the theory of a dense pair of o-minimal structures, is complete and in a model $\langle \tilde{M}, G \rangle$ of \tilde{T}_{dense} , every $\tilde{L}(U)$ -formula $\psi(\bar{y})$ has an equivalent which is a Boolean combination of formulas of the form

$$\exists \bar{x} \left(\bigwedge_i U(x_i) \right) \wedge \phi(\bar{x}, \bar{y})$$

with $\phi(\bar{x}, \bar{y}) \in \tilde{L}$.

A predicate for a discrete subgroup

Now add a predicate A to $\tilde{L}(U)$ to get the language $\tilde{L}(U, A)$ and let ω be a constant in \tilde{L} such that in $\tilde{\mathbb{R}}$, $\omega > 1$. Also add the following axioms to \tilde{T}_{dense} to get $\tilde{T}_{dense-discrete}$:

$\mathbb{T}_4)$ (A, \cdot) is a multiplicative group.

$\mathbb{T}_5)$ $\omega \in A$, $\omega > 1$, $\forall x \ [x \in (1, \omega) \rightarrow x \notin A]$

$$\mathbb{T}_6) \quad \forall x > 0 \quad \exists y \quad (A(y) \wedge y \leq x < \omega y)$$

$$\mathbb{T}_7) \quad A \subseteq G$$

We also add an auxiliary ‘floor’ function symbol and countably many predicates $\{P_n\}_{n \in \mathbb{N}}$ to our language and we interpret them by the following axioms:

$$\mathbb{T}_8) \quad \forall x > 0 \quad \lambda(x) \in A \text{ and } \forall x < 0 \quad \lambda(x) = 0.$$

$$\mathbb{T}_8) \quad \forall x \quad \lambda(x) \leq x < w\lambda(x).$$

$$\mathbb{T}_9) \quad \forall x \quad P_n(x) \leftrightarrow \exists y \in A \quad (x = y^n).$$

Note that $\langle A, \{P_n\}, \cdot, < \rangle$ is a model of Pressburger arithmetic, which is well-known to eliminate quantifiers (see for example [20]).

$\tilde{T}_{discrete} := \tilde{T} \cup \{\mathbb{T}_4, \mathbb{T}_5, \mathbb{T}_6\}$ is complete and model-complete (Chapter 2, van den Dries [8], Miller [23]). In fact, for a model of $\tilde{T}_{discrete}$, every formula $\psi(\bar{y})$ has an equivalent which is a Boolean combination of the formulas of the form

$$\exists \bar{x} \quad \left(\bigwedge_i A(x_i) \wedge \phi(\bar{x}, \bar{y}) \right)$$

with $\phi(\bar{x}, \bar{y})$ an \tilde{L} -formula.

Notation. We denote by \mathbb{T} the theory axiomatised by $\mathbb{T}_1, \dots, \mathbb{T}_9$. Let $\mathbb{L} = L(\mathbb{T})$. \mathbb{L} -structures are denoted by \mathbb{M} , etc., where \mathbb{M} stands for an \mathbb{L} -structure $\langle \tilde{M}, G, A, \lambda, P_n \rangle$. We may consider a structure \mathbb{M} and prove statements about \tilde{M}, M, G, A etc. where we have implicitly assumed that \mathbb{M} is the structure $\langle \tilde{M}, G, A, \lambda, P_n \rangle$.

By $\mathcal{M}\langle x \rangle$ we denote the structure generated over \mathcal{M} by x whenever \mathcal{M} is the model of a theory with a known pregeometry, say it is a real closed field or a model of \tilde{T} . In this case by $M\langle x \rangle$ we denote the universe of $\mathcal{M}\langle x \rangle$. Models of \tilde{T} are denoted \tilde{M} etc. The model generated over \tilde{M} by x is denoted by $\tilde{M}\langle x \rangle$ and its universe by $M\langle x \rangle$. If Λ is a set of elements and \mathcal{M} is model of a theory with a known pregeometry, we write $M\langle \Lambda \rangle$ for $\langle M \cup \Lambda \rangle$. If x is a single element we also write $M\langle x, \Lambda \rangle$ for $\langle M \cup \{x\} \cup \Lambda \rangle$. We usually write P_n both for P_n as a predicate and for $P_n(M)$.

Additional remarks

A

As noted before, if $\omega = 2$ then $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}}, \lambda, \{P_n\}_{n \in \mathbb{N}} \rangle$ is a model of \mathbb{T} , where P_n denotes $2^{n\mathbb{Z}}$ for each n , and for each x , $\lambda(x)$ is the largest integer power of two less than or equal to x . More precisely, we will prove that if $\tilde{T} = \text{Th}(\bar{R})$ and $\omega = 2$, then \mathbb{T} axiomatises the theory of this structure in the language \mathbb{L} . This structure was indeed our first motivation for this study as it simultaneously contains a discrete and a dense codense subset of \mathbb{R} and yet it is still not wild.

B

Let $\mathbb{M} \models \mathbb{T}$ (and $\omega = 2$). Then by our axioms

$$\forall x \quad \lambda(x) \leq x < 2\lambda(x).$$

Since for each x , $\lambda(x) \in A$, we have the following group homomorphism:

$$\Gamma(M) = \Gamma(G) \cong A/2^{\mathbb{Z}}.$$

Note that as \tilde{M} is a real closed field, $(\Gamma(M), \cdot)$ is a divisible abelian group and hence so is $A/2^{\mathbb{Z}}$.

C

Let \tilde{R} be some expansion of $\bar{\mathbb{R}}$. Let $\tilde{T} = \text{Th}(\tilde{R})$ and \tilde{L} be its language. The field of exponents of \tilde{R} is the following field K .

$$K = \{r \in \mathbb{R} : \text{the function } x \rightarrow x^r \text{ on } (0, \infty) \text{ is 0-definable in } \tilde{R}\}.$$

\tilde{R} is called polynomially bounded ([21]) if for every unary definable function f there exists some $N \in \mathbb{N}$ such that for all sufficiently large positive elements of \mathbb{R} we have $|f(x)| \leq x^N$.

In [21] Miller proved that if \mathcal{R} is an expansion of $\langle \bar{\mathbb{R}}, < \rangle$ then either \mathcal{R} defines e^x , or for every ultimately nonzero \mathcal{R} -definable $f : \mathbb{R} \rightarrow \mathbb{R}$ there exists a nonzero $c \in \mathbb{R}$ and a 0-definable real power function x^r such that $f(x) = cx^r + o(x^r)$.

D

In our proofs we will make use of the following proposition from valuation theory. For its proof one can refer to [9].

Proposition 4.1. *Suppose that $v : K \rightarrow \Lambda \cup \{\infty\}$ is a valuation of the field K , Λ is an ordered subgroup of an ordered group Λ' and $\gamma \in \Lambda' - \Lambda$ has the following property: if $n \in \mathbb{Z}$ satisfies $n\gamma \in \Lambda$, then $n = 0$. Then there is exactly one valuation w on $K(X)$ extending v with $w(X) = \gamma$. For this w , we have $\overline{K(X)} = \overline{K}$ (overline denoting the residue field), and $w(K(X)^*) = \Lambda \oplus \mathbb{Z}_\gamma$ with the order induced from Λ' .*

E does \mathbb{T} have algebraically prime models?

The first approach towards \mathbb{T} should be checking classical quantifier elimination tests: checking whether \mathbb{T} has algebraically prime models (explained in the next paragraph), and if it does, then checking if M is simply closed in N for all $\mathbb{M}, \mathbb{N} \models \mathbb{T}$ with $M \subseteq N$. (M being simply closed in N means that for every quantifier-free formula $\phi(x, \bar{y})$ and tuple $\bar{m} \in M$ the following happens: $\mathbb{N} \models \exists x \phi(x, \bar{m})$ if and only if $\mathbb{M} \models \exists x \phi(x, \bar{m})$.) As in [20] these two imply quantifier elimination. This approach fails, but it is worth seeing how far one can get towards proving that \mathbb{T} has algebraically prime models. This is addressed in this subsection.

Let us do the argument for when \tilde{T} is *RCF* and $\omega = 2$. Checking if \mathbb{T} has algebraically prime models means checking if the answer to the following question is positive: if $\mathbb{M} \models \mathbb{T}_\forall$, then is there any $\mathbb{M}' \supseteq \mathbb{M}$ such that $\mathbb{M}' \models \mathbb{T}$ and it can be embedded in all models $\mathbb{M}'' \supseteq \mathbb{M}$?

Let $\mathbb{M} = \langle M, G, \dots \rangle \subseteq \mathbb{M}_1, \mathbb{M}_2 \models \mathbb{T}$. Then M, G are universes of models of *RCF_∇* and real integral domains. By similar techniques to those employed in [7] (which we will explain shortly) FM , the fraction field of M , is closed under λ_1 and setting: $FA = \{\frac{a}{b} : a, b \in A\}$ and $FP_n = \{\frac{a}{b} : a, b \in P_n(M)\}$, the \mathbb{L} -structure $F\mathbb{M} := \langle FM, FM \cap G_1, \lambda_1|_{FM}, FA, FP_n \rangle$ extends \mathbb{M} . But, there is no guarantee that $FM \cap G_1 = FM \cap G_2$. Before resolving this, let us first prove the statement that FM is closed under λ_1 :

The reason why FM is closed under λ_1 . Let $\frac{a}{b}$ be a positive element in FM , then:

$$0 < \lambda(a) \leq a < 2\lambda(a)$$

$$0 < \lambda(b) \leq b < 2\lambda(b)$$

so

$$\frac{1}{2} \frac{\lambda(a)}{\lambda(b)} \leq \frac{a}{b} < 2 \frac{\lambda(a)}{\lambda(b)},$$

which implies that

$$\lambda_1(a/b) = \begin{cases} (\frac{1}{2}) \frac{\lambda(a)}{\lambda(b)} & \text{if } a/b < \frac{\lambda(a)}{\lambda(b)} \\ \frac{\lambda(a)}{\lambda(b)} & \text{if } a/b \geq \frac{\lambda(a)}{\lambda(b)}. \end{cases}$$

This means that FM is closed under λ_1 . We define $F\lambda$ as the restriction of λ_1 to FM . Clearly for each $x \in FM$, $F\lambda(x)$ is in the fraction field of G . Furthermore, $A_1 \cap FM = \{a/b : a, b \in A\} \subseteq FM \cap G_1$ and $P_{1n} \cap FM = \{a/b : a, b \in P_n\} \subseteq FM \cap G_1$. Note that the definition of $F\lambda$ comes naturally and not dependent on λ_1 . \square

Now add a predicate $P(x, y)$ to \mathbb{L} and the axiom

$$ax_p := P(x, y) \leftrightarrow \exists z \quad (U(z) \wedge x = yz)$$

to \mathbb{T} . Let $\mathbb{M} \subseteq \mathbb{M}_1, \mathbb{M}_2 \models \mathbb{T} \cup \{ax_p\}$. Now if $\frac{a}{b} \in FM \cap G_1$, then $\mathbb{M}_1, \mathbb{M}_2 \models p(a, b)$, so $\frac{a}{b} \in FM \cap G_2$. This shows that if we work with the theory $\mathbb{T} \cup \{ax_p\}$, then $F\mathbb{M}$ is prime (as a structure not a model) over \mathbb{M} .

Working with the theory $\mathbb{T} \cup \{ax_p\}$, Let \overline{FM} denote the real closure of FM inside M_1 (and its universe). Then $\overline{FM} \cap G_1$ is a real closed field and by valuation theory techniques, the structure $\overline{FM} = \langle \overline{FM}, \overline{FM} \cap G_1, \lambda_1|_{\overline{FM}}, \overline{FA}, \overline{FP_n} \rangle$ extends \mathbb{M} and embeds in \mathbb{M}_1 . By \overline{FA} and $\overline{FP_n}$ we have denoted the divisible hulls of FA and FP_n . Let us see the proof of this as well:

The reason why \overline{FM} is closed under λ_1 . Let $v : FM \rightarrow \Gamma(FM)$ be the standard valuation on FM (and on FG , where $v(FG)$ is a subgroup of $v(FM)$). The valuation v extends to $\bar{v} : \overline{FM} \rightarrow \Gamma(\overline{FM})$ where $\Gamma(\overline{FM})$ is the divisible hull of $\Gamma(FM)$. So for each $x \in \overline{FM}$, we have

$$\bar{v}(x) = \frac{1}{n} v(y)$$

for some $y \in FM$ and $n \in \mathbb{N}$. We can assume that $y \in P_{1n}$ (so it is in FG), and hence $y^{\frac{1}{n}} \in A_1$ (this is because y in above can be replaced by $\lambda_1(y)$ and then by $2^i \lambda_1(y)$,

where i —between 0 and n and existing by axioms—is such that $2^i \lambda_1(y) \in P_{1n}$. So $\bar{v}(x) = v(y^{\frac{1}{n}})$ and this implies the existence of a $k \in \mathbb{Z}$ such that $2^k y^{\frac{1}{n}} \leq x < 2^{k+1} y^{\frac{1}{n}}$. So $\lambda_1(x) = 2^k y^{\frac{1}{n}} \in \overline{FG}$. Note that this extension of $F\lambda$ is indeed independent from λ_1 . \square

Add to \mathbb{L} , predicates $q_{\phi(\bar{x}, \bar{y})}(\bar{y})$ for every L_{or} -formula $\phi(\bar{x}, \bar{y})$ and to \mathbb{T} , axioms

$$ax_{q_{\phi(\bar{x}, \bar{y})}} := q_{\phi(\bar{x}, \bar{y})}(\bar{y}) \leftrightarrow \exists \bar{x} \quad (U(\bar{x}) \wedge \phi(\bar{x}, \bar{y})).$$

Call the obtained language and theory $\mathbb{L}_{enriched}$ and $\mathbb{T}_{enriched}$ respectively. Now, if $\mathbb{M}_1, \mathbb{M}_2 \models \mathbb{T}_{enriched}$, then we can follow the same argument to get \overline{FM} but with the advantage that in this case, $\overline{FM} \cap G_1 = \overline{FM} \cap G_2$ are real closed fields and \overline{FM} is prime over \mathbb{M} .

However, $\overline{FM} \cap G_1$ may not be dense in \overline{FM} . \overline{FM} can be further extended to a model of $\mathbb{T}_{enriched}$, but there is no canonical way of embedding it into a model FM_{dense} of $\mathbb{T}_{enriched}$ which can be embedded into \mathbb{M}_1 and \mathbb{M}_2 . Let us summarise the above discussion in a corollary as follows:

Corollary 4.2. *\mathbb{T} does not have algebraically prime models. However, a model of $\mathbb{T}_{enriched \forall}$ can be extended to a model \mathbb{M}' of it, where M', G' are models of \tilde{T} , but G' may not be dense in M' . \mathbb{M}' embeds in all models of $\mathbb{T}_{enriched}$ that extend \mathbb{M} .*

4.2 Freeness and structures generated by elements

By the Fact 2.2. in [7], any elementary pair of o-minimal structures $\langle \mathcal{M}, G \rangle$ can be embedded into a dense elementary pair $\langle \mathcal{M}^*, G^* \rangle$. In corollary 4.2, we noted that there is no canonical way of finding a dense pair extension $\langle \mathcal{M}^*, G^* \rangle$ of $\langle \mathcal{M}, G \rangle$ that can be embedded in all dense pairs extending $\langle \mathcal{M}, G \rangle$.

Let $\langle \mathcal{M}, G \rangle$ be an elementary pair of real closed fields which is a substructure of the elementary pair $\langle \mathcal{M}_1, G_1 \rangle$, and $x \in G_1 - M$. We first need to work out the answer to this question: ‘what is the structure generated by x over $\langle \mathcal{M}, G \rangle$ in $\langle \mathcal{M}_1, G_1 \rangle$?’ We need to find a way of using the pregeometry of real closed fields in order to find the structure generated over a given elementary pair.

Clearly, in order for a structure like $\langle \mathcal{M}\langle x \rangle, G\langle x \rangle \rangle$ to be a substructure of $\langle \mathcal{M}_1, G_1 \rangle$ we need to have $M\langle x \rangle \cap G_1 = G\langle x \rangle$ (by $\mathcal{M}\langle x \rangle$ we mean the real closed field generated

by x over \mathcal{M} in \mathcal{M}_1). Also if we want this structure to be prime over $\langle \mathcal{M}, G \rangle$ we need to have $M\langle x \rangle \cap \hat{G} = M\langle x \rangle \cap G_1 = G\langle x \rangle$, for all dense elementary extensions $\langle \hat{\mathcal{M}}, \hat{G} \rangle$ of $\langle \mathcal{M}, G \rangle$. But unfortunately this is not always the case. The following proposition helps us understand why. This proposition is due to Tressl and its proof follows from Theorem 4.57 (also from Tressl) whose statement and proof is in the appendix of this chapter.

Proposition 4.3. *Let $R \subseteq S \subseteq S'$ be real closed fields with transcendence degree of S over R not greater than cardinality of R . Assume that $S' = S\langle x \rangle$ for some $x \notin S$. Then there is a real closed field $R' \subseteq S'$ containing R such that $R' \cap S = R$ and $S' = R'\langle x \rangle$. Consequently $S' = R'\langle \alpha \rangle$ for every $\alpha \in S - R$.*

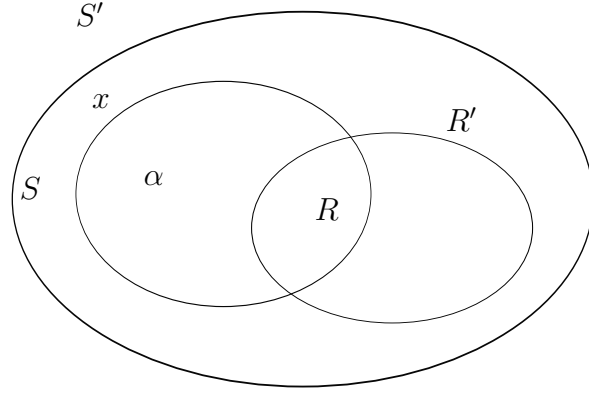


Figure 4.1: $R'\langle \alpha \rangle \cap S \neq R\langle \alpha \rangle$

In the above proposition $\langle S', S \rangle$ an extension of $\langle R', R \rangle$, and if transcendence degree of S over R is greater than 1, then for all $\alpha \in S - R$, $R'\langle \alpha \rangle \cap S = S \neq R\langle \alpha \rangle$. So $\langle R'\langle \alpha \rangle, R\langle \alpha \rangle \rangle$ is obviously not a substructure of $\langle S', S \rangle$ for any $\alpha \in S - R$.

This is why we need the following definition of **freeness**. This definition is essentially an adaptation of the similar definition in [7] to our notation. Freeness makes the following possible:

If the elementary pair $\langle \tilde{M}, G \rangle$ of models of \tilde{T} is a free substructure of the elementary pair $\langle \tilde{M}', G' \rangle$ of models of \tilde{T} and $x \in G' - M$, then $\langle \tilde{M}\langle x \rangle, G\langle x \rangle \rangle$ is a free substructure of $\langle \tilde{M}', G' \rangle$.

Definition 4.3.1. *Let $\mathbb{M} = \langle \tilde{M}, G, \dots \rangle \subseteq \mathbb{M}_1 \models \mathbb{T}$ be such that $\langle \tilde{M}, G \rangle$ is an elementary pair of models of \tilde{T} . We say \mathbb{M}_1 is free over \mathbb{M} , or \mathbb{M}_1 is a free extension of \mathbb{M} , or \mathbb{M} is a free substructure of \mathbb{M}_1 , if the following happens: if $Y \subseteq G_1$, then Y is*

independent over G (with respect to the pregeometry of \tilde{T}) in M_1 if and only if it is independent over M .

Under the conditions of the above definition, we have the following:

1. if $Y \subseteq M$ then Y is independent over G if and only if it is independent over G_1 .
2. If $Z \subseteq G_1$, then $M\langle Z \rangle \cap G_1 = G\langle Z \rangle$.

One can check that freeness is transitive. That is if \mathbb{M}_1 is a free extension of \mathbb{M}_0 and \mathbb{M}_2 a free extension of \mathbb{M}_1 , then \mathbb{M}_2 is a free extension of \mathbb{M}_0 .

The following proposition whose proof will be given at the end of this chapter provides examples of non-free extensions for the pairs of real closed fields.

Proposition 4.4. *Let $G \subseteq M$ be real closed fields. Let a_1, \dots, a_d be distinct elements of G . Let $\alpha_1, \dots, \alpha_d, \epsilon$ in $M - G$ be independent (with the pregeometry of real closed fields) over G . Set $t_i = \alpha_i(\epsilon + a_i)$. Then:*

$$G\langle t_1, \dots, t_d \rangle \cap G\langle \alpha_1, \dots, \alpha_d \rangle = G.$$

In our terms, the pair $\langle M, G\langle t_1, \dots, t_d \rangle \rangle$ is an extension of the pair $\langle G\langle \alpha_1, \dots, \alpha_d \rangle, G \rangle$ but is not free over it.

At some point in our proof of completeness of \mathbb{T} , we will need the following lemma whose statement and proof can be found in [7].

Lemma 4.5. *[from [7]] Let T be a complete o-minimal theory which extends the theory of ordered abelian groups. Then*

- i) If the dense pair $\langle \mathcal{M}, G \rangle$ of models of T is κ -saturated for $\kappa > |T|$, then $\dim_{\bar{G}}(\tilde{M}) \geq \kappa$.*
- ii) If $\langle \mathcal{M}, G \rangle$ is a dense pair of models of T , then $M - G$ is dense in M .*

In several parts of this thesis I have posed questions which naturally come to mind when dealing with the dense pairs, and whenever I have known the answer I have provided it. The following is among them.

Question 4.6. *Is a field dense in its real closure?*

The answer is no; a field may not be dense in its real closure. Consider the field $\mathbb{Q}(x)$ where x is an element larger than all of the elements in \mathbb{Q} . Then the interval $(\sqrt{x}, 2\sqrt{x})$ in the real closure of $\mathbb{Q}(x)$ contains no element of $\mathbb{Q}(x)$. However, every field is co-final in its real closure. This means that for every element in the real closure, there is an element in the field which is larger than it.

Another question along these lines is the following.

Question 4.7. *Suppose that a field is dense in a bigger field; does this imply that its real closure is dense in the real closure of the bigger field?*

As I was trying to prove this, Tressl pointed out to me that it is well-known. The following proposition is the answer.

Proposition 4.8 (Scott 1969). *Let $M \subseteq N$ be two ordered fields where M is dense in N . Then the real closure of M is dense in the real closure of N .*

One can find the proof in [30]. Tressl has a different proof in his notes [31] and I thank him for pointing out this to me.

Question 4.9. *Let M, N and K be real closed fields with $M \subseteq N \subseteq K$. Let $x \in K - N$. Then is the field generated by x over M dense in the field generated by x over N ?*

The answer to the above question is also no. As a counterexample consider $\mathbb{R}_{alg} \subseteq \mathbb{R}$. Let ϵ in some extension of \mathbb{R} be an infinitesimal element over \mathbb{R} . Then the field $\mathbb{R}_{alg}(\pi + \epsilon)$ is not dense in $\mathbb{R}(\pi + \epsilon)$. Because there is no element in $\mathbb{R}_{alg}(\pi + \epsilon)$ between π and $\pi + \epsilon$.

The negative answer to the above question was unfortunate to this thesis, and led to changes in some proofs. Those proofs were based on the wrong statement below and its wrong proof! we will point out why this will be unfortunate as we reach that point in the thesis. It will be in the proof of Lemma 4.43.

A wrong statement 4.10. *Let the field M be dense in the field N and K be a real closed field containing N . Then for each $x \in K - N$, $M\langle x \rangle$ (the real closure of $M \cup \{x\}$) is dense in $N\langle x \rangle$.*

A wrong proof. As $M(x)$, the field generated over M by x is dense in $N(x)$ and by Proposition 4.8. □

Now as the title of this section suggests, let us move towards identifying the structures generated by an element over a given structure, inside a model for which we defined freeness. The following remark is the first step.

Remark 4.10.1. *Let $\langle \tilde{M}, A \rangle \subseteq \langle \tilde{M}_1, A_1 \rangle$ be models of $\tilde{T}_{discrete}$ and $x \in A_1 - A$. Then*

1. *By Lemma 2.3, for all $t \in M\langle x \rangle$, $\lambda_1(t) \in M\langle x \rangle$, i.e. $M\langle x \rangle$ is closed under λ_1 .*
2. *For simplicity, we write $A\langle x \rangle$ for $M\langle x \rangle \cap A_1$.*
3. *$\langle \tilde{M}\langle x \rangle, A\langle x \rangle \rangle \models T_{discrete}$.*
4. *We also write $P_n\langle x \rangle$ for $M\langle x \rangle \cap P_{1n}$.*
5. *$\langle A\langle x \rangle, P_n\langle x \rangle, \cdot, \cdot, \cdot, \cdot \rangle$ is a model of Presburger arithmetic generated by x over $\langle A, P_n \rangle$ in $\langle A_1, P_{1n}, \cdot, \cdot, \cdot, \cdot \rangle$.*
6. *If Λ is a sequence of elements in A_1 , we write $A\langle \Lambda \rangle$ and $P_n\langle \Lambda \rangle$ respectively for $M\langle \Lambda \rangle \cap A_1$ and $M\langle \Lambda \rangle \cap P_{1n}$.*

The following lemma describes the structure generated by an element in the discrete group over a given structure, and in a free extension.

Lemma 4.11. *Let $\mathbb{M}_1 = \langle \tilde{M}_1, G_1, A_1, \lambda_1, P_{1n} \rangle$ be a free extension of \mathbb{M} and $x \in A_1 - M$. Then the structure $\langle \tilde{M}\langle x \rangle, G\langle x \rangle, A\langle x \rangle, \lambda_1|_{M\langle x \rangle}, P_n\langle x \rangle \rangle$ is a free substructure of \mathbb{M}_1 . We denote this structure by $\mathbb{M}\langle x \rangle$, and we call it the structure generated by x over \mathbb{M} .*

Note that in this lemma, $\tilde{M}\langle x \rangle$ and $\tilde{G}\langle x \rangle$ are \tilde{T} -structures generated by x over M and G , respectively, and, $A\langle x \rangle$ and $P_n\langle x \rangle$ are as in the above remark.

Proof. As \mathbb{M}_1 is free over \mathbb{M} , by item 2 after Definition 4.3.1, $M\langle x \rangle \cap G_1 = G\langle x \rangle$. Also, by the previous remark, $M\langle x \rangle \cap A_1 = A\langle x \rangle$, $M\langle x \rangle \cap P_{1n}(M_1) = P_n\langle x \rangle$ and $M\langle x \rangle$ is closed under λ_1 . So, $\mathbb{M}\langle x \rangle$ as described in the statement of the theorem is an \mathbb{L} -structure extending \mathbb{M} and a free substructure of \mathbb{M}_1 . \square

In the first draft of this thesis, in the definition 4.3.1 of freeness the condition that $\langle \tilde{M}, G \rangle$ is an elementary pair of models of \tilde{T} was not assumed. So, for lemmas like the above lemma to hold, we needed to keep predicates $q_{\phi(\bar{x}, \bar{y})}(\bar{y})$ in the language and

their corresponding axioms in the theory, and then by similar discussions to those before Corollary 4.2, we needed to get from a pair of integral domains to a pair of models of \tilde{T} . That made all the proofs longer but more general. Then it came to the author's attention that considering freeness (without assuming that $\langle \tilde{M}, G \rangle$ is an elementary pair of models of \tilde{T}) and doing the back and forth argument as we will see in the proof of completeness, Theorem 4.14 is equivalent to adding those predicates to the language and not considering freeness at the first place. This method is used for proving a similar result in section 4.6.

In the next lemma we will see the structure generated by an element of G_1 over \mathbb{M} can be more complex than what we saw in the above lemma.

Lemma 4.12. *Let \mathbb{M}_1 be a free extension of \mathbb{M} and $x \in G_1 - M$. Then there is a countable sequence $\Lambda = (a_i)_{i \in \mathbb{N}}$ in A_1 with the following property. $a_1 = \lambda_1(f(x))$ for some \tilde{L} -definable function f with parameters in M , and $a_{n+1} = \lambda_1(g_n(a_1, \dots, a_n, x))$ for some \tilde{L} -definable function g_n with parameters in M . Furthermore $\mathbb{M}\langle x, \Lambda \rangle := \langle \tilde{M}\langle x, \Lambda \rangle, G\langle x, \Lambda \rangle, A\langle \Lambda \rangle, P_n\langle \Lambda \rangle \rangle$, with $A\langle \Lambda \rangle$ and $P_n\langle \Lambda \rangle$ as in Remark 4.10.1, is an \mathbb{L} -structure which extends \mathbb{M} and is a free substructure of \mathbb{M}_1 .*

Proof. Consider the \tilde{L} -structure $\tilde{M}\langle x \rangle$. Two cases can occur. Case one, when $M\langle x \rangle$ has no more Archimedean classes than M does. In this case, $\langle \tilde{M}\langle x \rangle, G\langle x \rangle, \lambda_1 M\langle x \rangle, A, P_n \rangle$ is the \mathbb{L} -structure we are looking for, and we let $\Lambda = \emptyset$. Case two is when $M\langle x \rangle$ has Archimedean classes which are not represented in M . By the valuation inequality (2.5-B), there is only one new independent Archimedean class in $M\langle x \rangle$ which is not represented in M , say the class of some $f(x)$, and by the axioms, there is $a \in A_1$ which represents this new class in M_1 , i.e. $a = \lambda_1(f(x))$. Let $a_1 = a$, and consider the structure $\mathbb{M}\langle a_1 \rangle$ as described in the previous lemma. Now, by induction, if $M\langle x, a_1, \dots, a_n \rangle$ has an Archimedean class which is not represented in $A\langle a_1, \dots, a_n \rangle$, then let $a_{n+1} \in A_1 - M\langle a_1, \dots, a_n \rangle$ be the representative of this class in M_1 , otherwise let $a_{n+1} = a_n$. Let $\Lambda = (a_i)_{i \in \mathbb{N}}$. We claim that $M\langle x, \Lambda \rangle$ is closed under λ_1 and $M\langle x, \Lambda \rangle \cap A_1 = A\langle \Lambda \rangle$. Let $y \in M\langle x, \Lambda \rangle$, then $y \in M\langle x, a_{i_1}, \dots, a_{i_m} \rangle$ for $a_{i_1}, \dots, a_{i_m} \in \Lambda$. So by induction hypothesis, $\lambda_1(y)$ is in $M\langle x, a_1, \dots, a_k \rangle$ for $k > i_m$. By freeness of \mathbb{M}_1 over \mathbb{M} , $M\langle x, \Lambda \rangle \cap G_1 = G\langle x, \Lambda \rangle$, and the structure $\mathbb{M}\langle x, \Lambda \rangle$ described in the statement of the theorem is an \mathbb{L} -structure extending \mathbb{M} and a free substructure of \mathbb{M}_1 . \square

Now let \mathbb{M}_1 be a free extension of \mathbb{M} and $x \in M_1 - M\langle G_1 \rangle$. Then we have $M\langle x \rangle \cap G_1 = G$ and the following lemma:

Lemma 4.13. *Let \mathbb{M}_1 , \mathbb{M} and x be as in above (lines). Then there is a countable sequence $\Lambda = (a_i)_{i \in \mathbb{N}}$ in A_1 such that $M\langle x, \Lambda \rangle$ is closed under λ_1 and in this case $\mathbb{M}\langle x, \Lambda \rangle := \langle M\langle x, \Lambda \rangle, G\langle \Lambda \rangle, A\langle \Lambda \rangle, P_n\langle \Lambda \rangle \rangle$ is a free substructure of \mathbb{M}_1 .*

Sketch of the proof. The proof is by the same method as in the previous lemma and considering the comments preceding this lemma. \square

We are now ready for the proof of completeness and quantifier elimination for \mathbb{T} in the next section.

4.3 Completeness and quantifier elimination

Completeness of \mathbb{T} is proved in the following theorem with the background of the previous section.

Theorem 4.14. *\mathbb{T} is complete.*

Proof. Consider the following diagram:

$$\begin{array}{c}
 \mathbb{M}_1 = \langle M_1, G_1, A_1, \lambda_1, \{P_{1n}\}_{n \in \mathbb{N}} \rangle \\
 \uparrow \\
 \mathbb{M} = \langle M, G, A, \lambda, \{P_n\}_{n \in \mathbb{N}} \rangle \\
 f \parallel \\
 \mathbb{M}' = \langle M', G', A', \lambda', \{P'_n\}_{n \in \mathbb{N}} \rangle \\
 \downarrow \\
 \mathbb{M}_2 = \langle M_2, G_2, A_2, \lambda_2, \{P_{2n}\}_{n \in \mathbb{N}} \rangle
 \end{array} \tag{4.1}$$

In the above diagram assume that \mathbb{M}_1 and \mathbb{M}_2 are κ -saturated models of \mathbb{T} , for $\kappa > |\tilde{T}| + |\mathbb{L}| + \aleph_0$, and \mathbb{M}_1 is a free extension of \mathbb{M} , \mathbb{M}_2 is a free extension of \mathbb{M}' , $|M|, |M'| \leq \kappa$, and \mathbb{M} is isomorphic to \mathbb{M}' with the isomorphism map denoted by f in the diagram. In the sequel, we will prove that the collection of such isomorphisms between free substructures of \mathbb{M}_1 and \mathbb{M}_2 has the back and forth property. Clearly this will imply that \mathbb{M}_1 and \mathbb{M}_2 are elementarily equivalent.

Let $x \in M_1 - M$. What we need is a $y \in M_2 - M'$, and a structure containing x and extending \mathbb{M} which is isomorphic to a structure containing y and extending \mathbb{M}' , with an isomorphism that extends f and sends x to y .

According to ‘where’ in \mathbb{M}_1 the element x comes from, we have the following cases:

Case one; when $x \in A_1 - M$.

Consider the structure $\mathbb{M}\langle x \rangle$, generated over \mathbb{M} by x as described in Lemma 4.11. Let $y \in A_2 - M'$ be an element which satisfies the same Presburger arithmetic type over $\langle A', P'_n, \cdot, < \rangle$ as x does over $\langle A, P_n, \cdot, < \rangle$ (via f). Such a y exists by quantifier elimination of Presburger arithmetic. y realises the same cut over M' as x does over M , and one can simply verify that the two structures $\mathbb{M}\langle x \rangle$ and $\mathbb{M}\langle y \rangle$, both generated as in Lemma 4.11 are isomorphic.

Case two; when $x \in G_1 - M$ and $x \notin A_1$.

In this case, consider the structure $\mathbb{M}\langle x, \Lambda \rangle$ as described in Lemma 4.12 with $\Lambda = (a_i)_{i \in \mathbb{N}}$ a sequence in A_1 . Let $b_1 \in A_2 - A'$ be an element, as in case one, which realises the same Pr-type (=Presburger arithmetic type) over $\langle A', P'_n, \cdot, < \rangle$ as a_1 does over $\langle A, P_n, \cdot, < \rangle$. Let b_{n+1} be an element that realises the same Pr-type over $A'\langle b_1, \dots, b_n \rangle$ as does a_{n+1} over $A'\langle a_1, \dots, a_n \rangle$. Denote by Λ' the sequence $(b_i)_{i \in \mathbb{N}}$ obtained this way. Now let $y \in M_2$ be an element which realises the same cut in $M'\langle \Lambda' \rangle$ as does x in $M\langle \Lambda \rangle$. It is now easy to check that the two structures $\mathbb{M}\langle x, \Lambda \rangle$ and $\mathbb{M}'\langle y, \Lambda' \rangle$ are isomorphic.

Case three; when $x \in M\langle G_1 \rangle$ and $x \notin G_1$.

In this case, there are elements x_1, \dots, x_n in G_1 such that $x \in M\langle x_1, \dots, x_n \rangle$. Now one can combine the arguments for cases one and two to get the result.

Case Four; when $x \in M_1$ and $x \notin M\langle G_1 \rangle$.

In this case we use Lemma 4.13 to construct the structure $\mathbb{M}\langle x, \Lambda \rangle$. Now as in case two, we can find a sequence $\Lambda' = (b_i)_{i \in \mathbb{N}}$ of elements in A_2 such that $\mathbb{M}\langle \Lambda \rangle$ and $\mathbb{M}\langle \Lambda' \rangle$ are isomorphic. Now by Lemma 4.5, $M'\langle G_2 \rangle = G_2\langle M' \rangle \neq M_2$, and since $M_2 - G_2$ is dense in M_2 and M_2 is saturated, we can find $y \in M_2 - M'\langle G_2 \rangle$ which realises the same cut in $M'\langle \Lambda' \rangle$ as x does in $M\langle \Lambda \rangle$. The rest of the proof is similar to the previous cases.

The above four cases exhaust all possibilities and the completeness of \mathbb{T} results from the fact that by our argument \mathbb{M}_1 is elementarily equivalent to \mathbb{M}_2 . \square

Note that if $\tilde{T} = \text{Th}(\bar{\mathbb{R}})$ and $\omega = 2$, then the structure $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, \lambda, 2^{\mathbb{Z}}, (2^{n\mathbb{Z}})_{n \in \mathbb{N}} \rangle$, with $\lambda(x)$ the biggest integer power of two, less than or equal to x , is a model of \mathbb{T} . So, as \mathbb{T} is complete, $\mathbb{T} = \text{Th}(\bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}})$. As mentioned before, this structure is our main example as we go through this chapter. Also from here onwards when we say \mathbb{M}_1 is a ‘sufficiently’ saturated extension of \mathbb{M} , we mean saturated as in the setting of the proof of the above theorem.

Quantifier elimination for \mathbb{T} can be proved by slight modifications of the above proof. This is done in the next theorem and its proof.

Theorem 4.15. *Every \mathbb{L} -formula (with free variables \bar{y}) has an equivalent which is a Boolean combination of formulas of the form*

$$\exists \bar{x} = (x_1, \dots, x_n) \quad \left(\bigwedge_{i=1, \dots, n} U(x_i) \wedge \phi(\bar{x}, \bar{y}) \right) \quad (*)$$

where $\phi(\bar{x}, \bar{y})$ is in $\tilde{L} \cup \{\lambda\}$.

Proof. Consider the diagram 4.1 in the proof of Theorem 4.14. Let $\bar{a} = (a_1, \dots, a_n) \in M_1$ and $\bar{b} = (b_1, \dots, b_n) \in M_2$ realise the same formulas of the form (*). We will prove that then $tp_{M_1}(\bar{a}) = tp_{M_2}(\bar{b})$, and this is equivalent to the statement of the theorem. Suppose that $\text{rank}(G_1 \langle \bar{a} \rangle | G_1) = \text{rank}(G_2 \langle \bar{b} \rangle | G_2) = r \leq n$ and without loss of generality suppose that a_1, \dots, a_r are independent over G_1 and so are b_1, \dots, b_r over G_2 (that $\text{rank}(G_1 \langle \bar{a} \rangle | G_1) = \text{rank}(G_2 \langle \bar{b} \rangle | G_2)$ easily follows from the assumption that \bar{a} and \bar{b} realise the same formulas of the form (*)).

Since $\text{rank}(G_1 \langle \bar{a} \rangle | G_1) = r$, there is a tuple \bar{c} of elements in G_1 such that $\text{rank}(\langle \bar{a}, \bar{c} \rangle | \langle \bar{c} \rangle) = r$. Note that $\langle \bar{a}, \bar{c} \rangle$ is the \tilde{T} -closure of $\{\bar{a}, \bar{c}\}$ in \tilde{M}_1 .

Consider the type $\Phi(\bar{y})$ in M_2 defined as follows.

$$\Phi(\bar{y}) = \{ \phi(\bar{b}, \bar{y}) \wedge U(\bar{y}) : \phi(\bar{x}, \bar{y}) \in \tilde{L} \cup \{\lambda\} \text{ and } M_1 \models \phi(\bar{a}, \bar{c}) \}.$$

As \bar{a} and \bar{b} realise the same formulas of the form (*) and M_2 is saturated, this type is satisfied in M_2 by a tuple $\bar{d} \in G_2$. One can easily check that then $\text{rank}(\langle \bar{b}, \bar{d} \rangle | \langle \bar{d} \rangle) = \text{rank}(\langle \bar{a}, \bar{c} \rangle | \langle \bar{c} \rangle) = r$.

As elementary pairs, $\langle \langle \bar{a}, \bar{c} \rangle, \langle \bar{c} \rangle \rangle$ is isomorphic to $\langle \langle \bar{b}, \bar{d} \rangle, \langle \bar{d} \rangle \rangle$, via say a map i and $\text{rank} \langle \bar{b}, \bar{d} \rangle | \langle \bar{d} \rangle = r$. In the sequel we will find an \mathbb{L} -structure isomorphism between two free \mathbb{L} -substructures of M_1 and M_2 which extends this isomorphism.

Take a positive $x \in \langle \bar{a}, \bar{c} \rangle$ with $\lambda_1(x) \notin \langle \bar{a}, \bar{c} \rangle$ (if there is no such x then the proof is clear). Let $y = i(x) \in \langle \bar{b}, \bar{d} \rangle$. We need to show that $\lambda_2(y)$ realises the same cut in $\langle \bar{b}, \bar{d} \rangle$ as does $\lambda_1(x)$ in $\langle \bar{a}, \bar{c} \rangle$, via i , and hence we have the isomorphism: $\langle \langle \bar{a}, \bar{c}, \lambda_1(x) \rangle, \langle \bar{c}, \lambda_1(x) \rangle \rangle \cong \langle \langle \bar{b}, \bar{d}, \lambda_2(y) \rangle, \langle \bar{d}, \lambda_2(y) \rangle \rangle$ between two elementary pairs.

Suppose that $\lambda_1(x) < t$ for some $t \in \langle \bar{a}, \bar{c} \rangle$. We can take t as $f(\bar{a}, \bar{c})$ for some definable function $f : M_1 \rightarrow M_1$ with no parameters. We can also write $x = g(\bar{a}, \bar{c})$ for some definable function g . Then the formula $\psi(\bar{z}) := \exists u \ U(u) \wedge [u = \lambda(g(\bar{z}, \bar{c}))] \wedge [u < f(\bar{z}, \bar{c})]$, is satisfied by $\bar{a} \in M_1$. Since by definition \bar{d} satisfies the type $\Psi(\bar{y})$, the corresponding formula $\exists u \ U(u) \wedge [u = \lambda(g(\bar{z}, \bar{d}))] \wedge [u < f(\bar{z}, \bar{d})]$ is satisfied by \bar{b} in M_2 which means $\lambda_2(y) < i(t)$. Consequently $\lambda_1(x)$ and $\lambda_2(y)$ satisfy the same cuts in $\langle \bar{a}, \bar{c} \rangle$ and $\langle \bar{b}, \bar{d} \rangle$ respectively and we have two isomorphic L_2 -structures $\langle \langle \bar{a}, \bar{c}, \lambda_1(x) \rangle, \langle \bar{c}, \lambda_1(x) \rangle \rangle$ and $\langle \langle \bar{b}, \bar{d}, \lambda_2(y) \rangle, \langle \bar{d}, \lambda_2(y) \rangle \rangle$.

Iterating the same argument, we obtain two isomorphic \mathbb{L} -structures, one a substructure of \mathbb{M}_1 containing \bar{a} , and the other, a substructure of \mathbb{M}_2 containing \bar{b} . Then by the back and forth argument, the isomorphism between these two implies that \mathbb{M}_1 and \mathbb{M}_2 are elementarily equivalent by which $tp_{\mathbb{M}_1}(\bar{a}) = tp_{\mathbb{M}_2}(\bar{b})$, and this finishes the proof. \square

4.4 Consequences of quantifier elimination, description of definable sets

The proof of quantifier elimination provides us with a way of characterising types of elements. The following theorem is a compact statement of what we will prove in this section.

Theorem 4.16. *Let \mathbb{M} be a common elementary substructure of $\mathbb{M}_1, \mathbb{M}_2 \models \mathbb{T}$. Then*

- I) *if $\bar{a}_1 \in A_1$ and $\bar{a}_2 \in A_2$ realise the same Pr -types over $\langle A, P_n, <, \cdot \rangle$ then they realise the same \mathbb{L} -types over M in \mathbb{M}_1 and \mathbb{M}_2 .*
- II) *If $\bar{g}_1 \in G_1$ and $\bar{g}_2 \in G_2$ realise the same $\tilde{L} \cup \{\lambda\}$ -types over M (over G) in \mathbb{M}_1 and \mathbb{M}_2 , then they realise the same \mathbb{L} -types over M (over G) in \mathbb{M}_1 and \mathbb{M}_2 .*
- III) *If $\bar{m}_1 \in M_1 - M\langle G_1 \rangle$ and $\bar{m}_2 \in M_2 - M\langle G_2 \rangle$ realise the same $\tilde{L} \cup \{\lambda\}$ -types in M , then they realise the same \mathbb{L} -types over M in \mathbb{M}_1 and \mathbb{M}_2 .*

- IV) If $\bar{g}_1 \in G_1$ and $\bar{g}_2 \in G_2$ realise the same \tilde{L} -types over M and for all f definable in $\tilde{L}(M)$, $\lambda_1(f(\bar{g}_1)) \in A$, then they realise the same \mathbb{L} -types over M in \mathbb{M}_1 and \mathbb{M}_2 .
- V) If $m_1 \in M_1 - M\langle G_1 \rangle$ and $m_2 \in M_2 - M\langle G_2 \rangle$ realise the same cut in M and for all f definable in $\tilde{L}(M)$, $\lambda_1(f(m_1)) \in A$, then m_1 and m_2 realise the same \mathbb{L} -types over M in \mathbb{M}_1 and \mathbb{M}_2 .
- VI) If $\bar{g}_1 \in G_1$ and $\bar{g}_2 \in G_2$ realise the same \tilde{L} -types over M in \mathbb{M}_1 and \mathbb{M}_2 respectively, then they realise the same $\tilde{L}(U)$ -types over M in \mathbb{M}_1 and \mathbb{M}_2 .
- VII) If $m \in M_1 - G$ and $m_2 \in M_2 - G$ realise the same cut in M , then they realise the same $\tilde{L}(U)$ -types over M in \mathbb{M}_1 and \mathbb{M}_2 .

We will treat each item in the above theorem as a separate theorem as we go through this chapter. Let us first consider the following direct corollary of it. This corollary is a strengthened version of the above theorem:

Corollary 4.17. *Let $\mathbb{M}_1, \mathbb{M}_2 \models \mathbb{T}$ and $\langle A, P_n, \cdot, \langle \rangle \rangle \subseteq \langle A_1, P_{1n}, \cdot, \langle \rangle \rangle$, $\langle A_2, P_{2n}, \cdot, \langle \rangle \rangle$, and $\langle \tilde{G}, A, \lambda, P_n \rangle \subseteq \langle \tilde{G}_1, A_1, \lambda_1, P_{1n} \rangle, \langle \tilde{G}_2, A_2, \lambda_2, P_{2n} \rangle$, then:*

- I) *If $a_1 \in A_1 - A$ and $a_2 \in A_2 - A$ realise the same cut in A and $P_{1n}(a_1) \leftrightarrow P_{2n}(a_2)$, then a_1 and a_2 realise the same \mathbb{L} -types over M in \mathbb{M}_1 and \mathbb{M}_2 .*
- II) *if $a_1 \in M_1 - G$ and $a_2 \in M_2 - G$ realise the same $\tilde{L} \cup \{\lambda\}$ -types over G , then they realise the same \mathbb{L} -types over G in \mathbb{M}_1 and \mathbb{M}_2 .*
- III) *if $x \in G_1 - G$ and $y \in G_2 - G$ realise the same $\tilde{L} \cup \{\lambda\}$ types over G then they realise the same types in \mathbb{M}_1 and \mathbb{M}_2 over G .*
- IV) *If $a_1 \in M_1 - G_1$ and $a_2 \in M_2 - G_2$ realise the same cut in G and $\lambda_1(f(\bar{a})) \in A$ for all $\tilde{L}(G)$ -definable functions f , then a_1 and a_2 realise the same \mathbb{L} -types over G in \mathbb{M}_1 and \mathbb{M}_2 .*

In the following theorem, we prove Part II) of Theorem 4.16

Theorem 4.18. *Let \mathbb{M}_1 and \mathbb{M}_2 be two models of \mathbb{T} in the language \mathbb{L} , and $\mathbb{M} = \langle \tilde{M}, G, A, \lambda, \{P_n\}_{n \in \mathbb{N}} \rangle$ a common free substructure of them. Then if $\bar{a} \in G_1$ and \bar{b} in G_2 realise the same $\tilde{L} \cup \{\lambda\}$ -type over M in \mathbb{M}_1 and \mathbb{M}_2 , then they realise the same types over M in \mathbb{M}_1 and \mathbb{M}_2 respectively.*

Proof. We assume that \mathbb{M}_1 and \mathbb{M}_2 are κ -saturated, for κ a large cardinal (as in the proof of Theorem 4.14) and $|M| < \kappa$. Consider the two isomorphic $\tilde{L} \cup \{U\}$ -structures $\langle M\langle\bar{a}\rangle, G\langle\bar{a}\rangle \rangle \stackrel{f}{\cong} \langle M\langle\bar{b}\rangle, G\langle\bar{b}\rangle \rangle$, with f the isomorphism map between them. Let $x \in M\langle\bar{a}\rangle$ be such that $\lambda_1(x) \notin M\langle\bar{a}\rangle$. Then as \bar{a} and \bar{b} realise the same $\tilde{L} \cup \{\lambda\}$ -types over M , $\lambda_2(f(x))$ realises the same cut in $M\langle\bar{b}\rangle$ as $\lambda_1(x)$ does in $M\langle\bar{a}\rangle$ via f . Consider two \tilde{L} -isomorphic structures $M\langle\bar{a}, \Lambda_1\rangle$ and $M\langle\bar{b}, \Lambda'_1\rangle$ where Λ_1 contains $\lambda_1(x)$ for all $x \in M\langle\bar{a}\rangle$ and Λ'_1 is the isomorphic image of Λ_1 in M_2 , i.e $\Lambda' = \{\lambda_2(f(x)), x \in M\langle\bar{a}\rangle\}$. Let $M\langle\bar{a}, \Lambda_1, \dots, \Lambda_{n+1}\rangle$ be obtained by adding $\lambda_1(x)$ for all $x \in M\langle\bar{a}, \Lambda_1, \dots, \Lambda_n\rangle$. Let $M\langle\bar{b}, \Lambda'_1, \dots, \Lambda'_{n+1}\rangle$ be the isomorphic image of it in M_2 . Then $\bigcup_{n \in \omega} M\langle\bar{a}, \Lambda_1, \dots, \Lambda_n\rangle$ is closed under λ_1 and isomorphic to $\bigcup_{n \in \omega} M\langle\bar{b}, \Lambda'_1, \dots, \Lambda'_n\rangle$. Let $M_e = \bigcup_{n \in \omega} M\langle\bar{a}, \Lambda_1, \dots, \Lambda_n\rangle$ and $G_e = \bigcup_{n \in \omega} G\langle\bar{a}, \Lambda'_1, \dots, \Lambda'_n\rangle$. Let $M'_e = \bigcup_{n \in \omega} M\langle\bar{b}, \Lambda'_1, \dots, \Lambda'_n\rangle$ and $G'_e = \bigcup_{n \in \omega} G\langle\bar{b}, \Lambda'_1, \dots, \Lambda'_n\rangle$. Then \mathbb{M}_e , the structure $\langle \tilde{M}_e, G_e, \lambda_1(M_e), \lambda_1|_{M_e}, P_{1n}(M_e) \rangle$ is isomorphic to the similar structure \mathbb{M}'_e . This isomorphism as in the proof of Theorem 4.14 starts a back and forth argument to prove that \mathbb{M}_1 and \mathbb{M}_2 are elementarily equivalent and this proves the statement of the theorem. \square

The above theorem leads to the following two corollaries on definable subsets of the dense substructure.

Corollary 4.19 (definable subsets of G^n). *Let $\mathbb{M} = \langle \tilde{M}, G, A, \lambda, \{P_n\}_{n \in \mathbb{N}} \rangle$ be a model of \mathbb{T} and $Y \subseteq G^n$ be definable in \mathbb{M} . Then $Y = Z \cap G^n$ for some $Z \subseteq M^n$ definable by a formula in $\tilde{L} \cup \{\lambda\}$.*

Proof. Let $\phi(\bar{y})$ be the formula with parameters in M that defines Y . We will prove that there is an $\tilde{L} \cup \{\lambda\}$ -formula $\psi(\bar{y})$ such that $\mathbb{M} \models \forall \bar{x} (U(\bar{x}) \rightarrow (\phi(\bar{x}) \leftrightarrow \psi(\bar{x})))$. This is equivalent to the following: if \mathbb{M}_1 and \mathbb{M}_2 are two elementary extensions of \mathbb{M} and $\bar{a} \in G_1$ and $\bar{b} \in G_2$ realise the same $\tilde{L} \cup \{\lambda\}$ -type over M , then $\mathbb{M}_1 \models \phi(\bar{a})$ if and only if $\mathbb{M}_2 \models \phi(\bar{b})$. This is immediate from the previous lemma. \square

In the above corollary if Y is defined with parameters in G , then in $\langle \tilde{G}, A \rangle$ it is defined with an $\tilde{L} \cup \{\lambda\}$ -formula with parameters. This is explained in the following corollary.

Corollary 4.20. *Let $Y \subseteq G^n$ be definable in \mathbb{M} with parameters in G . Then in $\langle \tilde{G}, A \rangle$, Y is defined with an $\tilde{L} \cup \{\lambda\}$ -formula. Also if $Y \subseteq M^n$ is defined with parameters in G , then $Y \cap G^n$ is defined in $\langle \tilde{G}, A \rangle$ with an $\tilde{L} \cup \{\lambda\}$ -formula.*

Proof. By quantifier elimination we may suppose that Y is defined by a formula of the form $\exists \bar{x} \in G \ \phi(\bar{x}, \bar{y})$ for $\phi(\bar{x}, \bar{y})$ an $\tilde{L} \cup \{\lambda\}$ formula with parameters in G . So, in $\langle \tilde{G}, A \rangle$, Y is defined with the formula $\exists \bar{x} \ \phi(\bar{x}, \bar{y})$. \square

Here is a simple and yet useful observation. Let $Y \subseteq M^n$ be defined by a formula of the form $\exists \bar{x} \in G \ \phi(\bar{x}, \bar{y})$ where $\phi(\bar{x}, \bar{y})$ is an $\tilde{L} \cup \{\lambda\}$ -formula. Let $O_{\bar{y}}$ be the set $\{\bar{x} : \langle \tilde{M}, A \rangle \models \phi(\bar{x}, \bar{y})\}$. For a fixed \bar{y} , If $O_{\bar{y}}$ has interior then since G is dense in M , we have:

$$\langle \tilde{M}, G, A \rangle \models (\exists \bar{x} \in G \ \phi(\bar{x}, \bar{y})) \leftrightarrow \exists \bar{x} \ \phi(\bar{x}, \bar{y}).$$

Let Y' be the set $\{\bar{y} : O_{\bar{y}} \text{ has interior}\}$. Then Y' is obviously a subset Y .

Note that if in the above $Y \subseteq G$ and $\phi(\bar{x}, \bar{y})$ is a quantifier free $\tilde{L} \cup \{\lambda, P_n\}$ -formula with parameters in G , then

$$\langle \tilde{M}, A \rangle \models \exists \bar{x} \ \phi(\bar{x}, \bar{y}) \Leftrightarrow [\langle \tilde{G}, A \rangle \models \exists \bar{x} \ \phi(\bar{x}, \bar{y})] \Leftrightarrow [\mathbb{M} \models \exists \bar{x} \in G \ \phi(\bar{x}, \bar{y})],$$

that is

$$[\mathbb{M} \models \exists \bar{x} \in G \ \phi(\bar{x}, \bar{y})] \Leftrightarrow [\mathbb{M} \models \exists \bar{x} \ \phi(\bar{x}, \bar{y})]$$

which means that in this case, Y is $\tilde{L} \cup \{\lambda\}$ -definable.

Corollary 4.21. *If $Y \subseteq G$ is definable in \mathbb{M} , a model of \mathbb{T} , then Y is the union of an open subset of G (with order topology in G) and finitely many definable discrete subsets of G . The open set and discrete sets are defined with parameters in M in the language $\tilde{L} \cup \{\lambda\}$.*

Proof. Y is equal to $Z \cap G$ for a set Z defined by an $\tilde{L} \cup \{\lambda\}$ formula. By d-minimality of $\tilde{T}_{discrete}$, Z is the union of an open set and finitely many discrete sets and the result follows. \square

By the above theorem, \mathbb{Q} and hence \mathbb{Z} are not definable in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$.

In the next theorem we consider the type of a tuple in A , when $\langle \tilde{M}, G, A, \lambda, P_n \rangle$ is a model of \mathbb{T} . As noted before, $\langle A, P_n, \cdot, < \rangle$ is a model of Presburger arithmetic. In the following and in the rest, by Pr we mean Presburger arithmetic, and by Pr-type of a tuple in A we mean the Pr-formulas it satisfies.

Theorem 4.22. *Let \mathbb{M} be a model of \mathbb{T} and \mathbb{M}_1 and \mathbb{M}_2 be two sufficiently saturated models of \mathbb{T} extending \mathbb{M} and free over it. If $a_1 \in A_1$ and $a_2 \in A_2$ realise the same*

quantifier free Pr-types over $\langle A, \cdot, <, \{P_n\} \rangle$ in $\langle A_1, \cdot, <, \{P_{1n}\} \rangle$ and $\langle A_2, \cdot, <, \{P_{2n}\} \rangle$, then they realise the same types in \mathbb{M}_1 and \mathbb{M}_2 over M .

Sketch of proof. It is easy to check that a, b realise the same cuts in M . As M and G are real closed fields and by similar discussions and notation to the case one in the proof of Theorem 4.14, the structure $\mathbb{M}\langle a \rangle$ is isomorphic to $\mathbb{M}\langle b \rangle$ and back and forth starts to prove that \mathbb{M}_1 and \mathbb{M}_2 are elementarily equivalent, and hence a and b realise the same types over M in \mathbb{M}_1 and \mathbb{M}_2 . \square

Notice that the proof of the above theorem—with more care—is also the proof of the part I) of theorem.4.16.

Now let \mathbb{M}_1 and \mathbb{M}_2 be extensions of \mathbb{M} . Let $a \in A_1$ and $b \in A_2$ realise the same Pr-types over A . Then the field $\langle A, a \rangle$ (the universe of the \tilde{T} -closure of $A \cup \{a\}$), has representatives for all its Archimedean classes in $A\langle a \rangle$, the divisible hull of the abelian group generated by a over A . Also $\mathbb{A}\langle a \rangle := \langle \langle A, a \rangle, \langle A, a \rangle, A\langle a \rangle \rangle$ is isomorphic to $\mathbb{A}\langle b \rangle$ and they are free substructures of \mathbb{M}_1 and \mathbb{M}_2 respectively. Now by similar back and forth discussions, this means that a and b realise the same types over A in \mathbb{M}_1 and \mathbb{M}_2 . We have proved the following.

Theorem 4.23. *Let \mathbb{M}_1 and \mathbb{M}_2 be two models of \mathbb{T} and $\langle A, \cdot, P_n, < \rangle$ a common substructure of $\langle A_1, \cdot, P_{1n}, < \rangle$ and $\langle A_2, \cdot, P_{2n}, < \rangle$. If $a \in A_1$ and $b \in A_2$ realise the same Pr-types over A in \mathbb{M}_1 and \mathbb{M}_2 , then they realise the same types in \mathbb{M}_1 and \mathbb{M}_2 over $\langle A \rangle$.*

Corollary 4.24 (definable subsets of A). *If a subset of A is definable in \mathbb{M} with parameters in A , then it is definable in $\langle A, \cdot, \{P_n\}_{n \in \mathbb{N}} \rangle$.*

Proof. Let $Y \subseteq A$ be definable by a formula $\psi(y)$ in \mathbb{M} . Then by the above theorem there is a Pr-formula $\phi(y)$ with parameters in A such that $\mathbb{M} \models A(y) \rightarrow (\psi(y) \leftrightarrow \phi(y))$. So in A , Y is defined by the formula $\phi(y)$. \square

Theorem 4.25. *Let \mathbb{M}_1 and \mathbb{M}_2 be two models of \mathbb{T} . Let $\langle \tilde{G}, A \rangle \subseteq \langle \tilde{G}_1, A_1 \rangle, \langle \tilde{G}_2, A_2 \rangle$. Then if $x \in M_1 - G_1$ and $y \in M_2 - G_2$ realise the same $\tilde{L} \cup \{\lambda\}$ -types over G in \mathbb{M}_1 and \mathbb{M}_2 , then they realise the same types over G in \mathbb{M}_1 and \mathbb{M}_2 . If $x \in A_1 - G$ and $y \in A_2 - G$ realise the same Pr-types over A in $\langle A_1, <, \cdot, P_{1n} \rangle$ and $\langle A_2, <, \cdot, P_{2n} \rangle$, then they realise the same types in \mathbb{M}_1 and \mathbb{M}_2 over G . Finally if $x \in G_1 - G$ and*

$y \in G_2 - G$ realise the same $\tilde{L} \cup \{\lambda\}$ types over G then they realise the same types in \mathbb{M}_1 and \mathbb{M}_2 over G .

Proof. We assume that $\mathbb{M}_1, \mathbb{M}_2$ are sufficiently saturated. If $x \in M_1 - G_1$ and $y \in M_2 - G_2$ realise the same cuts in G , then we have two isomorphic elementary pairs $\langle \tilde{G}\langle x \rangle, G \rangle$ and $\langle \tilde{G}\langle y \rangle, G \rangle$. Let Λ be a sequence of elements in $A_1 - G$ such that $G\langle x, \Lambda \rangle$ is closed under λ_1 and $\mathbb{G}\langle x, \Lambda \rangle := \langle G\langle x, \Lambda \rangle, G\langle \Lambda \rangle, \lambda_1|_{G\langle x, \Lambda \rangle}, A\langle \Lambda \rangle, P_n\langle \Lambda \rangle \rangle$ is an \mathbb{L} -structure. Let Λ' be the corresponding sequence of elements in $A_2 - G$ whose elements are chosen, as in the proof of quantifier elimination, in such a way that $\mathbb{G}\langle x, \Lambda \rangle$ is isomorphic to $\mathbb{G}\langle y, \Lambda' \rangle$. Now by the back and forth argument on the class of isomorphic free substructures of \mathbb{M}_1 and \mathbb{M}_2 , as in the proof of Theorem 4.14, we are led to the conclusion that x and y realise the same types over G in \mathbb{M}_1 and \mathbb{M}_2 .

If $x \in A_1 - A$ and $y \in A_2 - A$ realise the same Pr-types over $\langle A, <, \cdot, \cdot, \{P_n\} \rangle$ in $\langle A_1, <, \cdot, \cdot, P_{1n} \rangle$ and $\langle A_2, <, \cdot, \cdot, P_{2n} \rangle$ then they realise the same cuts in G (If $g \in G$ and $x < g \neq \lambda(g)$, then $x < \lambda(g)$, since otherwise $\lambda_1(g) = x$; now as $\lambda(g) \in A_2$, we have $y < \lambda(g) < g$). Now the structure $\mathbb{G}\langle x \rangle := \langle G\langle x \rangle, G\langle x \rangle, \lambda_1|_{G\langle x \rangle}, A\langle x \rangle, P_n\langle x \rangle \rangle$ is isomorphic to a similar structure $\mathbb{G}\langle y \rangle$ and this starts the back and forth.

The argument for $x \in G_1 - G$ and $y \in G_2 - G$ is a combination of the previous paragraph with the proof of Theorem 4.18. \square

Although we did the argument for single elements, the above theorem holds for tuples.

Remark 4.25.1. *In Theorem 4.25, x and y can be tuples of elements (with the same length).*

In chapter 2 we saw the definition of d-minimality and its equivalents. A model of our theory, say \mathbb{M} , is simply not d-minimal as G itself is a definable set. So, it is also not ‘o-minimalistic’ even though our models satisfy ‘definable completeness’. Let us first see the definitions of the terms we just mentioned.

The following definition and properties are due to Schoutens in [29]. An ordered structure is called o-minimalistic if it has all the first order properties of an o-minimal structure. Equivalently an ordered structure is o-minimalistic if it is an elementary substructure of an ultraproduct of o-minimal structures. In particular, o-minimalistic

structures satisfy definable completeness and type completeness. Definable completeness is the property of an ordered structure that every definable subset of it has infimum which can be $-\infty$. Type completeness is the two following properties of an ordered structure. First, given a definable set Y and a point x , there exists $y < x$ such that either $(y, x) \subseteq Y$ or (y, x) is disjoint from Y ; second, there is a y such that $(-\infty, y)$ is either contained in Y or disjoint from it.

Every definable discrete subset of an o-minimalistic structure is closed and bounded. Every definable subset of an o-minimalistic structure is the disjoint union of a closed bounded discrete set and (possibly infinitely many) disjoint intervals, so o-minimalistic structures are d-minimal). An o-minimalistic ordered pure field is o-minimal.

As mentioned before, the theory \mathbb{T} is obviously not o-minimal, d-minimal or o-minimalistic; but, in the sequel, we will see that if $\mathbb{M} \models \mathbb{T}$ then definable subsets of M do have a similar pattern.

Lemma 4.26. *Every model of \mathbb{T} satisfies DC, definable completeness, i.e. every definable subset of a model of \mathbb{T} has an infimum which can be $-\infty$.*

Proof. Since \mathbb{T} is complete, $\mathbb{T} = \text{Th}(\mathbb{M})$ where M is an expansion of \mathbb{R} and a model of \tilde{T} . The result is clear since every definable subset of M satisfies this property. \square

Models of \mathbb{T} obviously do not satisfy TC , type completeness. Let $\tilde{T} = RCF$ then such a model will satisfy $DCTC$ and any definable discrete subset of it must have a maximum [29]. It is not possible, since $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ does not satisfy it and \mathbb{T} is complete.

The following is easy to check and is ‘proved in [7]. It simply says that freeness is guaranteed for elementary extensions.

Lemma 4.27. *If $\mathbb{M}_1 \preceq \mathbb{M}_2$ are models of \mathbb{T} then \mathbb{M}_2 is free over \mathbb{M}_1 .*

Proof. See [7], Lemma 2.3. \square

By the following lemma, if \mathbb{M}_1 and \mathbb{M}_2 are models of \mathbb{T} , then one is a free extension of the other if and only if it is an elementary extension of it.

Lemma 4.28. *Let $\mathbb{M}_1 \subseteq \mathbb{M}_2$ be models of \mathbb{T} and \mathbb{M}_2 be free over \mathbb{M}_1 . Then $\mathbb{M}_1 \preceq \mathbb{M}_2$.*

Proof. Let \mathbb{M}_3 and \mathbb{M}_4 be $|M_1|^+$ -saturated models of \mathbb{T} such that $\mathbb{M}_1 \subseteq \mathbb{M}_3$ and $\mathbb{M}_2 \subseteq \mathbb{M}_4$ and \mathbb{M}_4 is an elementary extension of \mathbb{M}_2 and \mathbb{M}_3 is an elementary extension of \mathbb{M}_1 .

$$\begin{array}{ccc} \mathbb{M}_3 & & \mathbb{M}_4 \\ \uparrow & & \uparrow \\ \mathbb{M}_1 & \rightarrow & \mathbb{M}_2 \end{array}$$

Consider \mathbb{M}_1 as a common free substructure of \mathbb{M}_3 and \mathbb{M}_4 .

$$\begin{array}{ccc} \mathbb{M}_3 & & \mathbb{M}_4 \\ & \swarrow \quad \searrow & \\ & \mathbb{M}_1 & \end{array}$$

By the proof of the main theorem, each tuple \bar{a} in M_1 realises the same types over M_1 in \mathbb{M}_3 and \mathbb{M}_4 . So for each formula $\phi(\bar{x})$, if $\mathbb{M}_1 \models \phi(\bar{a})$ then $\mathbb{M}_3, \mathbb{M}_4 \models \phi(\bar{a})$ and hence $\mathbb{M}_2 \models \phi(\bar{a})$. \square

A subset of G^n which is definable in \mathbb{M} is defined in the expansion of G by traces of M in G . We will prove this for the subsets of G and the proof for the subsets of G^n is similar to this with slight modifications.

Lemma 4.29. *If $Y \subseteq G$ is definable in \mathbb{M} , then it is definable in the structure $\langle \tilde{G}, A, G \cap (0, b) : b \in M \rangle$.*

Proof. By lemma 4.19, $Y = Z \cap G$ where Z is defined in $\tilde{L} \cup \{\lambda\}$. Z is then defined without quantifiers in the language $\tilde{L} \cup \{\lambda\} \cup \{P_n, n \in \mathbb{N}\}$ and is a finite union of sets defined by formulas of the form:

$$\begin{aligned} & [P_{n_1}(h_1(x)) \wedge \dots \wedge P_{n_k}(h_k(x))] \wedge \\ & [g_1(x) > 0 \wedge \dots \wedge g_m(x) > 0] \wedge \\ & [l_1(x) = 0 \wedge \dots \wedge l_q(x) = 0] \quad (4.2) \end{aligned}$$

where h_i and g_i and l_i are $\tilde{L} \cup \{\lambda\}$ -terms with parameters in M . Suppose that $\lambda(f(x, b))$ has appeared in this formula for f an \tilde{L} -term where b is a tuple (b_1, \dots, b_n) which is for simplicity denoted without a bar. First for a tuple $b = (b_1, \dots, b_n)$, an element c

and an \tilde{L} -term f , we introduce the following formula $\Phi_f(x, b, c)$:

$$\begin{aligned} \forall \epsilon > 0 (\epsilon \in G \rightarrow \exists t_1, \dots, t_n \in G [\bigwedge_{i=1}^n (t_i < b_i) \wedge \\ \forall t'_1, \dots, t'_n \in G (\bigwedge_{i=1}^n (t'_i \in (t_i, b_i) \rightarrow \\ |c - f(x, t'_1, \dots, t'_n)| < \epsilon))]] \end{aligned}$$

To get the desired result, we need to get rid of all occurrences of $f(x, b)$ by using a formula of the above form. This can be done in the following way. To deal with $\lambda(f(x, b))$ we set

$$t = \lambda(f(x, b)) \leftrightarrow R(x) \vee S(x)$$

where we have defined:

$$R(x) := \exists c \in A \quad \Phi_f(x, b, c) \wedge (t = c)$$

and

$$\begin{aligned} S(x) := [\forall c \in A \quad \neg \Phi_f(x, b, c) \wedge [\exists e \in A (\\ \forall e' (\Phi_f(x, b, e') \rightarrow e < e' < we) \wedge \\ (t = e))]] \end{aligned}$$

Now in the formula 4.2 we first get rid of terms of the form $\lambda(f(x, b))$ in the way just described and replace these terms with new variables and treat the obtained formula in the same way.

As an example, we can deal with the formula $P_n(\lambda(f(x, b)))$ as follows

$$P_n(\lambda(f(x, b))) \leftrightarrow (R_1(x) \vee S_1(x))$$

where

$$R_1(x) := (\exists c \in A \quad \Phi_f(x, b, c)) \wedge P_n(c)$$

and

$$\begin{aligned} S_1(x) := (\forall c \in A \quad \neg \Phi_f(x, b, c)) \wedge \\ \exists t < b \in G \quad \forall t' \in G \quad (t' \in (t, b) \rightarrow P_n(\lambda(f(y, t')))) \end{aligned}$$

□

As mentioned before the lemma, the above lemma holds for a definable subset of G^n , $n \in \mathbb{N}$ with a similar proof involving tuples of variables instead of a single variable.

Theorem 4.30. *Let $\mathbb{M} = \langle \tilde{M}, G, A, \lambda, P_n \rangle$ be a model of \mathbb{T} . Then G is definably closed in \mathbb{M} .*

Proof. Let $b \in M - G$. We need to show that b is not definable with parameters in G . Let $\mathbb{M}_1 = \langle \tilde{M}_1, G_1, A_1, \lambda_1, P_{1n} \rangle$ be a saturated model of \mathbb{T} which is a free extension of \mathbb{M} . Let $b_1 \in M_1 - G_1$ be an element which satisfies the same cut over G as does b and $b_1 \neq b$. We claim that b_1 satisfies the same type over G as does b . Clearly $\lambda(b_1) = \lambda(b) \in G$. Since for each $x \in M$ (where $M \supseteq G\langle b \rangle$), $\lambda(x) \in G$, the structure $\langle G\langle b \rangle, G, \lambda(G), \lambda|_{G\langle b \rangle}, P_n\langle G \rangle \rangle$ is a model of \mathbb{T} which, as we will prove, is isomorphic to $\langle G\langle b_1 \rangle, G, \lambda_1(G), \lambda_1|_{G\langle b_1 \rangle}, P_{1n}\langle G \rangle \rangle$. Note that since b_1 realises the same cut in G as b , for each \tilde{L} -definable function f with parameters in G , $f(b_1)$ and $f(b)$ realise the same cuts in G , and $\lambda_1(f(b_1)) = \lambda(f(b))$, that is $G\langle b_1 \rangle$ is also closed under λ_1 and for each $x \in G\langle b_1 \rangle$, $\lambda_1(x) \in G$. This isomorphism, as in the proof of quantifier elimination, implies that b_1 and b satisfy the same types over G . From $b_1 \neq b$, we get that b is not definable with parameters in G . \square

The following is what we proved alongside the main statement in the above theorem.

Corollary of the proof (of Theorem 4.30). *Let $\mathbb{M} \models \mathbb{T}$ and \mathbb{M}_1 be a free of \mathbb{M} . Then if $b_1 \in M_1 - G_1$ realises the same cut in G as $b \in M - G$ then b and b_1 realise the same types over G in \mathbb{M}_1 .*

The above Corollary is specially interesting and not expected (at least for me) as it determines the whole type of an element by its cut and not its $\tilde{L} \cup \{\lambda\}$ -type. The following corollary generalises this idea.

Corollary of the proof (of Theorem 4.30). *Let $\mathbb{M} \subseteq \mathbb{M}_1, \mathbb{M}_2$ and \mathbb{M}_1 and \mathbb{M}_2 be free extensions of \mathbb{M} . Let $a_1 \in M_1$ and a_2 in M_2 realise the same cuts over M and be such that for all $t \in M\langle a_1 \rangle$, $\lambda_1(t) \in G$ and for all $t \in M\langle a_2 \rangle$, $\lambda_2(t) \in G$. Then a_1 and a_2 realise the same types over M in \mathbb{M}_1 and \mathbb{M}_2 .*

Proof. A consequence of the proof of quantifier elimination and the previous Corollary of the proof. \square

We noted before that in $\mathbb{M} = \langle \tilde{M}, G, \lambda, A, P_n \rangle$ a model of \mathbb{T} , the structure $\langle \tilde{M}, A \rangle$ is a model of $\tilde{T}_{discrete}$. Every subset of M definable in $\langle \tilde{M}, A \rangle$ is, by d-minimality, the union of an open definable set and finitely many discrete definable sets. We now focus our attention on the discrete and open subsets of M which are defined in \mathbb{M} . Let us start simply with a discrete definable set which is defined with parameters in G .

Corollary 4.31. *Let $\mathbb{M} \models \mathbb{T}$ and D be a discrete set defined with parameters in G . Then $D \subseteq G$.*

Proof. Let $x \in D$. Then since D is discrete and G is dense in M , there are $a, b \in G$ such that $x \in (a, b)$ and x is the only point in this interval. That is the singleton $\{x\}$ is definable with parameters in G and as G is definably closed, $x \in G$. \square

The above corollary holds also for finite unions of discrete sets. Before stating this, we need the following.

Let X be a topological space and $A \subseteq X$. As in [23], for each ordinal α we define the set $A^{[\alpha]}$ as follows:

$$\begin{aligned} A^{[0]} &= A \\ A^{[\alpha+1]} &= A^{[\alpha]} - \text{Isol}(A^{[\alpha]}) \\ A^{[\alpha]} &= A - \bigcup_{\mu < \alpha} \text{Isol}(A^{[\mu]}) \end{aligned}$$

In above the symbol Isol indicates the set of isolated points of a set. If the set A is the union of finitely many discrete sets then there exists $n \in \mathbb{N}$ such that $A^{[n]} = \emptyset$. Moreover, if A the union of an open set and finitely many discrete sets then there exists $n \in \mathbb{N}$ such that $(A - \text{Int}(A))^{[n]} = \emptyset$. ([23], 2.3).

The following Lemma generalises the previous corollary. We will use this lemma in Chapter 6.

Lemma 4.32. *Let S be a set definable with parameters in G which is a finite union of discrete sets (not necessarily each of which definable with parameters in G). Then $S \subseteq G$.*

Proof. The set $\text{Isol}(S)$ of isolated points of S is definable with parameters in G and discrete and by the previous lemma a subset of G . According to the lines before this lemma, there exists $n \in \mathbb{N}$ such that $S^{[n]} = \emptyset$. Since each $S^{[i]}$ and the set of its isolated points is definable and by the previous lemma, $S \subseteq G$. \square

In Theorem 4.30 we showed that G is definably closed in \mathbb{M} when $\mathbb{M} \models \mathbb{T}$. The following theorem says that the universe of an elementary substructure of $\langle \tilde{G}, A \rangle$ is also definably closed.

Theorem 4.33. *Let $\mathbb{M} \models \mathbb{T}$ and $\langle \tilde{G}_0, A_0, \lambda_0, P_{0n} \rangle \subseteq \langle \tilde{G}, A, \lambda, P_n \rangle$. Then G_0 is definably closed in \mathbb{M} .*

Proof. By the previous theorem if $b \in M$ is definable with parameters in $G_0 \subseteq G$ then it is in G . On the other hand, by the quantifier elimination of \mathbb{T} (the proof of Theorem 4.15 and corollary 4.19), we can assume that b is defined over G by an $\tilde{L} \cup \{\lambda\}$ -formula of the form $\exists \bar{y} \phi(x, \bar{y})$ with parameters in G_0 . Now, by the quantifier elimination of $\text{Th}(\tilde{\mathbb{R}}, w^{\mathbb{Z}}, \lambda, P_n)$, since G_0 is an elementary substructure of G , we have $b \in G_0$. \square

In the above theorem we proved that the universe of a substructure of G is definably closed. In the following theorem we will prove that if \mathbb{M}_1 is a model of \mathbb{T} , then the universe of any of its free substructures is definably closed in \mathbb{M}_1 .

Theorem 4.34. *Let $\mathbb{M}_1 = \langle \tilde{M}_1, G_1, \lambda_1, A_1, P_{1n} \rangle \models \mathbb{T}$ be a free extension of \mathbb{M} , where \mathbb{M} is not necessarily a model. Then the set M is definably closed in \mathbb{M}_1 .*

Proof. Let x be in the definable closure of M . We first claim that x is an element of $M\langle G_1 \rangle$. This is the case, because as dense pairs, $\langle \tilde{M}\langle G_1 \rangle, G_1 \rangle$ is an extension of $\langle \tilde{M}, G \rangle$ and a free substructure of $\langle \tilde{M}_1, G_1 \rangle$. Clearly $M\langle G_1 \rangle$ is closed under λ_1 and $\langle \tilde{M}\langle G_1 \rangle, G_1, \lambda_1|_{M\langle G_1 \rangle}, A_1, P_{1n} \rangle$ is a model of \mathbb{T} and hence an elementary substructure of \mathbb{M}_1 . The rest of the proof is similar to the proof of van Den Dries for a similar result about dense pairs, but it is worth rewriting to preserve the shape of the thesis.

So there is a minimal n and elements $\bar{a} := a_1, \dots, a_n$ in G_1 such that $x \in M\langle a_1, \dots, a_n \rangle$ and hence there exists an \tilde{L} -definable function $f : M_1^n \rightarrow M_1$ with parameters in M , such that $f(\bar{a}) = x$. By minimality of n , these a_i 's are independent over M . We will prove that n is indeed zero. We also assume that \mathbb{M}_1 is a saturated model of \mathbb{T} .

M_1^n can be decomposed into \tilde{M} -cells D_1, \dots, D_m , such that if D_i is an open cell, then f is strictly increasing or strictly decreasing or constant with respect to its n 'th variable in D_i . Suppose, with no loss of generality that \bar{a} belongs to D_1 and D_1 is an open cell. Now consider each of these cases:

First case suppose that f is increasing in its last variable in D_1 . In a similar way to the proof of Lemma 4.12, there is a sequence $(b_i)_{i \in \mathbb{N}}$ of elements of $A_1 \subseteq G_1$ such

that $M\langle a_1, \dots, a_k, b_1, \dots \rangle$ is closed under λ_1 . Let $M' = M\langle a_1, \dots, a_{k-1}, b_1, \dots \rangle$ and $G' = G\langle a_1, \dots, a_{k-1}, b_1, \dots \rangle$. Let $\mathbb{M}' = \langle \tilde{M}', G', \lambda', A', P'_n \rangle$ for λ', A' and P'_n the corresponding \mathbb{L} -constituents of \mathbb{M}' . Now we have $\mathbb{M} \subseteq \mathbb{M}' \subseteq \mathbb{M}_1$. Let $s \neq a_n$ be an element of G_1 which realises the same cut over M' as does a_n . Then $s \in D_1$ and $f(a_1, \dots, a_n) = x = f(a_1, \dots, a_{k-1}, s)$, this is because x is definable with parameters in M and a_n and s realise the same type over M' . This is a contradiction to f being strictly increasing.

Second case if f is constant in the last variable then with a similar argument to above, the minimality of n is contradicted. □

Theorems 4.30 and 4.34 are extremely helpful in characterising definable functions in a model of \mathbb{T} . How this is the case is what we discuss in the next section.

Corollary 4.35. *Let X be a discrete definable set which is a subset of G^c , and is defined with parameters $\bar{a} \notin G$. Then $X \subseteq G\langle \bar{a} \rangle$.*

Proof. Let $\phi(x, \bar{a})$ define X . Then $\langle \tilde{G}\langle \bar{a} \rangle, G, A, \dots \rangle \subseteq \langle \tilde{M}, G, A, \dots \rangle$. $G\langle \bar{a} \rangle$ is dense in M . So, every element $x \in X$ is definable with parameters in $G\langle \bar{a} \rangle$. As $G\langle \bar{a} \rangle$ is definably closed in M , we have $x \in G\langle \bar{a} \rangle$. □

Corollary 4.36. *With the assumptions of the above corollary, $X = G\langle \bar{a} \rangle \cap Y$ for some Y defined with an $\tilde{L} \cup \{\lambda\}$ -formula.*

Proof. Consider the structure $\langle \tilde{G}\langle \bar{a} \rangle, G, A, \dots \rangle$. We need to prove that if \mathbb{M}_1 and \mathbb{M}_2 are elementary extensions of this structure, then if $x \in M_1 - G_1$ and $y \in M_2 - G_2$ realise the same $\tilde{L} \cup \{\lambda\}$ -types over $G\langle \bar{a} \rangle$, then they realise the same types.

Considering the fact that since $x, y \notin G_1, G_2$ and by the previous corollary, $x \in G_1\langle \bar{a} \rangle$ and $y \in G_2\langle \bar{a} \rangle$, we have $x, y \in \langle \bar{a} \rangle$. Now since x, y realise the same $\tilde{L} \cup \{\lambda\}$ -types over $G\langle \bar{a} \rangle$ and by the back and forth argument, the result follows. □

4.5 Definable functions and definable sets

To know our models better, the next natural thing to do is to study—in a model \mathbb{M} of \mathbb{T} — the definable functions F from M to M , from G to G and from A to A . In a

dense pair of o-minimal structures, definable functions are almost well-behaving. For $\langle \tilde{M}, G \rangle$ a dense pair of models of \tilde{T} and F a definable function from M to M , there is an \tilde{M} -definable function (an \tilde{L} -definable function with parameters in M) which is almost equal to F . Let us first clarify what this precisely means. Note that, in the next paragraph we are considering a dense pair as in [7] and rephrasing the proof of a theorem already existing there.

In [7] (with notation changed according to our setting) a definable subset X of M is called G -small if $X \subseteq f(G^n)$ for some \tilde{M} -definable function $f : M^n \rightarrow M$ for some $n \in \mathbb{N}$. Now, Let $\langle \tilde{M}^*, G^* \rangle$ be a κ -saturated elementary extension of $\langle \tilde{M}, G \rangle$ for $\kappa > |M|, |\tilde{T}|$ where \tilde{T} is our o-minimal theory M and G are whose models. In $\langle \tilde{M}^*, G^* \rangle$, the equivalent of x not being in any G -small set is $x \notin M\langle G^* \rangle$. For such an x , $\langle \tilde{M}\langle x \rangle, G \rangle$ is an elementary pair and hence an elementary substructure of $\langle \tilde{M}^*, G^* \rangle$. Since $M\langle x \rangle$ is definably closed, we have $F(x) \in M\langle x \rangle$. So we have the following implication in \tilde{M}^* :

$$x \notin M\langle G^* \rangle \Rightarrow F(x) \in M\langle x \rangle.$$

The left-hand side above is an infinite conjunction of first order formulas of the form $x \notin f(G^*)$, for f an \tilde{L} -definable function with parameters in M , while the right hand side is an infinite disjunction of first order formulas of the form $F(x) = g(x)$, for g an \tilde{L} -definable function with parameters in M . So finitely many of the formulas on the left hand side imply the disjunction of finitely many of those on the right:

$$x \notin f_1(G^*) \wedge \dots \wedge x \notin f_n(G^*) \Rightarrow (F(x) = g_1(x)) \vee \dots \vee (F(x) = g_n(x)).$$

Since finite unions of G -small sets are G -small, the above means that there is a G -small set X such that for all $x \notin X$, $F(x)$ is equal to the disjunction of finitely many \tilde{L} -definable functions (with parameters).

Let $S_i = \{x : F(x) = g_i(x)\}$. For $x, y \notin M\langle G^* \rangle$, by the proof of quantifier elimination, if x, y realise the same cut in M , then they realise the same type in $\langle \tilde{M}^*, G^* \rangle$ (this is clear from the proof of QE). So the set S_i is a definable set without resorting to the predicate U , by which the function F is equal to **one** \tilde{M} -definable function f .

We have a similar theorem for our models. Before beginning to establish that, we state a lemma which is not going to be referred to in the rest of our discussion, but since it is proved with similar arguments it is worth mentioning here.

Lemma 4.37. *Let A be a real closed field. Suppose that A can be expanded to be a substructure of B which is an $|A|^+$ -saturated model of a theory T extending the theory of real closed fields, in a language L' which extends L_{or} . Now let $G : B \rightarrow B$ be a function definable in L' with parameters in A such that $\forall x \quad G(x) \in A\langle x \rangle$. Then there are finitely many L_{or} -definable functions F_1, \dots, F_m with parameters in A such that for all x we have $(G(x) = F_1(x)) \vee \dots \vee (G(x) = F_m(x))$. Furthermore if for all x , $G(x) \in A$ then G is equal to a disjunction of constant functions.*

Proof. The following set of formulas

$$P := \{G(x) \neq F(x) : F \text{ is an } L_{or}\text{-definable function with parameters in } A\}$$

is not satisfiable in B , because by our assumptions, for all x there is a function definable with parameters in A such that $G(x) = F(x)$. So there are F_1, \dots, F_n all definable with parameters in A such that the set $\{G(x) \neq F_1(x) \wedge \dots \wedge G(x) \neq F_n(x)\}$ is not satisfiable. This means that for all x in B we have $G(x) = F_1(x) \vee \dots \vee G(x) = F_n(x)$. For the second part simply consider the type $P := \{G(x) \neq a : a \in A\}$. \square

Note that the following is *not* an example of the above situation. Consider the floor function from \mathbb{R} to \mathbb{R} . Although this function takes all its values in \mathbb{Q} , it is not equal to finitely many polynomial-like functions. The reason is that \mathbb{R} is not a saturated model (as a field or just as a model of $\text{Th}(\mathbb{Q}, <)$) and if we have a saturated extension of \mathbb{Q} , then the values of the floor function will be in an ultraproduct of \mathbb{Q} . It means and it is obvious that in a saturated extension of \mathbb{Q} the correspondent of the floor function can not take all its values in \mathbb{Q} .

We now get back to our models and definable functions in them. We begin the discussion with the following lemma.

Lemma 4.38. *Let $M \subseteq M_1 \models \mathbb{T}$. Let $F : M_1 \rightarrow M_1$ be a definable function with parameters in M . Then for every x in M_1 , there are elements y_1, \dots, y_n (various n 's) in A_1 such that $F(x) \in M\langle x, y_1, \dots, y_n \rangle$. Elements y_1, \dots, y_n can be chosen in such a way that $y_i = \lambda_1(f(x, y_1, \dots, y_{i-1}))$ for some \tilde{L} -definable function f with parameters in M .*

Proof. For each x , there is a sequence $(y_i)_{i \in \mathbb{N}}$ of elements in A_1 such that $M\langle x, y_1, \dots \rangle$ is closed under λ_1 and can be the universe of a submodel of \mathbb{T} . By Theorem 4.34,

$M\langle x, y_1, \dots \rangle$ is definably closed and since F is definable with parameters in M , we have $F(x) \in M\langle x, y_1, \dots \rangle$. So there is a finite number of elements y_{i_1}, \dots, y_{i_n} such that $F(x) \in M\langle x, y_{i_1}, \dots, y_{i_n} \rangle$. The second part is now clear. \square

The following are immediate consequences of the above lemma.

Corollary 4.39. *Let \mathbb{M}_1 be a model of \mathbb{T} and a free extension of \mathbb{M} (a substructure). Let $F : M_1 \rightarrow M_1$ be a definable function in M_1 with parameters \bar{m} in M . Then for each x , there are elements $\bar{a} \in A_1$ such that $F(x) \in \langle \bar{m}, x, \bar{a} \rangle$, the \tilde{T} -closure of $\{\bar{m}, x, \bar{a}\}$ in M_1 .*

Corollary 4.40. *Let \mathbb{M}_1 be a free extension of \mathbb{M} and $F : M_1 \rightarrow M_1$ be a definable function. Let x be such that $\lambda_1(M\langle x \rangle) \subseteq A$. Then $F(x) \in M\langle x \rangle$.*

Proof. Because then $M\langle x \rangle$ is the universe of the structure $\langle \tilde{M}\langle x \rangle, G, A, \lambda, P_n \rangle$ or the structure $\langle \tilde{M}\langle x \rangle, G\langle x \rangle, A, \lambda, P_n \rangle$ and is definably closed by Lemma 4.34. \square

We are now prepared for our characterisation of definable functions up to the *small* sets. In our definition of smallness, we have used \tilde{L} -definable functions instead of the naturally more expected $\tilde{L} \cup \{\lambda\}$ -definable functions. This is because of the nature of our proofs in which we are constantly dealing with closures in \tilde{L} rather than $\tilde{L} \cup \{\lambda\}$, or better to say, because of the close relation between these two notions. This will be made clearer in the sequel.

Definition 4.40.1. *Let $X \subseteq M$ be definable in \mathbb{M} . We call X , G -small if there is an $n \in \mathbb{N}$ and an \tilde{L} -definable (with parameters) function $f : M^n \rightarrow M$ such that $X \subseteq f(G^n)$.*

Notation. In the following, whenever we say a function f is equal to a disjunction of functions g_1, \dots, g_n , we mean $\forall x \quad \bigvee_{i=1, \dots, n} (f(x) = g_i(x))$.

The following theorem asserts that the definable sets and functions in a model of \mathbb{T} are *almost* $\tilde{L} \cup \{\lambda\}$ -definable; put in other words, they can be defined in a way that only a small part of them is left not defined by an $\tilde{L} \cup \{\lambda\}$ -formula.

Theorem 4.41. *We have the following characterisation of definable functions and sets up to small sets:*

1. If $\mathbb{M} \models \mathbb{T}$ and $F : M \rightarrow M$ is a definable function, then F equals to an $\tilde{L} \cup \{\lambda\}$ -definable function outside a G -small set.
2. Let $X \subseteq M$ be a definable set in $\mathbb{M} \models \mathbb{T}$. Then there is an $\tilde{L} \cup \{\lambda\}$ -definable set X' and a G -small set Y such that $X - Y = X' - Y$.

Proof. For 1, take a saturated elementary extension \mathbb{M}_1 of \mathbb{M} (which is as we proved before, free over it). For each $x \notin M\langle G_1 \rangle$, as in the previous lemma, there are elements $\bar{a} = a_1, \dots, a_n$ in A_1 such that $F(x) \in M\langle x, \bar{a} \rangle$. But each a_i is equal to $g(x)$ for some $\tilde{L} \cup \{\lambda\}$ -function g definable in M . That is for each x , there exists an $\tilde{L} \cup \{\lambda\}$ -definable function g with parameters in M such that $F(x) = g(x)$. By compactness, F is equal to the disjunction of finitely many $\tilde{L} \cup \{\lambda\}$ -definable functions. As $x \notin M\langle G_1 \rangle$, the $\tilde{L} \cup \{\lambda\}$ -type of x over M determines its whole type over M and by similar discussions as we did just before Lemma 4.38, F agrees with an $\tilde{L} \cup \{\lambda\}$ -definable function outside a G -small set.

For 2. take the characteristic function of X for F in 1.. □

The above theorem is particularly useful when dealing with definable function from A to A . It makes them definable with much simpler formulas. But as we will see, it is not going to be of much help when it comes to the definable functions from G to G .

Corollary 4.42. *Let $\mathbb{M} \models \mathbb{T}$ and $F : M \rightarrow M$ be a definable function. Then on A , the function F is equal to a disjunction of finitely many \tilde{L} -definable functions. Clearly this is also the case if $F : A \rightarrow A$ is a definable function in \mathbb{M} . In the latter case if F is definable with parameters in A , then it is equal to a Pr -definable function with parameters in A .*

Proof. Let $\mathbb{M}_1 \models \mathbb{T}$ be a $\max\{|M|, |\mathbb{T}|\}^+$ -saturated free extension of \mathbb{M} . Then we have:

$$\forall x \quad (x \in A_1 \Rightarrow F(x) \in M\langle x \rangle).$$

This is because for $x \in A_1$, as in the proof of QE, $M\langle x \rangle$ is closed under λ_1 and $\langle \tilde{M}\langle x \rangle, G\langle x \rangle, A\langle x \rangle, \lambda_1, P_n\langle x \rangle \rangle$ is an \mathbb{L} -structure extending \mathbb{M} . By Theorem 4.34, the universe $M\langle x \rangle$ of this structure is definably closed. The last part of the corollary comes from Theorem 4.22 which says if x and y in A_1 realise the same Pr -types over A , then they realise the same types in \mathbb{M} over A . □

The following argument removes an ambiguity about the valuation inequality. It explains that new Archimedean classes do not appear only because of adding very large or very small elements.

Let $M \subseteq M_1$ be real closed fields. Let $x \in M_1$ be such that $x \in (a, b)$ for $a, b \in M$, that is $x \in \{y : M_1 \models a < y < b\}$. The question is: ‘is it true that in this case, there are no more Archimedean classes in $M\langle x \rangle$ than there are already in M ?’ The reason for one to think of this to be true, could be that since elements in $M\langle x \rangle$ are algebraic over $M \cup \{x\}$, they can not be too large or too small as their size is determined by the coefficients of the polynomial they are roots of (see Proposition 4.58). But, this is not the case. Let $x \in (a, b)$ be such that $x = a + (\text{an infinitesimal element with respect to } M)$. Then $x - a \in M\langle x \rangle$ is an infinitesimal element with a new Archimedean class which is not already in M .

Let $\mathbb{M} \subseteq \mathbb{M}_1$ and $\mathbb{M}, \mathbb{M}_1 \models \mathbb{T}$. Also assume that \mathbb{M}_1 is free over \mathbb{M} . Let $x \in G_1 - G$. Let $\lambda_1(f(x))$ be in $A_1 - M$ for f an \tilde{L} -definable function with parameters in M . Given that as G is dense in M , they both have the same Archimedean classes, we may also ask this question: ‘is there any \tilde{L} -definable function g with parameters in G such that $\lambda_1(f(x)) = \lambda_1(g(x))$?’ Putting the same question in other words: If adding x to M produces a new Archimedean class for M does it necessarily add a new Archimedean class to G as well? Is it possible that $G\langle x \rangle$ have the same Archimedean classes as G does but $M\langle x \rangle$ have more Archimedean classes than M ? It seems, on the face of it, that the last question has a negative answer. But rather surprisingly it does not.

Let us consider the example of \mathbb{R} and \mathbb{R}_{alg} again. For ϵ an infinitesimal element over \mathbb{R} , consider $\mathbb{R}_{alg}\langle \pi + \epsilon \rangle$ and $\mathbb{R}\langle \pi + \epsilon \rangle$. $\mathbb{R}\langle \pi + \epsilon \rangle$ has more Archimedean classes than \mathbb{R} has. The class of ϵ is the new Archimedean class. Let us check whether or not this class is in $\mathbb{R}_{alg}\langle \pi + \epsilon \rangle$. By valuation inequality if there is a new Archimedean class in $\mathbb{R}_{alg}\langle \pi + \epsilon \rangle$, it needs to be of the form $v(\pi + \epsilon - a)$ for some $a \in \mathbb{R}_{alg}(\pi + \epsilon)$ (just the field structure). To get an infinitesimal element, we need $\pi < a < \pi + \epsilon$. Algebraic calculations reveal that such an a does not exist. In other words, as in our answer to Question 4.9, despite \mathbb{R}_{alg} being dense in \mathbb{R} , $\mathbb{R}_{alg}\langle \pi + \epsilon \rangle$ is not dense in $\mathbb{R}\langle \pi + \epsilon \rangle$. This, as we discussed after Question 4.9, is why we cannot find functions with parameters in G to satisfy the statement of the next theorem.

Lemma 4.43. *Let $\mathbb{M} \models \mathbb{T}$. Let $F : G^n \rightarrow G$ be a definable function in \mathbb{M} . Then there*

are finitely many $\tilde{L} \cup \{\lambda\}$ -definable functions f_1, \dots, f_m , $m \in \mathbb{N}$, defined over M and from G^n to G such that F is equal to the disjunction of f_1, \dots, f_m .

Proof. Consider a free extension \mathbb{M}_1 of \mathbb{M} and $\bar{a}_1 \in G_1^n$. There is a sequence $(g_i)_{i \in \mathbb{N}}$ of elements of A_1 such that $M\langle \bar{a}_1, (g_i)_{i \in \mathbb{N}} \rangle$ and $G\langle \bar{a}_1, (g_i)_{i \in \mathbb{N}} \rangle$ are closed under λ_1 and have the same Archimedean classes.

There exist elements d_1, \dots, d_k in A_1 such that $F(\bar{a}_1) \in M\langle \bar{a}_1, \bar{d} \rangle$ where \bar{d} are among the elements of the sequence $(g_i)_{i \in \mathbb{N}}$. Each d_i is of the form $\lambda(f(\bar{a}_1))$ for some $\tilde{L} \cup \{\lambda\}$ -definable function f with parameters in M . So we have $F(\bar{a}_1) \in G_1 \cap M\langle \bar{a}_1, \bar{d} \rangle$ which is $G\langle \bar{a}_1, \bar{d} \rangle$. By compactness and similar arguments to the proof of Theorem 4.41 part 1., F is of the form described in the theorem. □

Note that if our wrong statement 4.10, were not wrong, then we could finish the proof as following: since $G\langle \bar{a}_1, \bar{d} \rangle$ is dense in $M\langle \bar{a}_1, \bar{d} \rangle$, the Archimedean classes of $G\langle \bar{a}_1, \bar{d} \rangle$ and $M\langle \bar{a}_1, \bar{d} \rangle$ are the same, and hence $d_i = f(\bar{a}_1)$ also for an $\tilde{L} \cup \{\lambda\}$ -definable function f with parameters in ‘ G ’.

In our axioms, the discrete group A is a subset of G . We will see later in this chapter that we can also axiomatise a complete theory with all its axioms the same as \mathbb{T} apart from that $A \cap G = \emptyset$. To make it clearer, not only do we have a natural complete axiomatisation for $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$, with a slight change of axioms we have a complete theory for which $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, \pi^{\mathbb{Z}} \rangle$ is a model. So, given any discrete definable set D with parameters in G , a natural question is whether or not D needs always to be disjoint from G or a subset of G . We seek the answer to this question in the following.

Question 4.44. *If $\mathbb{M} \models \mathbb{T}$ and D is a discrete set definable in \mathbb{M} , then do we have this: either $D \cap G = \emptyset$ or $D \subseteq G$?*

If parameters are allowed then obviously not. For example the set $\pi 2^{\mathbb{Z}} \cup 2^{\mathbb{Z}}$ does not satisfy this in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$. If our discrete set D is defined without any parameters, or with parameters in G then as we proved in corollary 4.31, it is a subset of G .

According to a remark in part 8.6. in [23], for α in $\mathbb{R}_{>0}$, if \mathcal{R} is an o-minimal expansion of $\bar{\mathbb{R}}$ and has the field of exponents \mathbb{Q} , then every function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ definable in $(\mathcal{R}, \alpha^{\mathbb{Z}})$ is given piecewise by $L(\mathcal{R}) \cup \{\lambda\}$ -terms. Further, if $\text{Th}(\mathcal{R})$ is

universal then $\langle \mathcal{R}, \alpha^{\mathbb{Z}} \rangle$ admits countable \mathcal{R} -cell decomposition. In particular every set definable in $\langle \bar{\mathbb{R}}, \alpha^{\mathbb{Z}} \rangle$ is a countable disjoint union of semialgebraic cells.

Identifying λ by the floor function and $2^{\mathbb{Z}}$ by \mathbb{Z} , one can use Matlab or Maple to plot the graph of definable functions to grasp the *pattern* of cells.

The following two lemmas, are essentially generalisation of similar lemmas by van den Dries in [7] and making them applicable to our definable sets. Note the slight change in our setting that our models are expansions of $\bar{\mathbb{R}}$.

Lemma 4.45. *Let $\mathbb{M} \models \mathbb{T}$ be an expansion of \mathbb{R} . Let $S \subseteq M^n$ be a set definable in $\langle \tilde{M}, A \rangle$. Let the map $g : M^n \rightarrow M^k$, $k \in \mathbb{N}$, be \tilde{L} -definable with parameters in M . Then there are countably many sets $S_i \subseteq S$, $i \in \mathbb{N}$, all \tilde{L} -definable with parameters in M , such that $G^n \cap S \cap g^{-1}(G^k) = \bigcup_{i \in \mathbb{N}} (G^n \cap S_i)$.*

Proof. By what came before this lemma, S can be written as a countable union of \tilde{M} -cells, say $S = \bigcup_{i \in \mathbb{N}} C_i$. By lemma 4.2, van den Dries, [7], for each C_i , there is an \tilde{M} -definable set S_i such that $G^n \cap C_i \cap g^{-1}(G^k) = G^n \cap S_i$ and this gives the result we are looking for. \square

The above lemma is obvious when G is countable. However, it is not the case that if \mathbb{M} is an expansion of \mathbb{R} , then G needs to be countable. For example, since the transcendences degree of \mathbb{R} over \mathbb{R}_{alg} is $> \aleph_0$, if we add uncountably many c_i 's (and not all of them) all transcendental over \mathbb{R}_{alg} to it, and call the real closure of the obtained set \mathbb{R}' , then $\langle \mathbb{R}, \mathbb{R}', \dots \rangle$ can serve as a model in which $G \neq \mathbb{R}$ is not countable. With the help of the above lemma we can, as in the following lemma, find a general pattern for small sets.

Lemma 4.46. *Let $\mathbb{M} \models \mathbb{T}$ expand $\langle \mathbb{R}, < \rangle$ and $X \subseteq M$ be G -small. Then there are countably many sets of the form $f(G^m \cap E)$ —with E an open \tilde{M} -cell in M^m (where m may vary) and f an \tilde{L} -definable function in M continuous on E — whose union is X .*

Proof. There are an \tilde{L} -definable function f and $m \in \mathbb{N}$ such that $X \subseteq f(G^m)$. As X is definable in \mathbb{M} , $X = f(X')$ for some $X' \subseteq G^m$ definable in \mathbb{M} . By Theorem 4.19, $X' = S \cap G^m$ for some $S \subseteq M^m$ definable in $\langle \tilde{M}, A \rangle$.

The case $m = 0$ is trivial.

Take $m > 0$ and suppose that the statement holds for n 's smaller than m . Partition S into countably many \tilde{M} -cells on each of which f is continuous. This is possible since

f is an \tilde{L} -definable function and there is a finite partition of each cell to cells on each of which f is continuous. The open cells in the partition of S do contribute to the union we need, so we consider E , a non-open cell with dimension $d < m$. Put $E' = P(E)$ with P the projection of E onto the open cell E' in M^d . Then $f(G^m \cap E) = (f \circ \mu)(G^d \cap E' \cap \mu^{-1}(G^m))$ where $\mu : B^d \rightarrow B^m$ is a definable map such that for all $x \in E$, $\mu(P(x)) = x$. By the previous lemma, $(G^d \cap E' \cap \mu^{-1}(G^m))$ is a countable union $\cup_{i \in \mathbb{N}}(G^d \cap S_i)$ and $f(G^m \cap E)$ is a countable union of the sets of the form $f \circ \mu(G^d \cap S_i)$. By the inductive hypothesis each $f \circ \mu(G^d \cap S_i)$ has the form desired in the theorem and so does the set X . \square

Theorem 4.47. *Let $\mathbb{M} \models \mathbb{T}$ expand \mathbb{R} and $X \subseteq M$ be G -small. Then there is a countable set of points C and countably many disjoint intervals I_i , $i \in \mathbb{N}$ such that for each i , X is dense and codense in I_i and $X \subseteq C \cup \bigcup_{i \in \mathbb{N}} I_i$.*

Proof. By Lemma 4.46, X is a countable union of the sets of the form $f(G^m \cap E)$ for E an open cell and f an \tilde{L} -definable function continuous on E . Since f is continuous and E is open, $f(E)$ is an interval including possibly the endpoints, or it is simply a single-point. Since $G^m \cap E$ is dense in E , so is $f(G^m \cap E)$ in I . Since intervals are not G -small, the complement of $f(G^m \cap E)$ in I is also dense in I . \square

In above theorem, \mathbb{M} being an expansion of \mathbb{R} just assures that the number of intervals and the set C are countable. In general a similar result holds for all models of \mathbb{T} but C and the number of I 's are not countable. We try to prove this later in this chapter.

Corollary 4.48. *In the above theorem, if X is G -small and closed, then it is a countable discrete set.*

Proof. $X \subseteq \cup I_i \cup C$. If there is an I_i in which X is dense and co-dense, then the complement of X can not contain an interval. So $X = C$ and C is closed. \square

Theorem 4.47 can be refined as follows. Let X be a G -small set defined in \mathbb{M} . Then $X \subseteq F(G^n)$ for some \tilde{L} -definable function F and $n \in \mathbb{N}$. X is then equal to $F(X')$ for some $X' \subseteq G^n$. By theorem 4.19, X' is of the form $S \cap G^n$ for some $\tilde{L} \cup \{\lambda\}$ -definable $S \subseteq M^n$. Now if \mathbb{M} is an expansion of \mathbb{R} , there is a countable partition of M^n into \tilde{M} -cells (better to say $\tilde{L}(M)$ -cells) compatible with S . Among these cells are cells with

no intersection with S . The projections of these cells on M are either open intervals or points. Some of the cells contributing to the decomposition under consideration are subsets of S . Among these cells, some are open whose images under F are intervals, say I in which $F(S \cap G^n)$ is dense and codense. Some other cells are subsets of S but are not open. To these cells we apply the method used in the proof of Lemma 4.46 to be able to consider them also open. We have now proved the following theorem about G -small sets:

Theorem 4.49. *Let $X \subseteq M$ be G -small where \mathbb{M} expands $\langle \mathbb{R}, < \rangle$. Then M can be partitioned into countably many intervals I_i , $i \in \mathbb{N}$, and a countable set C of points, such that for each i , either the interval I_i is disjoint from X or X is dense-codense in it.*

As we proved in Theorem 4.41, every definable set in \mathbb{M} is can be defined, up to a small set, by an $\tilde{L} \cup \{\lambda\}$ -formula. Now that we know what small sets are like, we can see how any definable set looks. The points of a definable set, distribute in the universe of a model by accumulating in disjoint intervals or being part of a countable discrete set or by forming an interval.

Theorem 4.50. *Let S be a set definable in $\mathbb{M} \models \mathbb{T}$ where \mathbb{M} expands $\langle \mathbb{R}, < \rangle$. Then there is a decomposition of M into countably many disjoint intervals I_i , $i \in \mathbb{N}$, and a countable set C such that for each i , either $I_i \subseteq S$ or $I_i \cap S = \emptyset$ or S is dense and codense in I_i .*

Proof. By Theorem 4.41, $S = (S' - X) \cup Y$ for X, Y two G -small sets and S' an $\tilde{L} \cup \{\lambda\}$ -definable set. As S' is $\tilde{L} \cup \{\lambda\}$ -definable, there is a decomposition of M into cells compatible with it, i.e. there is a countable set C and countably many disjoint intervals I_i , $i \in \mathbb{N}$, such that $M = \bigcup I_i \cup C$ and for each i either $I_i \subseteq S'$ or $S' \cap I_i = \emptyset$. If $I_i \subseteq S'$, then $I_i \cap S = I_i \cap ((S' - X) \cup Y) = (I_i - X) \cup (I_i \cap Y) = I_i - (X - Y)$. So $I_i \cap S = I_i - Z$ for some G -small set Z . Now, as Z is G -small, by Theorem 4.49, there is a partition of I_i into a countable set and countably many intervals $I_{i,j}$, in each of which Z and hence $I_i - Z$ is dense and codense. If $I_i \cap S' = \emptyset$ then $I_i \cap S = I_i \cap Y$. Again, as Y is G -small there is a decomposition of I_i into a countable set $C \subseteq I_i$ and countably many intervals $I_{i,k}$ each of which, say $I_{i,l}$, either has no intersection with S , or is such that $I_{i,l} \cap S$ is dense and codense in $I_{i,l}$. \square

If we consider the set $2^{\mathbb{Z}}$ as S in the above theorem in the our example $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$, then $C = \{0\} \cup 2^{\mathbb{Z}}$ and the intervals in the decomposition of \mathbb{R} are $(-\infty, 0) \cup (2^{-n-1}, 2^{-n}) \cup (2^n, 2^{n+1})$, $n \in \mathbb{N}$.

As another example, consider the following definable set: $\mathbb{R}_{alg} \cap \bigcup_{k \text{ is even}} [2^k, 2^{k+1}]$. If we take $S' = \cup [2^k, 2^{k+1}]$ then $S' - \mathbb{R}_{alg}$ has the form expected in Theorem 4.41 and the statement of the above theorem is easy to check that is satisfied.

Corollary 4.51. *Let $\mathbb{M} \models \mathbb{T}$ expand $\langle \mathbb{R}, < \rangle$. Then if $S \subseteq \mathbb{R}$ is open and definable, then it is definable in $\tilde{L} \cup \{\lambda\}$.*

Proof. There are definable G -small sets X and Y and an $\tilde{L} \cup \{\lambda\}$ -definable set S' such that $S = (S' - X) \cup Y$. Since X is G -small, $X = f(S'')$ for some definable set $S'' \subseteq G^n$. So $X = f(S''' \cap G^n)$ for some $\tilde{L} \cup \{\lambda\}$ -definable $S''' \subseteq M^n$. By Theorem 3.4.1 in [23], S''' can be partitioned into finitely many definable special C^p -manifolds, so there are finitely many projections P in the proof of 4.46 and all of them are definable. Considering the same argument about Y and the proof of the above theorem, S is an $\tilde{L} \cup \{\lambda\}$ -definable union of countably many intervals I_i , $i \in \mathbb{N}$. \square

Note that in all the above lemmas and corollaries, we assumed that \tilde{M} expands $\langle \mathbb{R}, < \rangle$. This is because of the fact that (by [23], remark, page 26), if \mathcal{R} is an o-minimal expansion of $\bar{\mathbb{R}}$ having field of exponents \mathbb{Q} , and its theory is universal and admits QE, then $\langle \mathcal{R}, 2^{\mathbb{Z}} \rangle$ admits countable C^p -cell decomposition into \mathcal{R} -cells.

Before stating what we expect for the more general case (when our models of \tilde{T} are not expansions of $\langle \mathbb{R}, < \rangle$), we need to distinguish between the two following definitions of d-minimality in two different sources.

Miller in [23] defines an expansion \mathcal{R} of $\langle \mathbb{R}, < \rangle$ to be d-minimal if for every $\mathfrak{M} \equiv \mathcal{R}$, every subset of M definable in \mathfrak{M} is the union of an open set and finitely many discrete sets. By a compactness argument, \mathcal{R} is d-minimal if for every m and definable set $A \subseteq \mathbb{R}^{m+1}$ there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$, A_x has interior or is a union of N discrete sets. Note that if \mathcal{R} is d-minimal then so is every reduct of \mathcal{R} over $\langle \mathbb{R}, < \rangle$.

He then calls a d -dimensional C^p -submanifold M of \mathbb{R}^n , special, if there exists $\mu \in \Pi(n, d)$ (the set of projections from \mathbb{R}^n to \mathbb{R}^d) such that for each $y \in \mu(M)$ there is an open box B about y such that each connected component X of $M \cap \mu^{-1}(B)$ projects C^p -diffeomorphically onto B , i.e. $\mu|_M : M \rightarrow \mu M$ is a C^p -smooth covering

map.

A collection \mathcal{A} of subsets of \mathbb{R}^n is compatible with a collection \mathcal{B} of subsets of \mathbb{R}^n if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, either A is contained in B or A is disjoint from B .

The following are all proved in [23]. Assume that \mathcal{R} is d-minimal. Let \mathcal{A} be a finite collection of definable subsets of \mathbb{R}^n . Then there is a finite partition of \mathbb{R}^n into special C^0 -submanifolds, each of which is definable and compatible with \mathcal{A} . If \mathbb{R} expands $\bar{\mathbb{R}}$, then this holds with C^p instead of C^0 .

Every d-minimal expansion of $\langle \mathbb{R}, <, + \rangle$ admits countable cell decomposition. Every d-minimal expansion of $\bar{\mathbb{R}}$ admits countable C^p -decomposition.

But, more interestingly, as we pointed out before, (by [23], remark, page 26), if \mathcal{R} is an o-minimal expansion of $\bar{\mathbb{R}}$ having field of exponents \mathbb{Q} , and its theory is universal and admits QE, then $\langle \mathcal{R}, 2^{\mathbb{Z}} \rangle$ admits countable C^p -cell decomposition into ‘ \mathcal{R} -cells’.

In a recent paper by A. Fornasiero, [10], there is a more general definition of d-minimality. A definably complete (a structure with an order where every definable set has a supremum which may be ∞) expansion \mathbb{K} of an ordered field, is called d-minimal if for every $\mathbb{K}' \equiv \mathbb{K}$, every definable subset of \mathbb{K}' (which is written K by mistake in that preprint) is the union of an open set and finitely many discrete sets.

From the same reference:

Definition 4.51.1 ([10]). *Let $d \leq n$ and $\Pi(n, d)$ be the set of projections from \mathbb{K}^n onto d -dimensional coordinate space and $\mu \in \Pi(n, d)$. For $p \in \mathbb{N}$, let $\text{reg}_\mu^p(A)$ and $\text{reg}^p(A)$ be defined as in [23]. ($\text{reg}_\mu^p(A)$, for $A \subseteq \mathbb{R}^n$, is the set of all $a \in A$ such that for some neighbourhood U about A , the restriction of μ to $A \cap U$ maps $A \cap U$, C^p -diffeomorphically onto some open $V \subseteq \mathbb{R}^d$). As in the case where \mathbb{K} is an expansion of \mathbb{R} , $\text{reg}_\mu^p(A)$ is definable, open in A and a C^p -submanifold of \mathbb{K}^n of dimension d .*

For every $A \subseteq \mathbb{K}^n$, let, as before, $\text{Isol}(A)$ be the set of isolated points of A . Note that then $\text{Isol}(A)$ is discrete.

Definition 4.51.2. ([10]) *A definable d -dimensional C^p -submanifold M is weakly μ -special if for each $y \in \mu(M)$ and $x \in M_y$, there exists $U \subseteq \mathbb{K}^d$, open box around y , and $W \subseteq \mathbb{K}^{n-d}$ open box around x , such that $A \cap (U \times V) = \Gamma(f)$ for some definable C^p -map $f : U \rightarrow W$. M is μ -special if the box U in the above definition does not depend on x (but only on y). M is special if it is μ -special for some $\mu \in \Pi(n, d)$.*

Proposition 4.52. ([10]) *Assume that \mathbb{K} is d -minimal, let $p \in \mathbb{N}$, and \mathcal{A} be a finite collection of definable subsets of \mathbb{K}^n . Then there exists a finite partition of \mathbb{K}^n into weakly special definable C^p -submanifolds compatible with \mathcal{A} .*

That we can replace weakly special submanifolds with special submanifolds in the above proposition, is a conjecture in [10].

The following were two theorems in this thesis, but I became skeptical to their proofs. As I expect them to be true I leave them to be proved later. The true proof of these two should involve using the described cell-decomposition results in above.

1. Let $\mathbb{M} \models \mathbb{T}$ (and be not necessarily an expansion of \mathbb{R}), and $X \subseteq M$ be G -small. Then there are disjoint open intervals I_i , $i \in J$ an index set, and finitely many discrete sets D_i , $i = 1, \dots, n$, such that $M = \cup_{i \in J} I_i \cup D_1 \dots \cup D_n$ and for each $i \in J$, one of these happens: either X is disjoint from I_i or X is dense and condense in I_i .
2. If $S \subseteq \mathbb{M} \models \mathbb{T}$ is definable. Then there is a partition of M into finitely many discrete sets and infinitely many open intervals I_i such that, for each i , one of these happens: either S is dense and codense in I_i or $I_i \cap S = \emptyset$ or $I_i \subseteq S$.

We need to remind the reader of the fact that the union of finitely many discrete subsets of \mathbb{R} is not dense in it. This can be proved in the following way. Let $S = D_1 \cup \dots \cup D_n$ be a subset of \mathbb{R} , where D_i 's are discrete. We prove by induction on n that S can not be dense in \mathbb{R} . If $n = 1$ then $S = D_1$ and for each $x \in D_1$ there is an interval I_x with $I_x \cap D_1 = \{x\}$ and the result is obvious.

Let $S = D_1 \cup \dots \cup D_{n+1}$ where $D_1 \cup \dots \cup D_n$ is not dense in \mathbb{R} , and $I \subseteq \mathbb{R}$ is an interval such that $I \cap (D_1 \cup \dots \cup D_n) = \emptyset$. Then let $x \in D_{n+1} \cap I$. There is $I' \subseteq I$ with $x \in I'$ and $I' \cap D_{n+1} = \{x\}$. So there is $I'' \subseteq I$ with $I'' \cap D_{n+1} = \emptyset$. So $I'' \cap (D_1 \cup \dots \cup D_{n+1}) = \emptyset$.

As another reminder, notice that any discrete subset of Euclidean space is countable, since the isolation of each of its points (together with the fact the rational are dense in the reals) implies that it can be mapped 1-1 to a set of points with rational coordinates.

The reason for assuming that \mathbb{M} expands \mathbb{R} in some of our theorems and lemmas above, is that we consider $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ as our main example. To this point, we have

proved that the open subsets of \mathbb{R} definable in this structure are definable in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. In fact, this is indeed the case for definable subsets of \mathbb{R}^n for each $n \in \mathbb{N}$. We mention this in the following theorem discussions on whose proof we put off to the next chapter.

Theorem. *Every open subset of \mathbb{R}^n definable in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ is definable in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.*

Remark 4.52.1. *Towards the end of this work, Gareth Jones reminded me of relevant results Fornasiero has in [10] and [11]. In these papers he considers more generally, a dense pair of d -minimal structures and proves that if $\langle \mathbb{B}, \mathbb{A} \rangle$ is a dense pair of d -minimal structures, then the ‘open core’ of it is the structure \mathbb{B} . We will give the definition of the open core of a structure in the next chapter. However, knowing this, I stopped my further attempts in proving the above theorem and left my old proof of a special case of it to the next chapter.*

We continue our scrutiny of the structure $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ by the following observation. Given that even the functions from \mathbb{R} to \mathbb{R} which are defined in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ are quite wild in the geometric point of view, since their graph can be dense and codense in \mathbb{R}^2 , we lose our hopes in finding any tame behaviour for those functions which are definable in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$.

The following is an example in [7] of such a function:

$$f(x) = \begin{cases} r & \text{if } x \text{ is of the form } r + se, \text{ for necessarily unique algebraic reals} \\ & s \text{ and } r, \text{ where } e \text{ is the usual transcendental real number.} \\ 0 & \text{otherwise} \end{cases}$$

New setting. For the next theorem we assume that \tilde{T} is universal and admits elimination of quantifiers.

We now summarise the Theorems and proofs of this section in the following:

Theorem 4.53. *Let $\mathbb{M} \models \mathbb{T}$ be an expansion of $\langle \mathbb{R}, < \rangle$. Then we have the following:*

- I) If $X \subseteq M$ is G -small, then there is a decomposition of M into countably many disjoint intervals I_i 's and finitely many discrete sets D_1, \dots, D_n , of the form:*

$$M = \bigcup_{i \in \mathbb{N}} I_i \cup D_1 \cup \dots \cup D_n \quad (**)$$

such that for each $i \in \mathbb{N}$, either $I_i \cap X$ is dense co-dense in I_i or X is disjoint from I_i .

II) If $S \subseteq M$ is definable, then M has a similar decomposition as in (**) above, where for each i , either $I_i \subseteq S$, or $I_i \cap S = \emptyset$ or $I_i \cap S$ is dense co-dense in I_i .

III) If $S \subseteq M$ is open and definable, then it is $\tilde{L} \cup \{\lambda\}$ -definable.

Proof. Let X be a G -small set. Then $X \subseteq F(G^n)$ for some \tilde{L} -definable function F and $n \in \mathbb{N}$. So $X = F(X')$ for some \mathbb{L} -definable $X' \subseteq G^n$. By Corollary 4.19 X' of the form $S \cap G^n$ for some $\tilde{L} \cup \{\lambda\}$ -definable $S \subseteq G^n$. M^n can be decomposed into countably many cells and this decomposition is compatible with S . So

$$S = \bigcup_i C_i$$

where C_i are $\tilde{L}(\mathbb{R})$ -cells.

We claim that X is a countable union of sets of the form $f(G^m \cap E)$ (for various m 's), where E is an open cell in M^m and f is continuous on E and E and f are \tilde{L} -definable in M . We prove this by induction on n . The statement is obvious for $n = 0$, and assume it true for $d < n$.

The open cells in the decomposition of S contribute to the desired format. Let C_d be the collection of cells in $\cup C_i$ of dim d which form an $\tilde{L} \cup \{\lambda\}$ -definable manifold of dimension d . There is a map $\lambda \in \Pi(n, d)$ definable in $\tilde{L} \cup \{\lambda\}$ which projects C_d onto a collection of open cells, denoted by $C_{d\lambda}$. Note that λ projects each $\tilde{L}(\mathbb{R})$ -cell in C_d homeomorphically onto a cell in $C_{d\lambda}$. Let $\mu : M^d \rightarrow M^n$ be the inverse of λ , definable in $\tilde{L} \cup \{\lambda\}$. Then

$$f(G^n \cap (\bigcup_{c_i \in C_d} C_i)) = f \circ \mu((G^d \cap \bigcup_{c_i \in C_d} C_i) \cap \mu^{-1}(G^n))$$

We now apply (with due care), Lemma 4.2, van den Dries, in [7] to see that the set in the parentheses above, has the form

$$G^d \cap S'$$

where S' is $\tilde{L} \cup \{\lambda\}$ -definable and by induction hypothesis, it has the form we are looking for. So, we have proved that X is a countable union of sets of the form $f(G^m \cap E)$ for E open cell.

Now, for each open cell E , $f(E)$ is either an interval I or a singleton. If $f(E)$ is an interval I then $f(G^m \cap E)$ is dense in I . We can assume that the intervals I , obtained in this way are distinct. Also the set of singletons is $\tilde{L} \cup \{\lambda\}$ -definable and hence a finite union of discrete sets.

The projections of cells $D_i \not\subseteq S$ on M is also the union of countably many intervals and finitely many discrete sets. This proves part I) of the theorem.

Proof of II) Note that by Theorem 4.41 $S = (S' - X) \cup Y$ for S' , $\tilde{L} \cup \{\lambda\}$ -definable and X, Y , G -small definable sets. There is a partition of M into countably many intervals and finitely many discrete sets which is compatible with S . Let I be one of the intervals in this partition. If $I \cap S' = \emptyset$, then $S \cap I = Y \cap I$. As $Y \cap I$ is G -small, it has the property described in I). If $I \subseteq S'$, then $S \cap I = I - Z$ for some G -small set $Z \subseteq X \cup Y$, which is obtained by a Boolean combination of S, I, X, Y . Again, using part I) the result follows. Note that since X and Y are fixed, for each of them there are finitely many discrete sets. So in the proof, we are not confronted by infinitely many discrete sets. For III) note remark 4.52.1. \square

It remains a question whether or not a discrete definable set, needs the dense field for its definition:

Question 4.54. *Are the following statements true?*

- *If $S \subseteq M$ is definable and a finite union of discrete sets, then it is $\tilde{L} \cup \{\lambda\}$ -definable.*
- *If $S \subseteq M$ is definable and the union of an open set and finitely many discrete sets, then it is $\tilde{L} \cup \{\lambda\}$ -definable.*

In the next chapter we will study the topological properties of our models and in the final chapter we will prove that \mathbb{T} has NIP. But before these, we consider a slightly different theory, \mathbb{T}_π , and we will prove with similar methods that it is also complete. Models of \mathbb{T}_π have the same pattern as models of \mathbb{T} but they are different in the sense that their discrete part, A , has no intersection with their dense part G .

4.6 Replacing 2 with π , our last remarks

In the previous sections, our default example was always the structure $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$. A natural question is whether or not we can replace 2 with a non-algebraic element in \mathbb{R} say π . It is obvious that π and all its integer powers are in $\mathbb{R} - \mathbb{R}_{alg}$, but as we check below, small changes in our proofs work to prove the completeness of a theory which axiomatises this structure.

We add a symbol π to $\tilde{\mathcal{L}}$ and modify the axioms for A accordingly. We replace the axiom $A \subseteq G$ with $A \cap G = \emptyset$. We denote the obtained theory by \mathbb{T}_π and the corresponding theory with axioms $ax_{q_{\phi(\bar{x}, \bar{y})}}$ added to \mathbb{T}_π with \mathbb{T}'_π (these axioms are as in additional remarks, part E). We add to \mathbb{L} predicates for $q_{\phi(\bar{x}, \bar{y})}$ and call the obtained language \mathbb{L}' . For simplicity, We do the proof for when \tilde{T} is RCF, and the proof for the \tilde{T} as in our general setting is very similar.

As before, let \mathbb{M} stands for $\langle M, G, \lambda, A, \{P_n\}_{n \in \mathbb{N}} \rangle$. Let $\mathbb{M}_0 \subseteq \mathbb{M}_1 \models \mathbb{T}'_\pi$ be a model of $\mathbb{T}'_{\pi \vee}$ which implies that M_0 and G_0 are real integral domains. Then by slight modification of the arguments before Corollary 4.2, we can embed \mathbb{M}_0 into \mathbb{M}_2 with M_2 the fraction field of M and $G_2 = M_2 \cap G_1$. The \mathbb{L}' -structure \mathbb{M}_2 is then a substructure of \mathbb{M}_1 and is prime over \mathbb{M}_0 .

If in the above, M_0 and G_0 are two real fields, then again by the same discussions we can embed \mathbb{M}_0 into \mathbb{M}_3 in which M_3 is the real closure of M_0 and $G_3 = M_3 \cap G_1$.

Now let \mathbb{M}_1 and \mathbb{M}_2 be two κ -saturated models of \mathbb{T}'_π for a sufficiently large $\kappa > |M|, |M'|$. We claim that \mathbb{M}_1 is elementarily equivalent to \mathbb{M}_2 and hence \mathbb{T}'_π is complete. We sketch the proof of this claim as follows. Let \mathbb{M} be a substructure of \mathbb{M}_1 and $\mathbb{M}' \cong \mathbb{M}$ a substructure of \mathbb{M}_2 . We need to perform a back and forth process starting with the isomorphism between \mathbb{M} and \mathbb{M}' . In this argument we suppose that \mathbb{M}', \mathbb{M} are models of \mathbb{T}'_π rather than just structures in the language \mathbb{L} .

By the above comments, we assume that M and M' are real closed fields and so are G and G' .

Let $x = \lambda_1(x) \in M_1 - M$. Clearly $x \notin G_1$. Suppose also that $x \notin M \langle G_1 \rangle$. Then $\langle M \langle x \rangle, G \rangle$ is a substructure of $\langle M_1, G_1 \rangle$, as elementary pairs of real closed fields. As in case one before (in the proof of QE for \mathbb{T}), we can check that $M \langle x \rangle$ is closed under λ_1 , and then we have the \mathbb{L} -structure $\mathbb{M} \langle x \rangle = \langle M \langle x \rangle, G, \lambda_1|_{M \langle x \rangle}, A \langle x \rangle, \{P_n \langle x \rangle\} \rangle$, which

is isomorphic to $\mathbb{M}_2\langle y \rangle$ with a similar structure, for some y in $M_2 - M'\langle G_2 \rangle$.

Now let $x = \lambda_1(x)$ and $x \in M\langle G_1 \rangle$. As $x \notin G_1$ and the closure operator is a pregeometry, $x \in \text{cl}(M \cup G_1) - \text{cl}(G_1)$ implies that $x \in \text{cl}(M)$. Since M is real closed we have $x \in M$.

Let $x \in M_1 - M$ and $x \in G_1 - G$. Then obviously $\lambda_1(x) \neq x$. As in case two before, we can find sequences $\{a_i\}_{i \in \mathbb{N}}$ in A_1 and $\{b_i\}_{i \in \mathbb{N}}$ in A_2 such that $M\langle x, a_1, \dots \rangle$ is closed under λ_1 , and $M\langle a_1, \dots \rangle$ is isomorphic to $M'\langle b_1, \dots \rangle$. We can also find a $y \in M_2$ such that $\mathbb{M}_e = \langle M\langle x, \{a_i\} \rangle, G\langle x \rangle, \lambda_1|_{M\langle x, \{a_i\} \rangle}, A\langle \{a_i\} \rangle, \{P_n\langle \{a_i\} \rangle\}_{n \in \mathbb{N}} \rangle$ is isomorphic to a similar structure \mathbb{M}'_e with universe $M'\langle y, \{b_i\} \rangle$.

The two other cases $x \in M\langle G_1 \rangle, \lambda_1(x) \neq x$ and $x \notin M\langle G_1 \rangle, \lambda_1(x) \neq x$ can be analogously worked out.

We can also prove quantifier elimination for \mathbb{T}'_π in the corresponding language \mathbb{L}' , by the same argument as in the proof of quantifier elimination for $\mathbb{T}_{\text{enriched}}$, keeping in mind that in this case for $x \in \langle \bar{a} \rangle \langle \bar{b} \rangle$, when we add $\lambda_1(x)$ to $\langle \bar{a}, \bar{b} \rangle, \langle \bar{a} \rangle$, we get the elementary pair $\langle \langle \bar{a}, \bar{b}, \lambda_1(x) \rangle, \langle \bar{a} \rangle \rangle$.

Corollary 4.55. *The theory \mathbb{T}_π is complete and axiomatises $\text{Th}(\bar{\mathbb{R}}, \mathbb{R}_{\text{alg}}, \pi^{\mathbb{Z}})$. Also \mathbb{T}'_π eliminates quantifiers.*

There is subtlety in the above proof compared to the one we had before for the completeness of \mathbb{T} . In the above proof, we added all the necessary predicates for (the expected) quantifier elimination to our language, and hence we did not go through the ‘free extension’ discussion.

That the theory \mathbb{T} which axiomatises $\langle \bar{\mathbb{R}}, \mathbb{R}_{\text{alg}}, 2^{\mathbb{Z}} \rangle$ is decidable is clear. Indeed, we have:

Corollary 4.56. *The theory of $\langle \bar{\mathbb{R}}, \mathbb{R}_{\text{alg}}, \alpha^{\mathbb{Z}} \rangle$, for each $\alpha \in \mathbb{Q}$, is decidable and can be axiomatized by \mathbb{T} (considering the corresponding \tilde{T}).*

I am not sure of the situation, in terms of decidability, for the theory of $\langle \bar{\mathbb{R}}, \mathbb{R}_{\text{alg}}, \alpha^{\mathbb{Z}} \rangle$ for a general $\alpha \in \mathbb{R}$, or even for $\langle \bar{\mathbb{R}}, \mathbb{R}_{\text{alg}}, \pi^{\mathbb{Z}} \rangle$. I expect the following to be true.

The theory of $\langle \bar{\mathbb{R}}, \mathbb{R}_{\text{alg}}, \alpha^{\mathbb{Z}} \rangle$, for each computable $\alpha \in \mathbb{R}$, is decidable.

Let us finish this section by a rather interesting fact about the dense pairs. Let $\langle \tilde{M}, G \rangle$ be a dense pair. Consider a formula of the form $\exists y \in G \ \phi(\bar{x}, y)$. Then there

are circumstances under which

$$\langle \tilde{M}, G \rangle \models \exists y \in G \quad \phi(\bar{a}, y) \Leftrightarrow \tilde{M} \models \exists y \quad \phi(\bar{a}, y)$$

for $\bar{a} \in M$. In fact the following is true.

Let $S = \{y : M \models \phi(\bar{a}, y)\}$. Then if S is infinite then it contains an interval in M and since G is dense in M , it intersects with S . If S is finite and the $\bar{a} \in G$ then since G is an elementary substructure of M , all of the elements of S are in G .

4.7 Appendix to chapter 4

We finish this chapter by the proof of a proposition we stated in the first section.

Let K and L be fields. A map $\phi : K \rightarrow L \cup \{\infty\}$ is a *place* of K if for all $x, y \in K$:

1. $\phi(x + y) = \phi(x) + \phi(y)$,
2. $\phi(xy) = \phi(x) \cdot \phi(y)$,
3. $\phi(1) = 1$.

By Exercise 2.5.4 in [9], $O = \phi^{-1}(L)$ is a valuation ring of K with maximal ideal $M = \phi^{-1}(\{0\})$ and residue class field $\bar{K} \cong \phi(K)$.

On the other hand, for every valuation ring O of K with maximal ideal M , the map ϕ defined as follows: $\phi(x) = x + M$ for all $x \in O$, and $\phi(x) = \infty$ for all $x \in K - O$, defines a place $\phi : K \rightarrow L \cup \{\infty\}$ with $L = O/M$.

The proof of the following theorem is due to Tressl in unpublished notes.

Theorem 4.57. *Let k be a field and a_1, \dots, a_d be mutually distinct elements of $\text{AC}(k)$ (the algebraic closure of k). Let $\alpha_1, \dots, \alpha_d, \epsilon$ be algebraically independent over k . Let $t_i = \alpha_i(\epsilon + a_i)$. Then*

$$\text{AC}(k(t_1, \dots, t_d)) \cap \text{AC}(k(\alpha_1, \dots, \alpha_d)) = \text{AC}(k).$$

Proof. Let $\xi \in \text{AC}(k(t_1, \dots, t_d)) \cap \text{AC}(k(\alpha_1, \dots, \alpha_d))$. It is enough to prove that ξ is algebraic over $\text{AC}(k(\alpha_j, j \neq i))$ for every $i \in \{1, \dots, d\}$. We assume that $i = d$. Let V be the valuation ring of $\text{AC}(k(\alpha_1, \dots, \alpha_d))$ lying over the local ring $\text{AC}(k(\alpha_1, \dots, \alpha_d))[\epsilon]_{(\epsilon + a_d)}$ with corresponding place:

$$\lambda : \text{AC}(k(\alpha_1, \dots, \alpha_d, \epsilon)) \rightarrow k \cup \{\infty\}.$$

So λ embeds $\text{AC}(\alpha_1, \dots, \alpha_d)$ into k and $\lambda(\epsilon) = -a_d$. Since $\xi \in \text{AC}(k(\alpha_1, \dots, \alpha_d, \epsilon))$, we have $\lambda(\xi) = \xi$. Let $\mu(X) \in k[t_1, \dots, t_d][X]$ be a polynomial of degree $n > 0$ with $\mu(\xi) = 0$. Let

$$\mu(X) = f_n(t_1, \dots, t_d)X^n + \dots + f_0(t_1, \dots, t_d)$$

with polynomials $f_0, \dots, f_n \in k[Y_1, \dots, Y_d]$. Let $f_j(Y_1, \dots, Y_d) = Y_d^{l_j} h_j(Y_1, \dots, Y_d)$ with polynomials $h_j(Y_1, \dots, Y_d)$ such that $h_j(Y_1, \dots, Y_{d-1}, 0) \neq 0$ and let $l = \min\{l_0, \dots, l_n\}$. Finally let $\tilde{f}_j(Y_1, \dots, Y_d) = Y_d^{l_j - l} h_j(Y_1, \dots, Y_d)$ and

$$\tilde{\mu}(X) = \tilde{f}_n(t_1, \dots, t_d)X^n + \dots + \tilde{f}_0(t_1, \dots, t_d).$$

Then $\tilde{\mu}(X).(t_d)^l = \mu(X)$, so $\tilde{\mu}(\xi) = 0$ and there is some $j \in \{0, \dots, n\}$ such that $\tilde{f}_j(Y_1, \dots, Y_{d-1}, 0) = h_j(Y_1, \dots, Y_{d-1}, 0) \neq 0$.

Now we apply the place λ to the equation $\tilde{\mu}(\xi) = 0$ and get from $\lambda(\epsilon) = -a_d$,

$$\tilde{f}_n(\alpha_1(a_1 - a_d), \dots, \alpha_{d-1}(a_{d-1} - a_d), 0)\xi^n + \dots + \tilde{f}_0(\alpha_1(a_1 - a_d), \dots, \alpha_{d-1}(a_{d-1} - a_d), 0) = 0.$$

Because there is one $j \in \{0, \dots, n\}$ with $\tilde{f}_j(Y_1, \dots, Y_{d-1}, 0) = h_j(Y_1, \dots, Y_{d-1}, 0) \neq 0$ and $\alpha_1(a_1 - a_d), \dots, \alpha_{d-1}(a_{d-1} - a_d)$ are algebraically independent over k , we conclude that ξ is algebraic over $k(\alpha_1, \dots, \alpha_{d-1})$ as desired. \square

We also referred to the following proposition before Lemma 4.43.

Proposition 4.58 (Continuity of roots in complex analysis). *Let $\sum a_v x^v$ be a polynomial. Then given a very small ϵ , there is $\delta > 0$ such that for every polynomial $\sum b_v x^v$, if $|b_v - a_v| < \delta$, then in the distance ϵ of any root of f with multiplicity m , g has exactly m roots.*

See [26] for the proof.

Chapter 5

A survey of the open core

5.1 Introduction

That the interior and closure of a definable set in each structure is definable is the main motivation behind the the definition of the ‘open core’ of a structure.

Let $\mathcal{R} = \langle \mathbb{R}, <, \dots \rangle$ be an L -structure which is an expansion of $\langle \mathbb{R}, < \rangle$. Let

$$O_{\mathcal{R}} := \{O \subseteq \mathbb{R}^n : n \in \mathbb{N}, O \text{ is open and definable in } L(\mathbb{R})\}.$$

Then the structure $\mathcal{R}^\circ := \langle \mathbb{R}, \{O\}_{O \in O_{\mathcal{R}}} \rangle$ (in the language $L^\circ = \{<\} \cup \{P_O\}_{O \in O_{\mathcal{R}}}$ with P_O a predicate for O) is called the open core of \mathcal{R} .

This notion was introduced by Miller and Speissegger in [24]. Tameness of the open core of a structure, as we will see, guarantees the topological tameness of the structure itself.

Their study in [24] suggests that a structure \mathcal{R} which expanding $\langle \bar{\mathbb{R}}, < \rangle$ which is not o-minimal and does not admit cell-decomposition, can be ‘topologically close to being o-minimal’. This happens when the open core of such a structure is o-minimal. If this is the case, then given a definable set A in \mathcal{R} which is a subset of \mathbb{R}^n , there is a finite decomposition of \mathbb{R}^n into cells (in the sense o-minimal structures) C_i , some of which lay inside A and some other contain a part of A dense and codense in them.

Therefore, studying the open core of a structure gives us insights on what our structure is topologically like. The precise record of this is the following proposition.

Proposition 5.1 ([24], cell decomposition when the open core is o-minimal). *Let \mathcal{R} be an expansion of $\langle \mathbb{R}, < \rangle$ and suppose that \mathcal{R}° is o-minimal. Let A be a definable (in*

\mathcal{R}) subset of \mathbb{R}^n . Then there is a finite partition of \mathbb{R}^n into cells definable in \mathcal{R}° , such that for each of these cells C , either $A \cap C = \emptyset$ or $C \subseteq A$ or $A \cap C$ is dense codense in C .

In the previous chapter we noticed that in an expansion \mathbb{M} of $\bar{\mathbb{R}}$ which is a model of \mathbb{T} , the open definable sets in one variable are defined in $\langle \tilde{M}, A \rangle$. The main aim in writing this chapter was to prove the following, using similar methods to those in [24].

‘If an expansion $\mathbb{M} \models \mathbb{T}$ of $\bar{\mathbb{R}}$ has the same one variable definable open sets as does $\langle \tilde{M}, A \rangle$ then its open core is $\langle \tilde{M}, A \rangle$ ’.

I believe this is true but time did not permit proving it. There are parts of the proof in [24] which fail for our structure (for example the dimension of the frontier of a zero dimensional set may not be less than zero—the frontier of $2^{\mathbb{Z}}$ is $\{0\}$ — so we face difficulties even in the first step of induction) but I think these can be amended to have the following:

Theorem. *The open core of $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ is $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.*

However, that the above theorem is the case, also results from Fornasiero’s results in [11] and [10] (see remark 4.52.1).

It is proved in [24] that the open core of a dense pair of o-minimal structures is an o-minimal structure. This, together with techniques in [7] leads to the following: ‘The open core of the structure $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is the structure $\bar{\mathbb{R}}$ ’. This means that every open definable set in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is defined in $\bar{\mathbb{R}}$. The proof of the above statement involves both topological and model theoretical arguments and we will briefly discuss the outline of it in the next section.

Note that despite the fact that we know that the open core of $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is $\bar{\mathbb{R}}$, as the proof of this involves compactness theorem, we can not find any relation between the parameters of the formula that defines an open set in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ with the parameters of the formula that defines the same open set in $\bar{\mathbb{R}}$. Indeed for the above theorem to have a direct proof, we would need statements like the following to be true:

‘In $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$, let $O_{\bar{y}}$ be a family of open sets defined in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ where \bar{y} are in $2^{\mathbb{Z}}$. Then this collection is uniformly defined in $\bar{\mathbb{R}}$ over the index set $2^{\mathbb{Z}}$ ’.

5.2 The open core of $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is $\bar{\mathbb{R}}$

In [24] Miller and Speissegger proved the following interesting theorem about the open core of a structure:

Theorem 5.2. *Let $U_\gamma \subseteq \mathbb{R}^{n(\gamma)}$ be a collection of open sets. Then the structure $\langle \mathbb{R}, <, (U_\gamma)_{\gamma \in \Gamma} \rangle$ is o-minimal if and only if every definable subset of \mathbb{R} in this structure, is finite or uncountable.*

A corollary of this theorem is the following:

Proposition 5.3. *Let \mathcal{R} be an expansion of $\langle \mathbb{R}, < \rangle$. Then*

1. *If every definable subset of \mathbb{R} in \mathcal{R} is finite or uncountable, then \mathcal{R}° is o-minimal.*
2. *If \mathcal{R} expands $\langle \mathbb{R}, +, \cdot \rangle$, and every **open** definable subset of \mathbb{R} in \mathcal{R} has finitely many connected components, then \mathcal{R}° is o-minimal.*

In [7], van den Dries proved that if $\langle \mathcal{B}, A \rangle$ is dense pair of o-minimal structures, then every open definable subset of B in $\langle \mathcal{B}, A \rangle$ has finitely many connected components. Now clearly by item 2 above, this means that if $\langle \mathcal{B}, A \rangle$ is a dense pair of o-minimal structures, then the open core of it is o-minimal.

But van den Dries proved something more: If in $\langle \mathcal{B}, A \rangle$, \mathcal{B} is an expansion of $\langle \mathbb{R}, +, \cdot \rangle$, then the open core of $\langle \mathcal{B}, A \rangle$ is the structure \mathcal{B} . So the open core of the structure $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg} \rangle$ is $\bar{\mathbb{R}}$.

To prove this statement, van den Dries used the result of Miller and Speissegger just referred to, that the open core of $\langle \mathcal{B}, A \rangle$ is o-minimal. Then he argued as follows to prove that the open core is indeed \mathcal{B} . We present the main features of his argument below:

Statement. *If $\langle \mathcal{B}, A \rangle$ is a dense pair which expands $\bar{\mathbb{R}}$, then $\langle \mathcal{B}, A \rangle^\circ = \mathcal{B}$.*

Outline of the proof.

1. If $\langle \mathcal{B}, A \rangle$ is a dense pair, then if $F : B \rightarrow B$ is definable and continuous at all but finitely many points of B , then F is definable in \mathcal{B} .

2. Let $\mathcal{R} = \langle R, \dots \rangle$ be an o-minimal expansion of an ordered abelian group. Also let $\hat{\mathcal{R}} = \langle R, \dots \rangle$ be an \aleph_0 -saturated expansion of \mathcal{R} such that all functions $F : R \rightarrow R$ that are definable in $\hat{\mathcal{R}}$ are definable in \mathcal{R} . Then all sets $S \subseteq R^n$ definable in $\hat{\mathcal{R}}$ are definable in \mathcal{R} .
3. If $\langle \mathcal{B}, A \rangle$ expands $\bar{\mathbb{R}}$ then its open core is o-minimal (because then every open definable subset of B has finitely many connected components and we can use Proposition 5.3).
4. Let $\langle \mathcal{B}^*, A^* \rangle$ be an \aleph_0 -saturated elementary extension of $\langle \mathcal{B}, A \rangle$. Then $\langle \mathcal{B}^*, A^* \rangle^\circ$ is an \aleph_0 -saturated elementary extension of $\langle \mathcal{B}, A \rangle^\circ$ and hence o-minimal. So $\langle \mathcal{B}^*, A^* \rangle^\circ$ and \mathcal{B}^* have the same definable subsets of $(B^*)^n$ for each n (since each function $f : B^* \rightarrow B^*$ definable in $\langle \mathcal{B}^*, A^* \rangle^\circ$ is definable in \mathcal{B}^* and by the first item).
5. Now let $S \subseteq \mathbb{R}^n$ be open and definable in $\langle \mathcal{B}, A \rangle$ by a formula $\phi(\bar{y})$. We want to show that it is definable in \mathcal{B} .
Let S^* be the subset of $(B^*)^n$ defined by the same formula. Since S^* is open, and since \mathcal{B}^* and $\langle \mathcal{B}^*, A^* \rangle^\circ$ have the same definable subsets of $(B^*)^n$, there is a tuple of elements $\bar{b} \in B^*$ and an L -formula— L being the language of \mathcal{B} —say $\phi'(\bar{x}, \bar{y})$, such that $\phi'(\bar{b}, \bar{y})$ defines the same set as $\phi(\bar{y})$ does in \mathcal{B}^* .
6. Since $\langle \mathcal{B}, A \rangle \preceq \langle \mathcal{B}^*, A^* \rangle$, there is a $\bar{c} \in \mathbb{R}$ such that $\phi(\bar{y})$ and $\phi'(\bar{c}, \bar{y})$ are equivalent in $\langle \mathcal{B}, A \rangle$. So S is definable in \mathcal{B} by the formula $\phi'(\bar{c}, \bar{y})$.

We now give a brief sketch of the proof of Theorem 5.2. This proof is mainly based on topological arguments. We can summarise it stepwise as follows.

We do need to know the definitions of locally closedness and D_σ sets. These definitions are given after the sketch of the proof.

step 1 Under the conditions of the statement of the theorem, we have: a definable subset of \mathbb{R} is locally closed, if and only if, it has finitely many connected components.

step 2 We need to prove that the collection of all *finite unions* of locally closed subsets of \mathbb{R}^n , for a fixed n , is a Boolean Algebra. The proof of this part is straightforward and not difficult.

step 3 What is more difficult to prove is that the projection of any locally closed definable subset of \mathbb{R}^{n+1} on the first n coordinates is again a *finite union of* locally closed definable sets. This, together with the fact that under the assumptions of the theorem, every definable subset of \mathbb{R} is a finite union of locally closed definable sets, is equivalent to the following: ‘The collection of definable sets, in our structure, $\langle \mathbb{R}, (U_\gamma)_{\gamma \in \Gamma} \rangle$, is the collection of finite unions of locally closed definable sets’.

However, proving that the projection of a locally closed definable set is a finite union of locally closed definable sets is the painstaking part. But there is an easier thing to prove:

step 4 projections of a D_σ set are D_σ .

We also know that

step 5 Every locally closed definable set is D_σ .

So, the only thing remaining to prove is the following:

last step, 6 Every D_σ set is a finite union of locally closed definable sets.

The proof of the last step is by induction on the dimension of our D_σ set by using the sets of regular points of a given D_σ set as a set with a lower dimension on which the induction runs.

The following are the definitions we needed in the above argument. A set A is locally closed if for each $x \in A$ there is an open neighborhood U of x such that $A \cap U = \text{cl}(A) \cap U$. Equivalently A is locally closed if $A = \text{cl}(A) \cap U$ for some open set U . A subset of \mathbb{R}^n is called D_σ if it is definable and a countable increasing union of definable compact subsets of \mathbb{R}^n .

Proposition 5.4 ([24]). *Every set $S \subseteq \mathbb{R}^n$ definable in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ is a finite union of locally closed definable sets in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.*

5.3 Open sets defined in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ with special formulas

In this section we will prove that the open sets in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ which are defined by ‘special’ formulas with parameters in \mathbb{R}_{alg} can be defined in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. This is the furthest the author could prove towards the proof of the Theorem in the introduction of this chapter.

Theorem 5.5. *Let O be an open definable set in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ with parameters in \mathbb{R}_{alg} which is defined by a formula of the form*

$$\exists \bar{x} \in \mathbb{R}_{alg} \quad \exists \bar{y} \in 2^{\mathbb{Z}} \quad \phi(\bar{x}, \bar{y}, \bar{z}) \quad (***)$$

with free variables \bar{z} and $\phi(\bar{x}, \bar{y}, \bar{z})$ an L_{or} -formula. Then O can be defined in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.

Proof. Clearly we can change the order of quantifiers and assume that O is defined by the following formula:

$$\exists \bar{y} \in 2^{\mathbb{Z}} \quad \exists \bar{x} \in \mathbb{R}_{alg} \quad \phi(\bar{x}, \bar{y}, \bar{z}).$$

For a fixed \bar{y} , let $O_{\bar{y}}$ be the following set

$$\{\bar{z} \in \mathbb{R} : \exists \bar{x} \in \mathbb{R}_{alg} \quad \phi(\bar{x}, \bar{y}, \bar{z})\}$$

and $O'_{\bar{y}}$ the following:

$$\{\bar{z} \in \mathbb{R} : \exists \bar{x} \in \mathbb{R} \quad \phi(\bar{x}, \bar{y}, \bar{z})\}.$$

Consider the family of sets $\{O'_{\bar{y}}\}_{\bar{y} \in 2^{\mathbb{Z}}}$. There is a decomposition of each member of this family into cells, say:

$$O'_{\bar{y}} = [C_1 \cup \dots \cup C_{n(\bar{y})}](\bar{y}).$$

Note that if $\bar{z} \in O'_{\bar{y}}$ and $\bar{z} \in \mathbb{R}_{alg}$, then $\bar{z} \in O_{\bar{y}}$. This is because $\phi(\bar{z}, \bar{x}, \bar{y})$ is an L_{or} -formula and \mathbb{R}_{alg} is an elementary substructure of $\bar{\mathbb{R}}$. So, there is *no* open cell C_i in the decomposition of $O'_{\bar{y}}$ with $C_i \cap O_{\bar{y}} = \emptyset$.

Therefore, the open cells in the decomposition of $O'_{\bar{y}}$ are comprised of those cells C_i such that $C_i \cap O_{\bar{y}}$ is dense in C_i , and those C_i with $C_i \subseteq O'_{\bar{y}}$.

We claim that as $O = \bigcup_{\bar{y} \in 2^{\mathbb{Z}}} O_{\bar{y}}$ is open, the cells which could have contributed to this union are:

- Cells $C_i \subseteq O'_y$ with dimension n , (n for the number of z_i 's), or
- Cells C_i with lower dimensions which are on the boundary of some open cell, in which \mathbb{R}_{alg} is dense.

Call the cells C_i with $C_i \subseteq O'_y$, solid-in, and the cells C_i with $C_i \cap O'_y = \emptyset$, solid-out.

Let C_i be a non-open cell. Suppose that C_i is solid-out and separated from (with no point on the boundary of) other $C_{i'}$'s. Let \bar{z} be a point in C_i and U an open box with endpoints in \mathbb{R}_{alg} around \bar{z} . Outside C_i we have:

$$\forall \bar{x} \in \mathbb{R}_{alg} \quad \neg \phi(\bar{x}, \bar{y}, \bar{z}).$$

Consider the formula $\forall \bar{x} \quad \neg \phi(\bar{x}, \bar{y}, \bar{z})$. It is not possible for the cells in the decomposition obtained for this formula in \mathbb{R}_{alg} to have $C_i \cap U$ as their boundary (because C_i is solid-out and it has no intersection with \mathbb{R}_{alg}). So $C_i \cap U$ is inside one such cell C' (taken in \mathbb{R}^m , for m the number of z_i 's). Since $\bar{\mathbb{R}}_{alg}$ is an elementary substructure of $\bar{\mathbb{R}}$ this means that in $C_i \cap U$ there are points \bar{z} such that $\forall \bar{x} \quad \neg \phi(\bar{x}, \bar{y}, \bar{z})$ and this is obviously impossible. This means that we do not have separated solid-out cells in our decomposition of O'_y .

Now let C_i be a non-open cell which is on the boundary of some open cell $C_{i'}$. Let $\bar{z} \in C_i$ and U be an open box around \bar{z} with endpoints in \mathbb{R}_{alg} . Then C_i is not solid-out since then the boundary in \mathbb{R}_{alg} (and hence in \mathbb{R}) of the cell

$$\exists \bar{x} \quad \phi(\bar{x}, \bar{y}, \bar{z})$$

can not be C_i . So C_i is defined in \mathbb{R}_{alg} and \mathbb{R}_{alg}^m , m the number of z_i 's, is dense in it.

Now we know that the union of O'_y 's is open and there is no solid-out cells in this decomposition. This means that $\cup O'_y$ is equal to $\cup O'_y$ and hence definable in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. \square

Note that if we were dealing with the situation that O were defined with parameters outside \mathbb{R}_{alg} , then we would have no way of distinguishing which cells do contribute to the openness of O and which of them do not. For example the graph of the function $x = \pi y$ could be on the boundary of a cell and we would not find a way of knowing whether or not it contributed to the openness of a subset of \mathbb{R}^2 . So the assumption that our formula takes parameters from \mathbb{R}_{alg} plays a crucial role in the above argument.

Corollary 5.6. *Let C be a closed definable set in $\langle \bar{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$ which is defined by $\neg\psi(\bar{z})$ for $\psi(\bar{z})$ a formula of the form (***) . Then C is defined in $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$.*

Chapter 6

\mathbb{T} has NIP: Not the Independence Property

Stability, the order property, the strict order property and the independence property are closely related features of a first order theory. In this section, after briefly defining some of these properties, we will prove that our theory \mathbb{T} is dependent (or has ‘not the independence property, NIP’).

A complete theory T is called stable if it is κ -stable for some cardinal κ , which means that it has the least number of types over a given subset of size κ in any of its models. So if $M \models T$ is κ -stable and $A \subseteq M$, and $|A| = \kappa$, then the number of complete types over A in M is κ (the definition can be found in many text books like [20]). Having ‘strict order’ or a formula which defines a strict order increases the number of types and makes a theory unstable. But instability can also be as a result of *independence*. Before going further, let us review these concepts itemwise.

- T has the order property if there are a formula $\phi(\bar{x}, \bar{y})$ and sequences $(\bar{a}_i)_{i < \omega}$ and $(\bar{b}_i)_{i < \omega}$ such that (in the monster model of T):

$$\models \phi(\bar{a}_i, \bar{b}_j) \text{ if and only if } i < j.$$

(see for example [2] for this definition and the next ones).

- T has the strict order property, if there is a formula $\phi(\bar{x}, \bar{y})$, and a sequence $(\bar{b}_i)_{i \in \mathbb{N}}$ (in the monster model M) such that the following chain of sets is infinite.

$$\phi(\bar{M}, \bar{b}_1) \subseteq \dots \subseteq \phi(\bar{M}, \bar{b}_n) \subseteq \dots$$

where $\phi(\bar{M}, \bar{b}_i)$ denotes the set $\{\bar{x} : M \models \phi(\bar{x}, \bar{b}_i)\}$.

- T has the independence property, [2], if there are a formula $\phi(\bar{x}, \bar{y})$ and sequences $(\bar{a}_i)_{i < \omega}$ and $(\bar{b}_\sigma)_{\sigma \in 2^\omega}$ such that (in the monster model M of T):

$$\models \phi(\bar{a}_i, \bar{b}_\sigma) \text{ if and only if } \sigma(i) = 0.$$

The finite version of the same definition is as follows. A formula $\phi(\bar{x}, \bar{y})$ is called independent (for T) if the following happens. In every model M of T , for each $n < \omega$, there is set $X = \{b_1, \dots, b_n\}$ of elements of M such that for each subset Y of X , there is an element $\bar{a}_Y \in M$ such that

$$M \models \phi(\bar{a}_Y, \bar{b}_i) \text{ if and only if } b_i \in Y.$$

T is then called independent if some formula is independent for T .

- For a theory T the following are equivalent:

$$\left\{ \begin{array}{l} \text{Instability} \\ \text{Order property} \\ \text{Strict order property or independence property} \end{array} \right.$$

Note that none of the two in the third item implies the other.

- If T is such that no atomic formula of the form $\phi(x, \bar{y})$ has the independence property then T is NIP ([1] page 5 Corollary 10). A very special case of this remark shows that strongly minimal theories are NIP. C -minimal theories also have NIP.

Figure 6.1 that I have borrowed from Pablo Kobeda, provides a good description of these notions (definitions of some of the items in the diagram are not given here). Although having informally defined independence makes the definition of *dependence* redundant, we give the following definition of dependence because it is easier to work with in this context.

Definition 6.0.1. *Let T be a complete theory in the language L and \mathbf{M} a monster model of T . Let $\phi(x_1, \dots, x_m, y_1, \dots, y_p)$ be an L -formula, (\bar{a}_i) an indiscernible sequence and \bar{b} an element. Let X and Y be the following definable sets:*

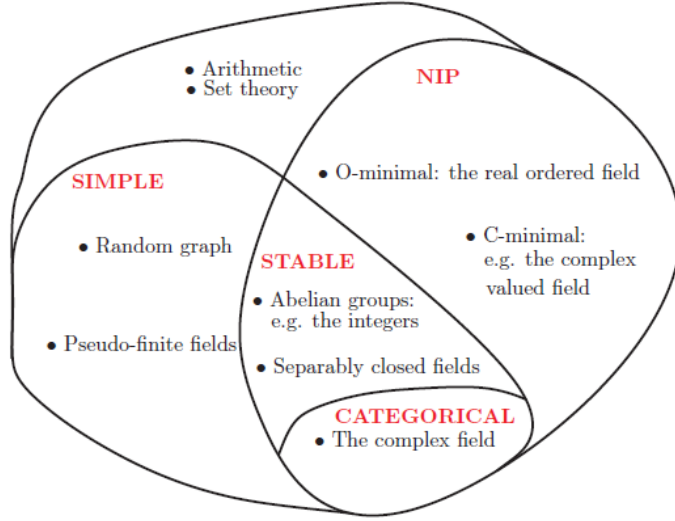


Figure 6.1: Classification of first order theories, figure borrowed from Pablo Kobeda

- $X = \{\bar{x} : \mathbf{M} \models \phi(\bar{x}, \bar{b})\}$
- $Y = \{\bar{x} : \mathbf{M} \models \neg\phi(\bar{x}, \bar{b})\}$

We say $\phi(\bar{x}, \bar{y})$ is dependent for (\bar{a}_i) and (\bar{b}) if there is an $N \in \mathbb{N}$ such that one of the following happens:

- for all $i > N$, $\bar{a}_i \in X$.
- for all $i > N$, $\bar{a}_i \in Y$.

Definition 6.0.2. Let T be a complete theory in the language L and \mathbf{M} a monster model of T . Let $\phi(x_1, \dots, x_m, y_1, \dots, y_p)$ be an L -formula. We say that $\phi(\bar{x}, \bar{y})$ is dependent (in T) if for every indiscernible sequence $(\bar{a}_i)_{i \in \mathbb{N}}$ and every tuple \bar{b} , ϕ is dependent for (\bar{a}_i) and \bar{b} . A theory T is dependent if every L -formula is dependent in T .

The following proposition is from [14] and we have maintained their notation. The assumptions under which the statement of the proposition holds are as follows:

$\mathcal{A} = \langle A, <, \dots, \rangle$ is an o-minimal structure in the language L and B a subset of A . The structure $\langle \mathcal{A}, B \rangle$ is considered in the language $L(U) = L \cup \{U\}$, where U is a unary relation symbol not in L . We let T_B denote the $L(U)$ -theory of $\langle \mathcal{A}, B \rangle$.

Proposition 6.1 ([14]). *The theory T_B of dense pairs $\langle \mathcal{A}, B \rangle$ is dependent if in every model (\mathcal{M}, N) of it, the following conditions holds:*

- Every subset of N^n definable in $\langle \mathcal{M}, N \rangle$ is a boolean combination of the sets of the form $S \cap K$, where $S \subseteq M^n$ is definable in \mathcal{M} and $K \subseteq M^n$ is \emptyset -definable in $\langle \mathcal{M}, N \rangle$.
- Every subset of M definable in $\langle \mathcal{M}, N \rangle$ is a boolean combination of subsets of M defined by

$$\exists \bar{y} \quad (U(\bar{y}) \wedge \phi(x, \bar{y}))$$

for ϕ a quantifier free L -formula.

- Every open subset of M definable in $\langle \mathcal{M}, N \rangle$ is a finite union of intervals.

The conditions of the above proposition, as we know, are all satisfied by a dense pair of o-minimal structures as in the setting of van den Dries in [7], hence the following corollary.

Corollary 6.2. *The theory of a dense pair of o-minimal structures (as described in Chapter 3) is dependent.*

In [14] they also provide us with conditions under which the theory of a structure with a predicate for a discrete group is dependent. Note that in the following proposition the left and right angle symbols indicate the T -structure generated by elements they enclose.

Proposition 6.3. *Let T be an o-minimal theory extending the theory of ordered abelian groups in the language L . Let λ be a unary function symbol and suppose that $T(\lambda)$ is a complete $L(\lambda)$ theory extending T . If the following conditions hold, then $T(\lambda)$ is dependent.*

- $T(\lambda)$ has quantifier elimination.
- For every $(A, \lambda) \models T(\lambda)$, $B \preceq A$ with $\lambda(B) \subseteq B$ and every $c_1, \dots, c_n \in A$, there are $d_1, \dots, d_n \in A$ such that

$$\lambda(B\langle c_1, \dots, c_n \rangle) \subseteq \text{the universe of } \langle \lambda(B), d_1, \dots, d_n \rangle.$$

- Let M be a monster model. Let f and g be L -terms of arities $m+k$ and $n+l$ respectively, (\bar{a}_i) an indiscernible sequence in M^m with $a_{i,1}, \dots, a_{i,n} \in \lambda(M)$ for

every $i \in \omega$, $\bar{b}_1 \in M^k$ and $\bar{b}_2 \in \lambda(M)^l$. Then the following set is finite or co-finite:

$$\{i \in \omega : M \models \lambda(f(\bar{a}_i, \bar{b}_1)) = g(a_{i,1}, \dots, a_{i,n}, \bar{b}_2)\}.$$

The clear corollary of the above Proposition is the following.

Corollary 6.4 ([14]). *The theory of $\langle \bar{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ is dependent.*

We are now at the point to prove that as a result of (the proofs of) Corollaries 6.2 and 6.4, the theory \mathbb{T} is also dependent.

Main Theorem of this chapter. *The theory \mathbb{T} is dependent.*

Proof. Let $\mathbb{M} = \langle \tilde{M}, G, A, \lambda, P_n \rangle$ be a monster model of \mathbb{T} . Let $(a_i)_{i \in \mathbb{N}}$ be an indiscernible sequence. Let $\phi(a_i, \bar{b}) = \exists \bar{z} (U(\bar{z}) \wedge \psi(a_i, \bar{b}, \bar{z}))$ be a formula in \mathbb{L} with parameters \bar{b} .

What we need to prove is that the following set $J \subseteq \mathbb{N}$ is finite or co-finite:

$$J := \{i \in \mathbb{N} : \mathbb{M} \models \phi(a_i, \bar{b})\}.$$

We break the proof of this down to the following cases.

Case 1. Let all a_i 's be in G . Let X be the set $\{x \in G : \phi(x, \bar{b})\}$. Then, by Corollary 4.19, $X = Y \cap G$ for Y a definable subset (possibly with other parameters than \bar{b}) in M in the language $\tilde{L} \cup \{\lambda\}$. So we have:

$$\begin{aligned} \mathbb{M} \models \phi(a_i, \bar{b}) &\leftrightarrow a_i \in X \\ &\leftrightarrow a_i \in G \cap Y \\ &\leftrightarrow a_i \in Y \end{aligned}$$

Since $a_i \in Y$ is an $\tilde{L} \cup \{\lambda\}$ formula, by the dependency of the theory of $\langle \tilde{M}, \alpha^{\mathbb{Z}} \rangle$, only finitely or cofinitely many a_i 's can be in Y .

Case 2. Let a_i 's all lie outside G and $\bar{b} \in G$. Fix $\bar{z} \in G$. Define $A_{\bar{z}} = \{x : \mathbb{M} \models \psi(x, \bar{b}, \bar{z})\}$. Then $A_{\bar{z}}$ is the union of an open set and finitely many discrete sets:

$$A_{\bar{z}} = O \cup D_1 \cup \dots \cup D_n$$

Let $a \in (a_i)_{i \in \mathbb{N}}$. If $a \in D_1 \cup \dots \cup D_n$ then by lemma 4.32, $a \in G$ which is contradictory with our assumption that $a_i \notin G$. So, for each $\bar{z} \in G$,

$$a \in A_{\bar{z}} \leftrightarrow a \in \text{Int}(A_{\bar{z}}).$$

We now have:

$$\begin{aligned} \mathbb{M} \models \exists \bar{z} \in G \quad \psi(a, \bar{b}, \bar{z}) &\Leftrightarrow \\ a \in \bigcup_{\bar{z} \in G} A_{\bar{z}} &\Leftrightarrow a \in \bigcup_{\bar{z} \in G} \text{Int}(A_{\bar{z}}). \end{aligned}$$

As $\bigcup_{\bar{z} \in G} \text{Int}(A_{\bar{z}})$ is an open definable set, by Remark 4.52.1 in it is defined by an $\tilde{L} \cup \{\lambda\}$ -formula. Since by Proposition 6.3, the theory of $\langle \tilde{M}, \alpha^{\mathbb{Z}} \rangle$ is dependent there are only finitely or cofinitely many a_i 's in this set and the statement of the theorem in this case is proved.

Before proceeding with the other cases, we need the following lemma from [14].

Lemma 6.5 ([14]). *Let M be a monster model of a theory T and $(\bar{a}_i)_{i \in \mathbb{N}}$ an indiscernible sequence. Let $\phi(\bar{x}, \bar{y})$ be a formula such that $M \models \exists \bar{y} \quad \phi(\bar{a}_i, \bar{y})$ for some i . Then there is an indiscernible sequence (\bar{b}_i) such that for each i , $M \models \phi(\bar{a}_i, \bar{b}_i)$.*

Now we can continue the rest of the proof.

Case 3. Let $(a_i)_{i \in \mathbb{N}}$ be an indiscernible sequence of elements not in G where the set $\{a_i : i \in \mathbb{N}\}$ is dcl dependent over G in the pregeometry of \tilde{T} . Then for some i , there exists an i_0 such that $a_i \in \text{dcl}(G, a_0, \dots, a_{i_0})$. Fixing this i , there exists an \tilde{L} -definable function $f : M \rightarrow M$ such that

$$\exists \bar{c} \quad \bar{c} \in G \wedge a_i = f(\bar{c}, a_0, \dots, a_{i_0}).$$

Since $(a_i)_{i \in \mathbb{N}}$ is indiscernible the above holds for all i 's (and the same set $\{a_0, \dots, a_{i_0}\}$). So by Lemma 6.5 there is an indiscernible sequence $(\bar{g}_i)_{i \in \mathbb{N}}$ of elements of G such that

$$\forall i \quad f(\bar{g}_i, a_0, \dots, a_{i_0}) = a_i.$$

Hiding the parameters a_0, \dots, a_{i_0} in the notation of f , we have

$$\mathbb{M} \models \phi(a_i, \bar{b}) \leftrightarrow \phi(f(\bar{g}_i), \bar{b}).$$

Since (\bar{g}_i) is an indiscernible sequence of elements in G , and by a very similar proof as for the first case, there are finitely or cofinitely many (\bar{g}_i) 's for which $\mathbb{M} \models \phi(f(\bar{g}_i), \bar{b})$. So there are finitely or cofinitely many a_i 's for which $\mathbb{M} \models \phi(a_i, \bar{b})$.

Case 4. Consider the case where $(a_i)_{i \in \mathbb{N}}$ are dcl-independent from G . In this case, for the reason which follows, we have for each i , $a_i \notin \text{dcl}(G, \bar{b})$. As $\text{rank}\{a_i, i \in \mathbb{N}\} | G$ is

infinite, if any of $a_i \in \text{dcl}(G, \bar{b})$, then we have $\text{rank}\{a_i, i \in \mathbb{N}\} | G \leq \text{rank}(\bar{b} | G)$ and this is impossible.

$G(\bar{b})$ is closed under λ since G contains λ of all elements in M . So $\langle \tilde{M}, G(\bar{b}), \dots \rangle$ is a model of \mathbb{T} and by Theorem 4.30, $G(\bar{b})$ is definably closed. The rest of the proof is as in case 2:

Let $A_{\bar{z}} = \{x : \mathbb{M} \models \psi(x, \bar{b}, \bar{z})\}$ for a fixed $\bar{z} \in G$. Then $A_{\bar{z}} = O \cup D_1 \dots \cup D_n$ where $D_1 \cup \dots \cup D_n$ is a finite union of discrete sets where this finite union is definable with parameters in $G(\bar{b})$. If $a_i \in A_{\bar{z}}$ then $a_i \in O$ (since as discussed in case 2, $D_1 \cup \dots \cup D_n \subseteq G(\bar{b})$). So $\mathbb{M} \models \phi(a_i, \bar{b}) \leftrightarrow a_i \in \bigcup_{\bar{z} \in G} \text{Int}(A_{\bar{z}})$. The set $\bigcup_{\bar{z} \in G} \text{Int}(A_{\bar{z}})$ is an open definable set and hence is definable in $\tilde{L} \cup \{\lambda\}$ and again by dependency of the theory of $\langle \tilde{M}, \alpha^{\mathbb{Z}} \rangle$ the result follows. \square

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