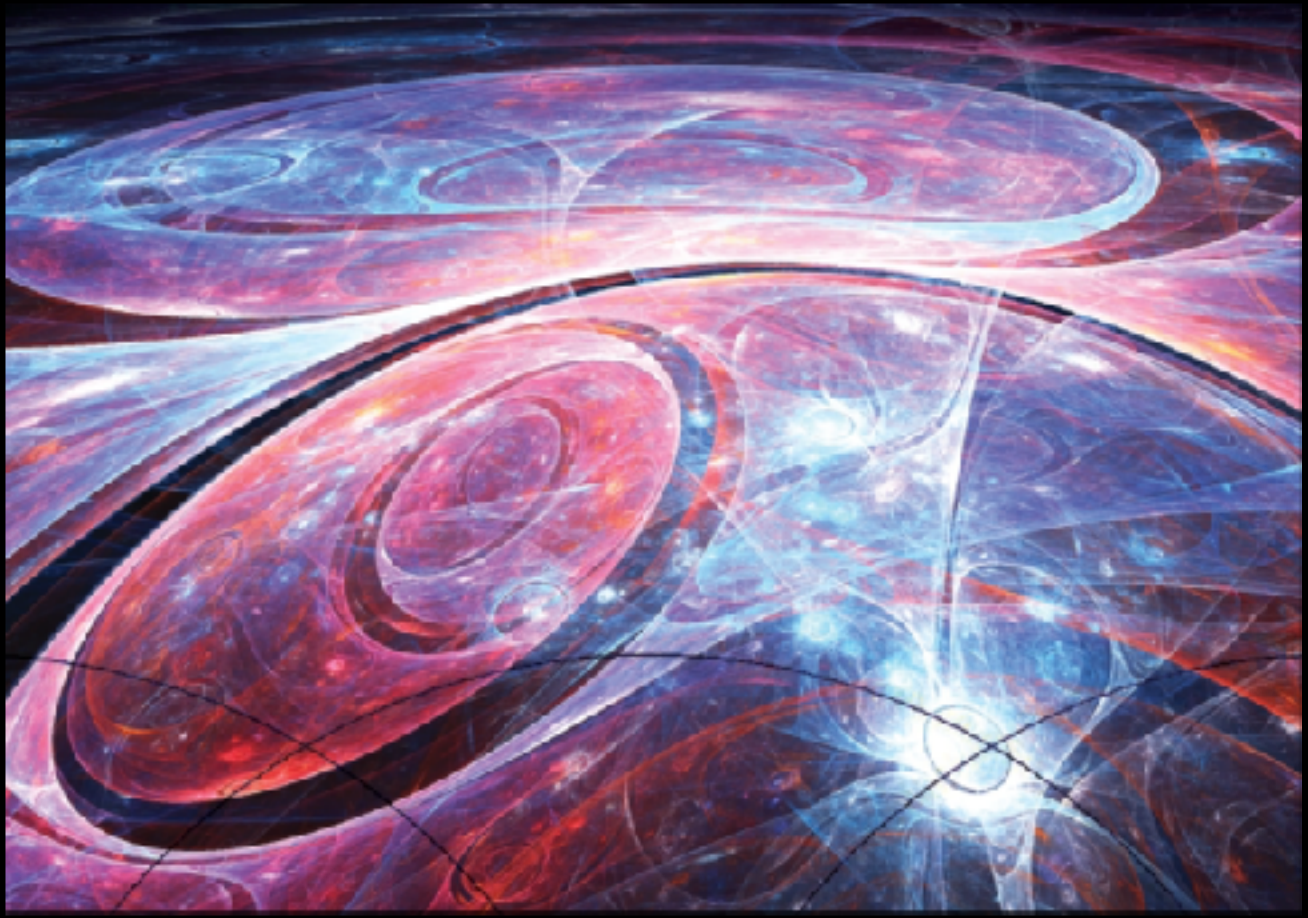
BPS-states and small automorphic representations

Daniel Persson
Chalmers University of Technology



Darmstadt-Erlangen-Freiburg seminar on CFT
Freiburg
January 19, 2018

Mainly based on our recent papers

Small automorphic representations and degenerate Whittaker vectors

w/ Gustafsson, Kleinschmidt; J. Num. Theory **166** (2016), 344-399

Eisenstein series and automorphic representations - with applications in string theory

w/ Fleig, Gustafsson, Kleinschmidt

CUP, Cambridge Studies in Advanced Mathematics

to appear on ***June 25, 2018!***

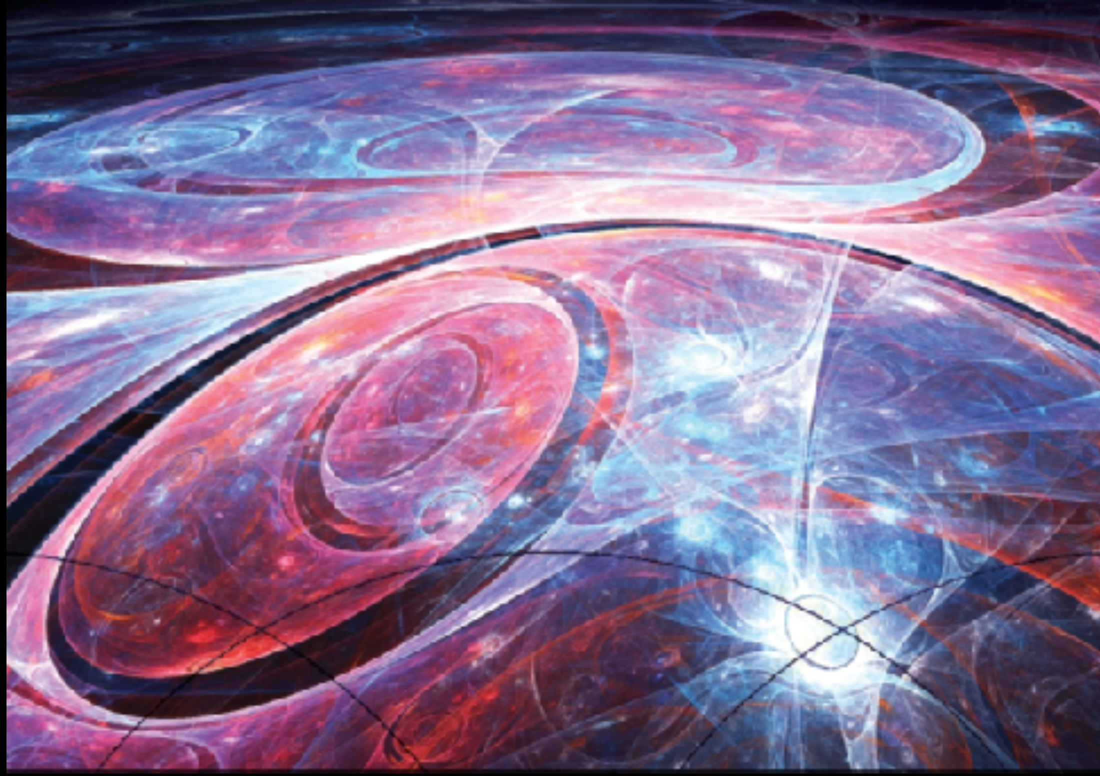
Fourier coefficients attached to small automorphic representations of $SL_n(\mathbb{A})$

w/ Ahlén, Gustafsson, Kleinschmidt, Liu; submitted to J. Num. Theory

(preprint: [arXiv:1707.08937])

and work in progress with Gourevitch & Sahi

Outline



1. Motivation

2. Automorphic forms and representation theory

3. Small representations: main results

4. Outlook

1. Motivation

Fourier coefficients of modular forms encode interesting **arithmetic information**

- **Classical theta function** $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2}$ $\quad \tau \in \mathbb{H}$

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Fourier expansion

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$$\frac{G_{2w}(\tau)}{2\zeta(2w)} = 1 + \frac{2}{\zeta(1-2w)} \sum_{k=1}^{\infty} \sigma_{2w-1}(k) e^{2\pi i k\tau}$$

divisor sum:

$$\sigma_{2w-1}(k) = \sum_{d|k} d^{2w-1}$$

Higher rank automorphic forms live on

$$G(\mathbb{Z})\backslash G/K$$

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What do **Fourier coefficients** encode in this case?

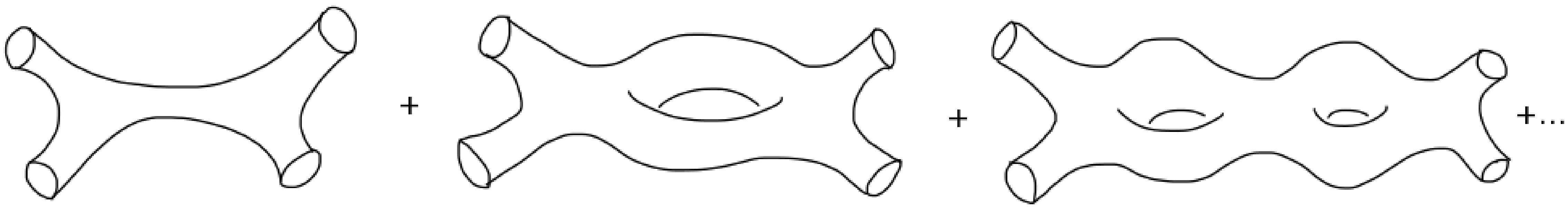
- By the **Langlands-Shahidi method**, the Fourier coefficients of Eisenstein series on simple Lie groups G give rise to **automorphic L-functions**.

- Fourier coefficients attached to **small automorphic representations** play a crucial role in establishing examples of functoriality using **theta correspondences**

- The Fourier coefficients of Eisenstein series also encode **non-perturbative effects in string theory**!

String amplitudes

Understand the structure of **string interactions**



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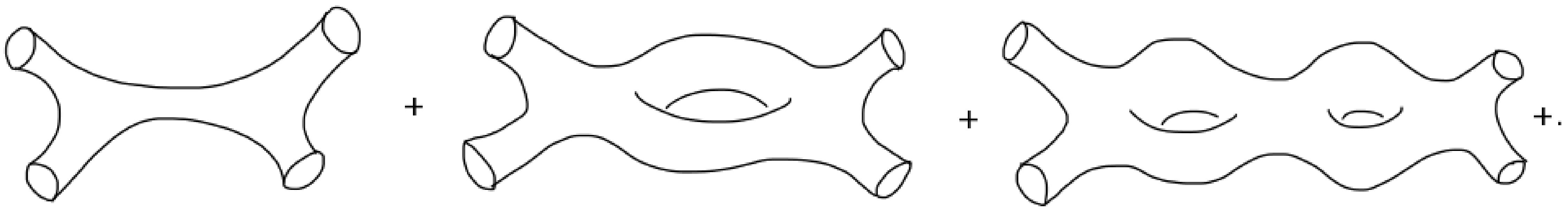

Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

$\longrightarrow$ amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

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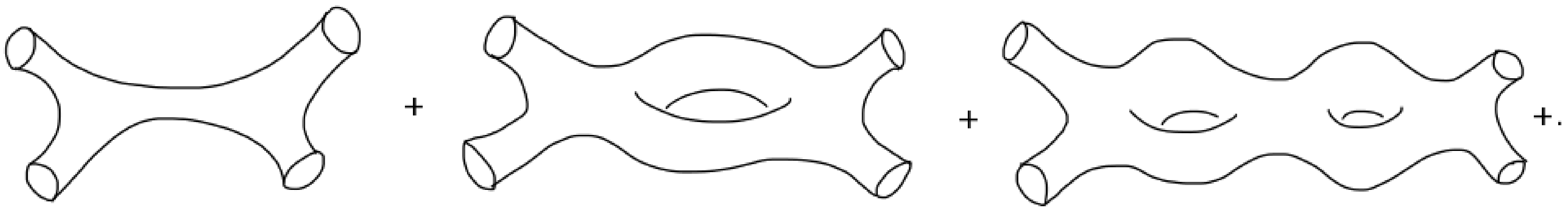

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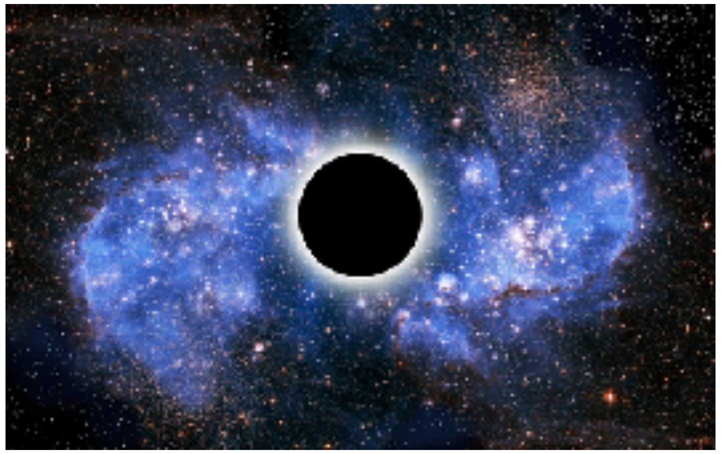
$\longrightarrow$ amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

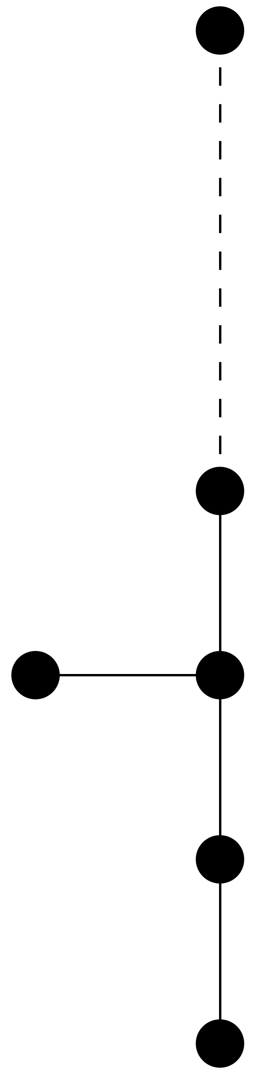
Symmetry constrains interactions, leads to insights about:

- ultraviolet properties of gravity
- non-perturbative effects (black holes, instantons)
- novel mathematical predictions from physics



Toroidal compactifications yield the chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]



D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$(\mathrm{Spin}(5) \times \mathrm{Spin}(5))/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Physical couplings are given by **automorphic forms** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Green, Gutperle, Sethi, Vanhove, Kiritsis, Pioline, Obers, Kazhdan, Waldron, Basu, Russo, Cederwall, Bao, Nilsson, D.P., Lambert, West, Gubay, Miller, Fleig, Kleinschmidt,…

Examples

$$\int d^{11-n}x\sqrt{G}f_0(g)\mathcal{R}^4$$

$$\int d^{11-n}x\sqrt{G}f_4(g)D^4\mathcal{R}^4$$

These partition functions are Eisenstein series attached to **small automorphic representations** of G.

[Green, Miller, Vanhove][Pioline]

minimal automorphic representation π_{min}	**next-to-minimal automorphic representation** π_{ntm}

1/2 - BPS **1/4 - BPS**

2. Automorphic forms and representation theory

Data:

- $G(\mathbb{R})$ real simple Lie group (e.g. $SL(n,\mathbb{R})$)
- $G(\mathbb{Z}) \subset G$ arithmetic subgroup (e.g. $SL(n,\mathbb{Z})$)

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Definition:

An **automorphic form** is a smooth function $\varphi : G \longrightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}), \;\; \varphi(\gamma g) = \varphi(g)$

2. φ is an eigenfunction of the ring of inv. diff. operators on G

3. φ has well-behaved growth conditions

Example: Eisenstein series on $SL(2,\mathbb{R})$

$$E(s,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|m\tau+n|^{2s}} \qquad s\in\mathbb{C}$$

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→ a function on $\mathbb{H} = \{\tau = x+iy \in \mathbb{C} \,|\, y>0\}$

→ invariant under $\tau \mapsto \gamma\cdot\tau = \dfrac{a\tau+b}{c\tau+d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

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→ invariant under $\tau \mapsto \gamma\cdot\tau = \dfrac{a\tau+b}{c\tau+d}$

→ converges absolutely for $\Re s > 1$

→ $\Delta_{\mathbb{H}} E_s = s(s-1)E_s$

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But to fit the definition of an automorphic form we must try to view this as a **function on the group** $SL(2,\mathbb{R})$

Iwasawa decomposition: Any element $g \in SL(2,\mathbb{R})$ can be represented in the form

$$g = nak = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

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Using this fact we can lift the Eisenstein series to a function on $SL(2,\mathbb{R})$ via:

$$E \longmapsto \varphi_E(g) = \varphi_E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = E(s, x+iy)$$

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$\Rightarrow$ $\varphi_E(\gamma g) = \varphi_E(g)$ $\qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

Suppose that instead we start with a **holomorphic modular form** of weight k

$$f\left(\frac{a\tau+b}{c\tau+b}\right)=(c\tau+d)^k f(\tau)$$

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section of a line bundle on $\mathbb{H}$: f	$\longleftrightarrow$	function on $SL(2,\mathbb{R})$: φ_f

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Under the action of $\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\in SO(2)$ we now pick up a phase:

$$\varphi_f\left(g\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\right)=e^{ik\theta}\varphi_f(g)$$

first hint of some representation theory underlying modular forms

Automorphic representations

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$$\bigcup$$

$$L^2(G(\mathbb{Z})\backslash G(\mathbb{R}))$$

Automorphic representations

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The group G acts on this space via the **right-regular representation**:

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for $\varphi \in \mathcal{A}$ and $h, g \in G$

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Definition: *An* ***automorphic representation*** π *of* G *is an irreducible representation in the decomposition of* $\mathcal{A}$ *under the right-regular action.*

[Gelfand, Graev, Piatetski-Shapiro][Langlands]…

Toy model: Fourier analysis on $\mathbb{Z}\backslash\mathbb{R} \cong S^1$

Any function $f \in C^\infty(\mathbb{Z}\backslash\mathbb{R})$ can be decomposed into a Fourier series:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \psi_k(x)$$

$\psi_k \,:\, \mathbb{Z}\backslash\mathbb{R} \to U(1)$ $\qquad$ $\psi_k(x) = e^{2\pi i k x} \quad k \in \mathbb{Z},\, x \in \mathbb{R}$

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Moderate growth: restrict to **square integrable functions**

$$L^2(\mathbb{Z}\backslash\mathbb{R}) = \{f \in C^\infty(\mathbb{Z}\backslash\mathbb{R}) \mid \sum_{k\in\infty} |c_k|^2 < \infty\}$$

$G = \mathbb{R}$ acts on $L^2(\mathbb{Z}\backslash\mathbb{R})$ via the regular representation

$$(\rho(y)f)(x) = f(x+y)$$

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"automorphic representation"

Automorphic representations

$$\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R})) = \mathcal{A}_{discrete} \oplus \mathcal{A}_{continuous}$$

- $\mathcal{A}_{discrete}$: generated by cusp forms
  (and residues of Eisenstein series)

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)du = 0$$

all unipotents
$U \subset G$

- $\mathcal{A}_{continuous}$: generated by Eisenstein series

Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

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$$
\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \arrow[d, "\text{Restrict}"] \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"] & \text{Fourier coefficient}
\end{tikzcd}
$$

Adelic framework

For each **prime number** p

$$\begin{tikzcd} & \mathbb{R} \\ \mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[dr, "p\text{-adic norm}"'] & \\ & \mathbb{Q}_p \end{tikzcd}$$

real numbers

p-adic numbers

$$\mathbb{Q}_\infty = \mathbb{R}$$

completions of $\mathbb{Q}$

Adelic framework

For each **prime number** *p*

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real numbers

p-adic numbers

$$\mathbb{Q}_\infty = \mathbb{R}$$

The adeles are then defined as

$$\mathbb{A} = \mathbb{R} \times \prod_{p\,\mathrm{prime}\,<\infty}{}' \mathbb{Q}_p$$

global

local

$$x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

Adelic framework

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$\mathbb{R}$ *real numbers*

$\mathbb{Q}_p$ p-adic numbers

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global local

$$x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

$$\mathbb{Q} \hookrightarrow \mathbb{A}$$

$$\mathbb{Q} \subset \mathbb{A}$$ **discrete embedding**

$$q \mapsto (q; q, q, q, \ldots)$$

analogous to: $\mathbb{Z} \subset \mathbb{R}$

Adelic framework

For each **prime number** *p*

$$\mathbb{Q} \xrightarrow{\text{Euclidean norm}} \mathbb{R} \quad \textit{real numbers}$$

$$\mathbb{Q} \xrightarrow{p\text{-adic norm}} \mathbb{Q}_p \quad p\text{-adic numbers}$$

$$\mathbb{Q}_\infty = \mathbb{R}$$

The adeles are then defined as

global local

$$\mathbb{A} = \mathbb{R} \times \prod_{p \,\mathrm{prime} < \infty}' \mathbb{Q}_p \qquad x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

$$\mathbb{Q} \hookrightarrow \mathbb{A} \qquad \boxed{\mathbb{Q} \subset \mathbb{A}}$$

much easier to work with since $\mathbb{Q}$ is a **field**.

$$q \mapsto (q; q, q, q, \ldots)$$

analogous to: $\mathbb{Z} \subset \mathbb{R}$

Adelic framework

It is desirable to formulate all this instead in terms of the adeles

$$\mathbb{A} = \mathbb{R} \times \prod_{p<\infty}' \mathbb{Q}_p$$

restricted
direct product

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$$\mathbb{A} = \mathbb{R} \times {\prod_{p<\infty}}' \mathbb{Q}_p$$

restricted direct product

- $\mathbb{Q} \subset \mathbb{A}$ diagonal embedding is discrete $(\text{c.f. } \mathbb{Z} \subset \mathbb{R})$
- $G(\mathbb{A}) = G(\mathbb{R}) \times {\prod_{p<\infty}}' G(\mathbb{Q}_p)$
- $K_{\mathbb{A}} = K_\infty \times \prod_{p<\infty} G(\mathbb{Z}_p)$ maximal compact subgroup
- $\mathcal{A}\big(G(\mathbb{Z})\backslash G(\mathbb{R})\big) \longrightarrow \mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$

Adelic framework

(completed) **Riemann zeta function**:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Adelic framework

(completed) **Riemann zeta function**:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-s/2}\Gamma(s/2) \prod_{p \,\mathrm{prime} < \infty} \frac{1}{1-p^{-s}}$$

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$$= \int_{\mathbb{A}} \gamma_{\mathbb{A}}(x)|x|_{\mathbb{A}}^s dx$$

In his famous thesis, Tate gave elegant new proofs of the **functional equation and analytic continution** of $\xi(s)$ using these techniques

Let (π, V) be an automorphic representation.

Let σ be an irreducible representation of $K_{\mathbb{A}}$

Definition: (π, V) is called ***admissible*** if $\dim V[\sigma] < \infty$

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Theorem [Flath]**:** *For* (π, V) *admissible we have the Euler product:*

$$(\pi, V) = \bigotimes_p (\pi_p, V_p)$$

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- The action of Hecke operators naturally encoded in the local representations π_p of $G(\mathbb{Q}_p)$
- Many calculations reduce to simpler local calculations
- Functional relations more natural in adelic picture

(c. f. Tate's adelic treatment of the Riemann zeta function)

Eisenstein series

- G simple Lie group over $\mathbb{Q}$ $\qquad (G = SL(n, \mathbb{Q}))$
- $B = AN$ **Borel subgroup**

$$A = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \qquad N = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

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- quasi-character: $\quad \chi : B(\mathbb{Q})\backslash B(\mathbb{A}) \to \mathbb{C}^{\times}$

$$\chi(na) = \chi(a) := e^{\langle \lambda+\rho | H(a)\rangle} = \prod_p \chi_p(a_p)$$

$$H : A \to \mathfrak{h} \qquad\qquad \lambda \in \mathfrak{h}^{\star} \otimes \mathbb{C}$$

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- $\chi(g) = \chi(nak) = \chi(a)$ $\qquad$ spherical

induced representation (principal series)

$$I(\chi) = \{f : G(\mathbb{A}) \to \mathbb{C} \mid f(bg) = \chi(b) f(g),\, b \in B(\mathbb{A})\}$$

$$= \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi = \prod_p I_p(\chi_p)$$

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Gelfand-Kirillov (functional) dimension

$$\mathrm{GKdim}(I_\infty) = \dim_{\mathbb{R}} G(\mathbb{R}) - \dim_{\mathbb{R}} B(\mathbb{R}) = \dim_{\mathbb{R}} N(\mathbb{R})$$

Eisenstein series

The theory of Eisenstein series gives a $G(\mathbb{A})$ -equivariant embedding

$$E \,:\, I(\lambda) \longrightarrow \mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$$

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It **converges absolutely** in the Godement range

$$\langle \lambda | H_\alpha \rangle > 1, \;\; \forall \alpha \in \Pi$$

Can be continued to a **meromorphic function** on $\mathfrak{h}^\star \otimes \mathbb{C}$ [Langlands]

Whittaker-Fourier coefficients

Introduce a **unitary character** $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$

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- $\psi([N,N]) = 1$ since it is a group homomorphism
- ψ is thus only non-trivial on the abelianization $[N,N]\backslash N$
- $\psi(n) = \psi(\exp[\sum_{\alpha>0} u_\alpha E_\alpha]) = e^{2\pi i \sum_{\alpha\in\Pi} m_\alpha u_\alpha}$ $\quad m_\alpha \in \mathbb{Q}$

  $u_\alpha \in \mathbb{A}$

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- We say that ψ is:

generic if $\quad m_\alpha \neq 0 \quad \forall \alpha \in \Pi$

degenerate if $\quad m_\alpha \neq 0 \quad \text{for some (but not all) } \alpha \in \Pi$

Whittaker-Fourier coefficients

For such a **unitary character** $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$

we have the **Whittaker-Fourier coefficient**

$$W_\psi(\lambda, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, ng)\,\overline{\psi(n)}\,dn$$

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- $$W_\psi(nak) = \psi(n) W_\psi(a)$$ determined by its restriction to A

- $$W_\psi(\lambda, g) \in Wh_\psi(\lambda) \subset \mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})} \psi$$ *unique* Whittaker model $I(\lambda) \cong Wh_\psi(\lambda)$

[Gelfand, Graev][Jacquet, Langlands][Piateski-Shapiro][Shalika][Rodier]

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- When ψ is generic we say that W_ψ is a **generic** coefficient
- When ψ is degenerate we say that W_ψ is a **degenerate** coefficient

[Moeglin, Waldspurger][Matumoto]

Holomorphic modular form $f(\tau)$ $\tau \in \mathbb{H} \cong SL(2,\mathbb{R})/U(1)$

$$\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(e^{xE_\alpha}) = e^{2\pi i m x}$$

$$x \in \mathbb{R} \qquad m \in \mathbb{Z}$$

ψ generic $\longleftrightarrow$ $m \neq 0$

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$$W_m(\tau) = \int_0^1 f(\tau+1) e^{-2\pi i m u}\, du$$

$x \in \mathbb{R}$ $m \in \mathbb{Z}$

$$= c(m) e^{2\pi i m \tau}$$

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Non-holomorphic Eisenstein series

$$E(s,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)=1}} \frac{y^s}{|m+n\tau|^{2s}}$$

$$s \in \mathbb{C}$$

$$\tau = x + iy \in \mathbb{H}$$

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$s \in \mathbb{C}$

$\tau = x + iy \in \mathbb{H}$

$$W_m(\tau) = \int_0^1 E(s,\tau+u) e^{-2\pi i m u} du = \frac{\sqrt{y}}{\xi(2s)} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}$$

$$\sigma_{1-2s}(m) = \sum_{d|m} d^{1-2s}$$

(modified) Bessel function

Theorem [Jacquet, Langlands]**:** *The generic Whittaker coefficient is Eulerian*

$$W_\psi(\lambda, g) = \int_{N(\mathbb{A})} \chi(w_0 n g)\, \overline{\psi(n)}\, dn = \prod_p W_{\psi_p}$$

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$w_0 =$ longest element of $W(\mathfrak{g})$

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$$W_{\psi_\infty} = \int_{N(\mathbb{R})} \chi_\infty(w_0 n a_\infty)\overline{\psi_\infty(n)}\, dn$$

$$W_{\psi_p} = \int_{N(\mathbb{Q}_p)} \chi_p(w_0 n a_p)\overline{\psi_p(n)}\, dn$$

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Theorem [Shintani, Casselman-Shalika]:

The (unramified) p-adic Whittaker function W_{ψ_p} *is given by the Weyl character formula of the Langlands dual group* ${}^L G$

Example: $G = SL(n, \mathbb{Q}_p)$ $\quad {}^LG = SL(n, \mathbb{C})$

$$a^J = \begin{pmatrix} p^{j_1} & & \\ & \ddots & \\ & & p^{j_n} \end{pmatrix} \in A(\mathbb{Q}_p)/A(\mathbb{Z}_p) \qquad J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$$

$$\chi(a^J) = \prod_{i=1}^{n} p^{-s_i j_i} \qquad s_i \in \mathbb{C}$$

$$A_\chi = \begin{pmatrix} p^{-s_1} & & \\ & \ddots & \\ & & p^{-s_n} \end{pmatrix} \in {}^LA(\mathbb{C})$$

Satake-Langlands parameter

$$W_\psi(\chi, a^J) = \begin{cases} \sharp\, \mathrm{ch}_J(A_\chi) & j_1 \geq \cdots \geq j_n \\ 0 & \text{otherwise} \end{cases}$$

3. Small representations: main results

General Fourier coefficients

- $P = LU$ **standard parabolic** of G

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Example:

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\} = LU$$

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General Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$

General Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$
- For any automorphic form φ we have the U ***-coefficient***

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

Also known as "unipotent period integrals".

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

- These are **not Eulerian** in general (no CS-formula)
- $F_{\psi_U}(ug) = \psi_U(u)F_{\psi_U}(g) \qquad \forall u \in U$
- Very difficult to compute in general
- Idea: consider **special types** of automorphic representations

Character variety orbits

Crucial obervation: For any $\gamma \in L(\mathbb{Q})$ we have

$$F_{\psi_U}(\gamma g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(\gamma^{-1} u \gamma g)\overline{\psi_U(u)} du$$

$$= \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(\gamma u \gamma^{-1})} du =: F_{\psi_U^\gamma}(g)$$

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Hence, Fourier coefficients are organized into **orbits** under the adjoint action of the Levi $L(\mathbb{Q})$ on the unipotent $U(\mathbb{Q})$

Sufficient to determine the coefficient for **one representative** in each **Levi orbit** of ψ_U

Each such Levi orbit can be embedded into a **nilpotent** G **-orbit**.

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Wavefront set: $WF(\pi) = \{\mathcal{O} \,|\, F_{\mathcal{O}} \neq 0\}$

The wavefront set of a representation is the set of **nilpotent orbits** which have **non-zero Fourier coefficients**

Minimal automorphic representations

Definition: *An automorphic representation*

$$\pi = \bigotimes_{p \leq \infty} \pi_p$$

is minimal if each factor π_p *has smallest non-trivial Gelfand-Kirillov dimension.*

[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]....

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Automorphic forms $\varphi \in \pi_{min}$ are characterised by having **very few non-vanishing Fourier coefficients**.

[Ginzburg, Rallis, Soudry]

The **wavefront set** of the minimal representation is

$$WF(\pi_{min}) = \overline{\mathcal{O}_{min}}$$

where $\mathcal{O}_{min}$ is the **smallest non-trivial nilpotent orbit**:

$$\mathcal{O}_{min} = G \cdot E_\alpha$$

α
simple root

The **Gelfand-Kirillov dimension** is:

$$\mathrm{GKdim}(\pi_{min}) = \frac{1}{2}\dim(\mathcal{O}_{min})$$

Example: Theta series

The minimal representation is a generalization of the **Weil representation**, associated with classical theta series

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2}$$

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$$\rho \, : \, SL(2, \mathbb{R}) \to L^2(\mathbb{R}) \qquad \rho\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) \cdot f(r) = \lambda \widehat{f(r)}$$

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Fix: $f(r) = e^{-\pi r^2}$ Then we have

$$\sum_{n\in\mathbb{Z}} \rho\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix}\right)\cdot f(n) = y^{1/4}\sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2} = y^{1/4}\theta(\tau)$$

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Very few Fourier coefficients!

$$\theta(\tau) = \sum_{k=1}^{\infty} R_2(k) e^{2\pi i k\tau}$$

Minimal automorphic representations of $SL(n, \mathbb{A})$

[w/ Ahlén, Gustafsson, Kleinschmidt, Liu]

$\mathrm{GKdim}(\pi_{min}) = n - 1$

Borel subgroup $B = NA$

F number field

$\mathbb{A} = \mathbb{A}_F$ adeles

$(F = \mathbb{Q}, \mathbb{A} = \mathbb{A}_{\mathbb{Q}})$

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Theorem: For any $\varphi \in \pi_{min}$ we have the Fourier expansion:

$$\varphi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)dn + \sum_{i=1}^{n-1} \sum_{\gamma\in\Gamma_i} \int_{N(F)\backslash N(\mathbb{A})} \varphi(n\gamma g)\overline{\psi_{\alpha_i}(n)}dn$$

$$\Gamma_i = \mathrm{Stab}_{\hat{e}_i}\backslash SL(n-i,F)$$

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This is the **complete** expansion, including all **non-abelian coefficients.**

Analogue to the **Piatetski-Shapiro-Shalika expansion** of cusp forms.

$P = LU \subset SL(n, \mathbb{A})$ maximal parabolic

What can we say about the U **-coefficient?**

$$F_{\psi_U}(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi_U(u)}du$$

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Theorem [AGKLP]: For $\varphi \in \pi_{min}$ we have

$\mathrm{rank}(\psi_U) > 1$ $\qquad F_{\psi_U} = 0$

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Next-to-minimal representations

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$$\mathrm{rank}(\psi_U) = 2 \qquad F_{\psi_U}(g) = \int_{C(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} \varphi(nwcg)\overline{\psi_{\alpha,\beta}(n)} dn\, dc$$

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Theorem [AGKLP]: For $\varphi \in \pi_{ntm}$ we have

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$\mathrm{rank}(\psi_U) = 2$

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$$= \prod_{p \leq \infty} F_p$$

This generalizes earlier results of [Miller, Sahi]

Example for $SL(5, \mathbb{A})$

$$P = LU \qquad L = SL(4) \times GL(1) \qquad U = \left\{ \begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\}$$

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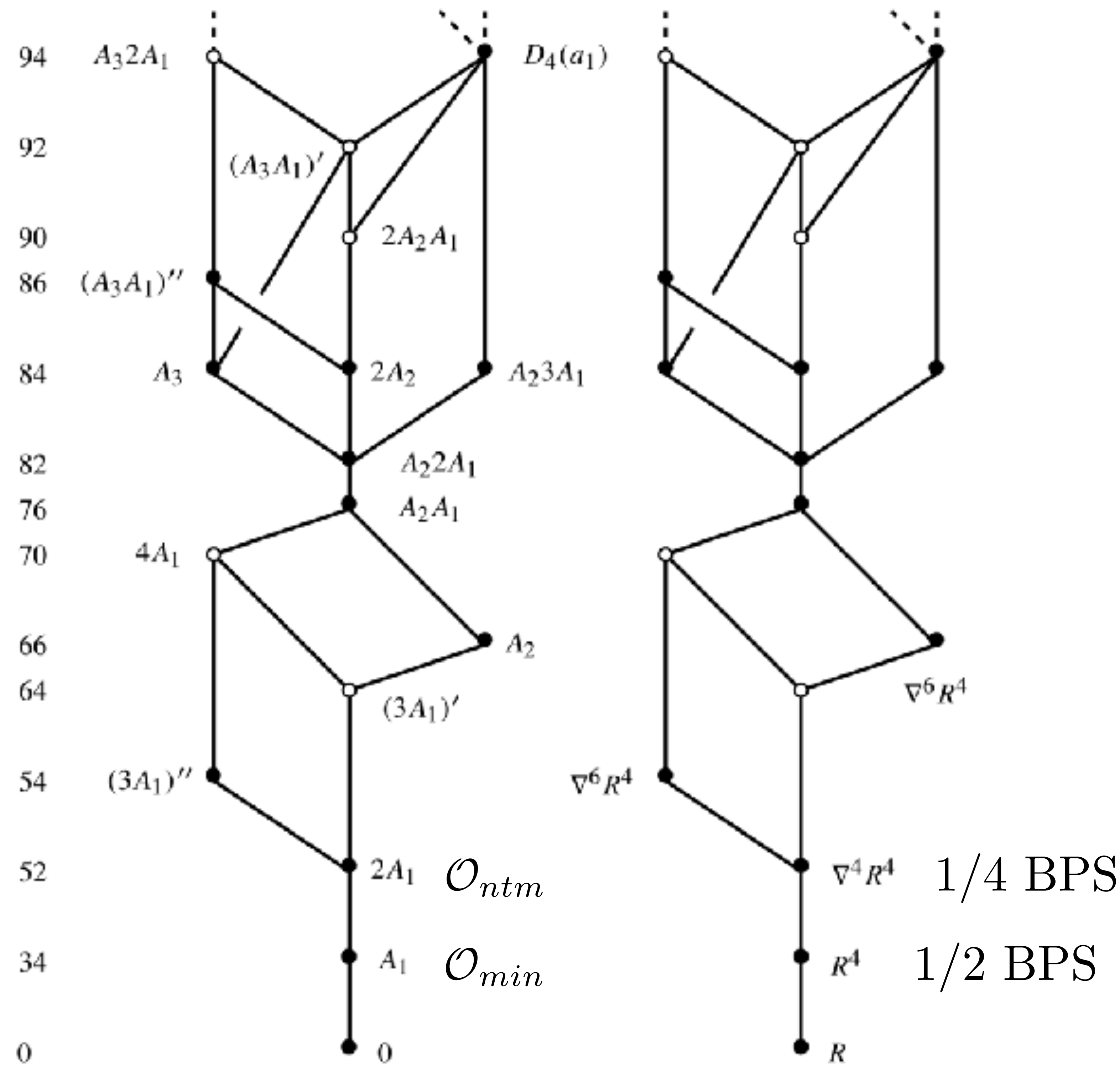
For $s = 3/2$ this captures the contributions from **M2-brane instantons** in M-theory compactified on T^4 [Green, Miller, Vanhove]

Instanton measure: $$\sigma_{2s-4}(k) = \sum_{d|k} d^{2s-4}$$

4. Outlook

Hasse diagram for E_7



Theorem: In progress w/ **[Gustafsson, Gourevitch, Kleinschmidt, Sahi]**

Let G be a ***semisimple, simply laced Lie group****.*

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

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where (α, β) are commuting simple roots.

This generalises earlier results of [Ginzburg, Rallis, Soudry][Miller, Sahi]

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This allows to extract instanton effects to 1/4-BPS couplings

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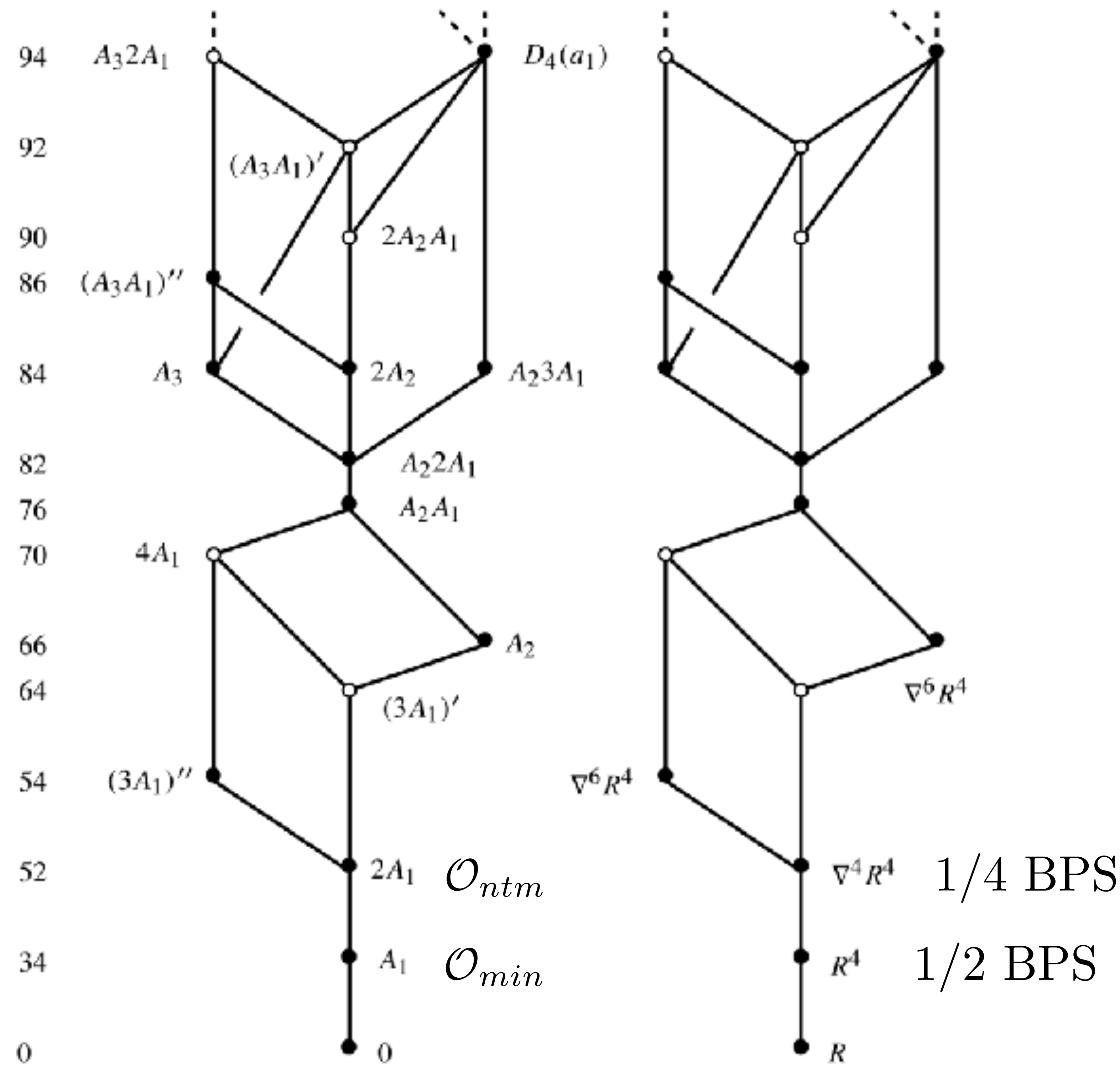

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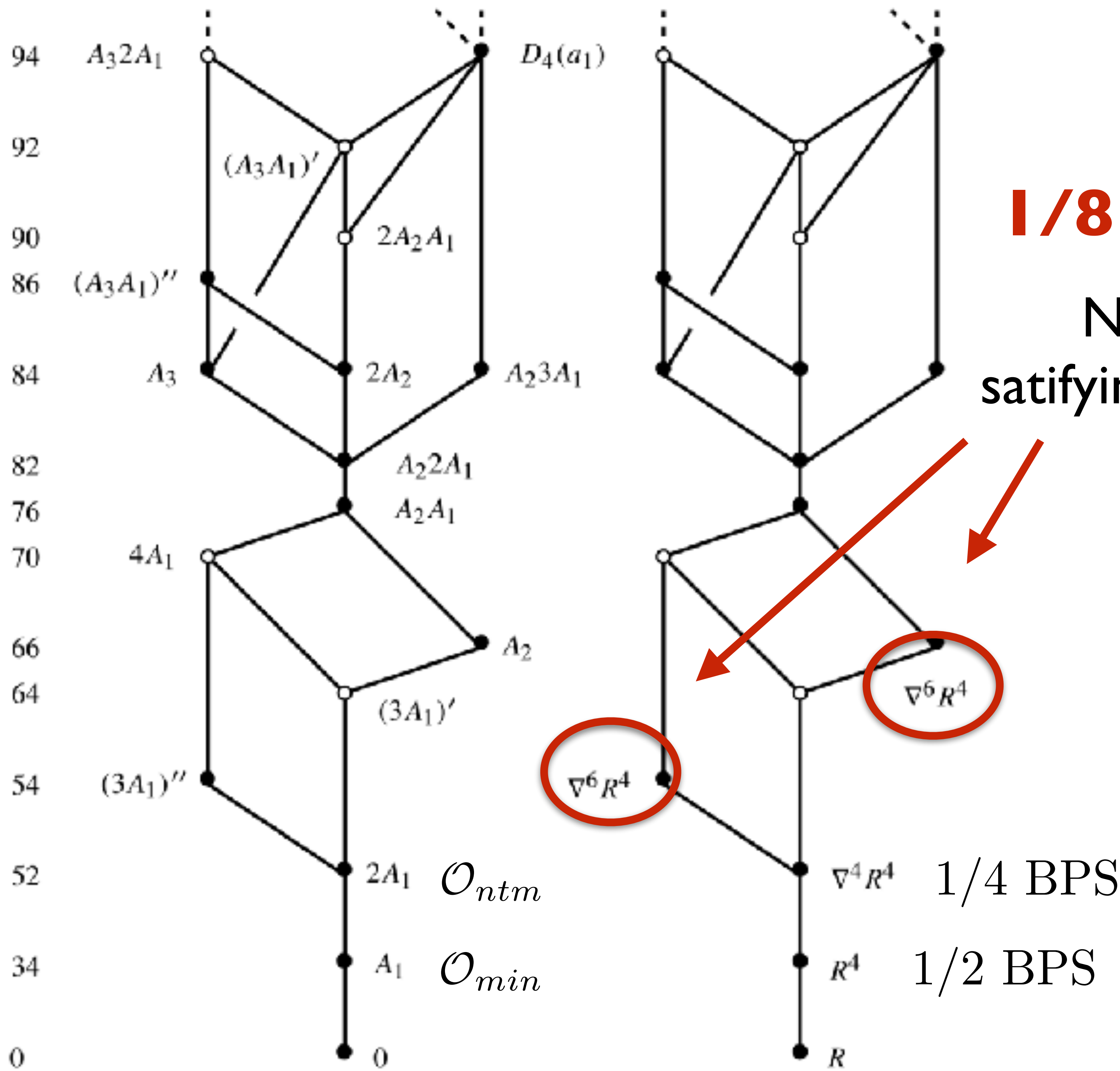

1/8 BPS-contributions:

New automorphic objects, satifying **Poisson-type** equations

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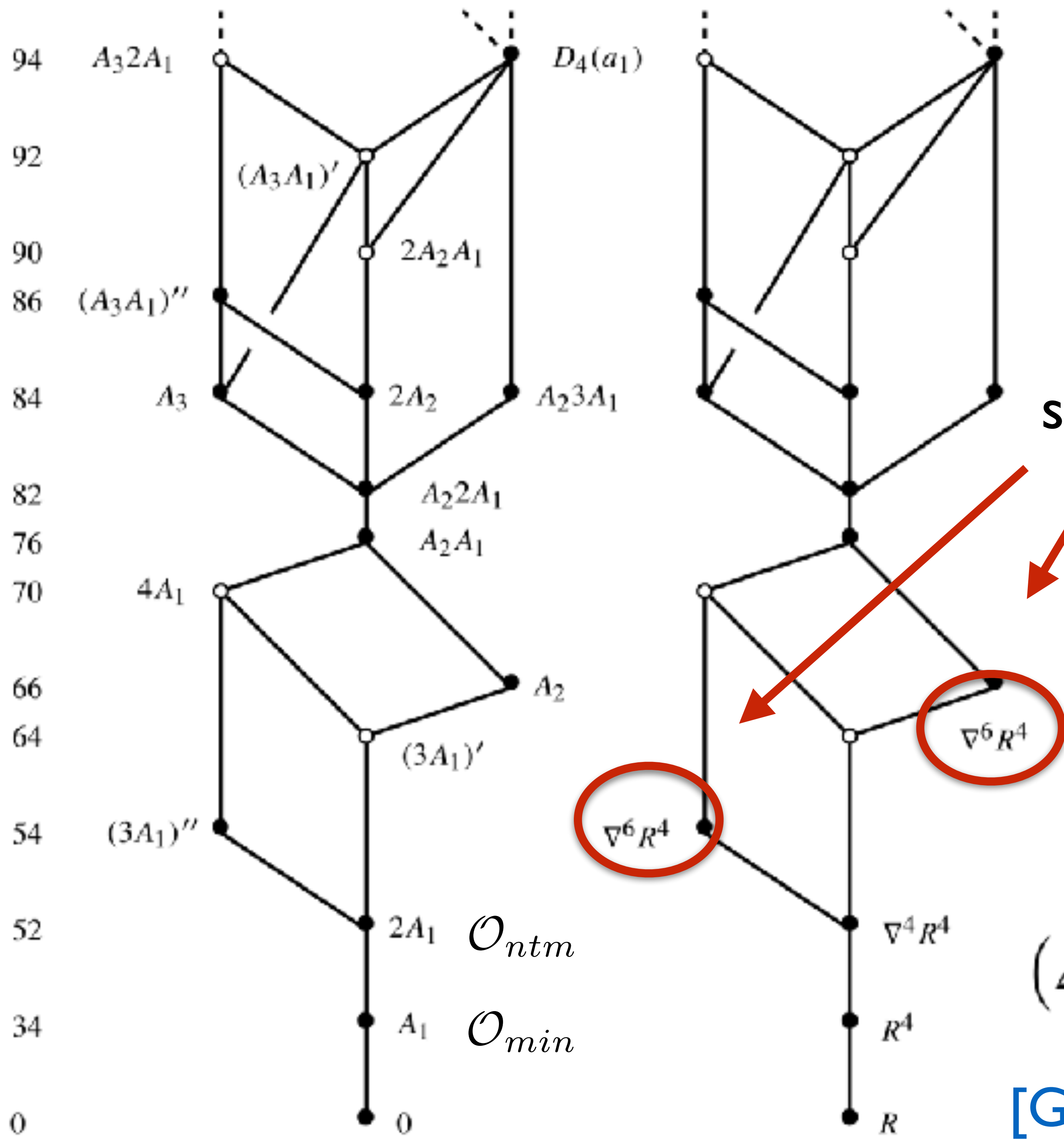

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For example, in D=10:

$$\left(\Delta_\tau - 12\right)\mathcal{F}(\tau) = -\left(E_{3/2}(\tau)\right)^2$$

[Green, Vanhove][Green, Miller, Vanhove]

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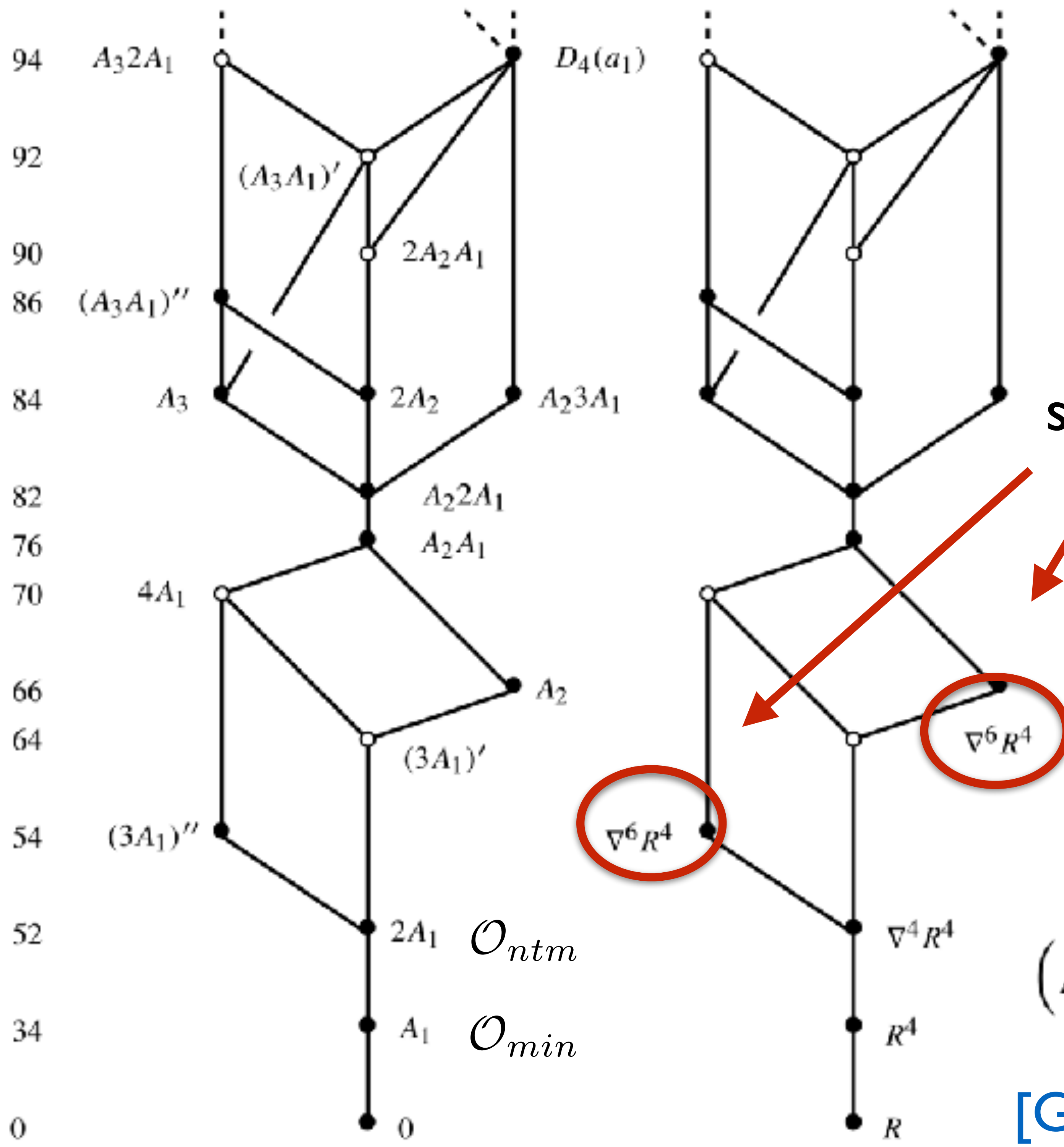

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How do they fit into the representation theory?

Thank you!