Topological strings, matrix models and nonperturbative effects

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Abstract

These are lecture notes for my course at University of Warwick.

Contents

1 Introduction

This is an introduction to the relation between matrix models and topological strings.

Motivation: intersection theory on Deligne–Mumford moduli space=Kontsevich integral, double-scaled matrix models. This makes manifest the connection to integrability.

What about the natural generalization, Gromov–Witten theory?

In physics: topological strings in $d < 1$/bosonic strings in $d < 1$=matrix model realizations.

2 Topological strings and GW invariants

2.1 Gromov–Witten invariants

Let X be a complex manifold, and let $Q \in H_2(X, \mathbb{Z})$ be a two-homology class. We want to "count" the number of holomorphic maps

$$x : \Sigma_g \to X, \quad x_*([\Sigma_g]) = Q \tag{2.1}$$

The modern way to do this counting is to construct a (virtual) moduli space of such maps $\mathcal{M}_g(X, Q)$.

Remark 2.1. When $X = \text{pt}$,

$$\mathcal{M}_g(\text{pt}, Q) = \overline{M}_g, \tag{2.2}$$

is the Deligne–Mumford moduli space of stable Riemann surfaces of genus g.

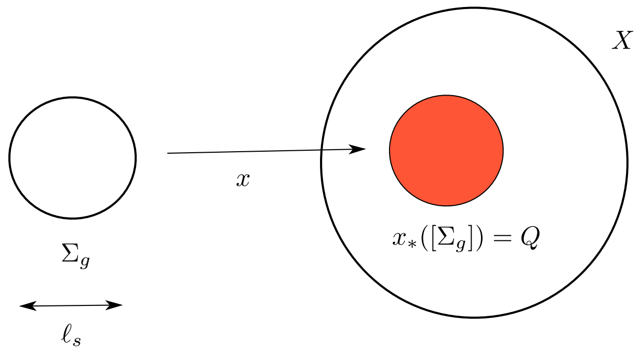

Figure 1: A map from a Riemann surface Σ_g to a complex manifold X.

We will focus on the case in which X is a *Calabi–Yau threefold*. i.e. X is Kähler with Kähler form ω, and has

$$c_1(X) = 0. \tag{2.3}$$

In general, $\mathcal{M}_g(X, Q)$ has (virtual) dimension

$$\dim \mathcal{M}_g(X, Q) = (1 - g)(d - 3) + Q \cdot c_1(X), \tag{2.4}$$

where d is the (complex) dimension of X and

$$Q \cdot c_1(X) = \int_Q c_1(X). \tag{2.5}$$

This dimension vanishes precisely when X is a CY threefold, and it is possible to define then

$$N_{g,Q} = \int_{\mathcal{M}_g(X,Q)} 1 \tag{2.6}$$

which is the *Gromov–Witten invariant* at genus g and for the class Q. It will be extremely useful to define a *generating functional* for such numbers

$$F_g(t) = \sum_{Q \in H_2(X)} N_{g,Q} \mathrm{e}^{-Q \cdot t} \tag{2.7}$$

where t are called *(complexified) Kähler parameters* (one for each class in $H_2(X, \mathbb{Z})$). They correspond to a choice of Kähler form and measure areas of two-cycles. More precisely, denoting the complexified Kähler class by

$$\mathcal{J} = \omega + \mathrm{i}B, \tag{2.8}$$

we have

$$Q \cdot t = \int_Q \mathcal{J} = \sum_{i=1}^{b_2(X)} Q_i t_i, \qquad t_i = \int_{\Sigma_i} \mathcal{J}, \tag{2.9}$$

where $\{\Sigma_i\}_{i=1,\cdots,b_2(X)}$ is a basis of $H_2(X, \mathbb{Z})$, and Q_i are nonnegative integers (negative integers have $N_{g,Q} = 0$). For $g = 0$, the generating function is usually called the *prepotential* and it is convenient to add to it a "classical" piece

$$F_0(t) = \frac{1}{3!}\kappa_{ijk} t_i t_j t_k + \sum_{Q \in H_2(X)} N_{0,Q} \mathrm{e}^{-Q \cdot t}, \tag{2.10}$$

where

$$\kappa_{ijk} = \int_X \eta_i \wedge \eta_j \wedge \eta_k, \tag{2.11}$$

and the η_i are 2-forms satisfying

$$\int_{\Sigma_i} \eta_j = \delta_{ij}. \tag{2.12}$$

2.2 Connections to physics

It turns out that the quantities $F_g(t)$ can be computed in string theory. More precisely, there is a "toy model" of string theory, called *topological string theory*, where $F_g(t)$ is a partition function at genus g of a *closed string* (no boundaries).

For fixed g, the free energy $F_g(t)$ is obtained by summing over all possible instanton sectors, which are classified topologically by $Q \in H_2(X, \mathbb{Z})$. Anti-instantons do not contribute, therefore only non-negative degrees appear. The action of such an instanton is

$$A_Q = \frac{1}{\ell_s^2} Q \cdot t, \tag{2.13}$$

where ℓ_s corresponds physically to the lenght of the string, so that the action is dimensionless. In fact, ℓ_s^2 plays the role of a Planck constant and we will write

$$\hbar_{\mathrm{ws}} = \ell_s^2. \tag{2.14}$$

In principle, at any given genus, we should have an expansion around a given instanton of the form

$$\sum_{\ell \geq 0} c_{g,Q,\ell} \hbar_{\mathrm{ws}}^{\ell} \mathrm{e}^{-A_Q}. \tag{2.15}$$

If we assume, for simplicity, that $b_2(X) = 1$, then the total $F_g(t)$ would look like

$$F_g(t) = \sum_{d\geq 0} \mathrm{e}^{-dt/\hbar_{\rm ws}} \sum_{\ell\geq 0} c_{g,Q,\ell} \hbar_{\rm ws}^{\ell}. \tag{2.16}$$

This is an example of what some people call a *trans-series.* It is a series in *two* small parameters,

$$\hbar_{\rm ws}, \qquad \mathrm{e}^{-t/\hbar_{\rm ws}} \tag{2.17}$$

This kind of series appear for example in the study of asymptotics of nonlinear differential equations (like for example the WKB method). But in fact, in topological string theory/Gromov–Witten theory, we have that

$$c_{g,Q,\ell} = N_{g,Q} \delta_{\ell,0}. \tag{2.18}$$

In physics, this means that the perturbation theory around a given instanton is trivial, and the reason is (worldsheet) supersymmetry. The regime in which the above expansion makes sense (i.e. where weak coupling/semiclassical methods/instanton calculus can be trusted) is

$$\mathrm{e}^{-t/\hbar_{\rm ws}} \ll 1 \quad \text{i.e.} \quad \frac{t_i}{\hbar_{\rm ws}} \gg 1, \tag{2.19}$$

which means large distances/"sizes". Strong coupling occurs at *small size*, and the instanton expansion breaks down. We will see however that $F_g(t)$ as a function of t has a finite radius of convergence, and a nontrivial analytic structure. The GW invariants are recovered as expansions around the limit $t \to \infty$.

We can put all genera together by defining the so-called *total free energy*

$$F(t, g_s) = \sum_{Q,g} N_{g,Q} \mathrm{e}^{-Q\cdot t/\hbar_{\rm ws}} g_s^{2g-2}$$

This theory has now *two quantum parameters.* We get a new Planck constant in *spacetime*

$$\hbar_{\rm st} = g_s. \tag{2.20}$$

Two natural questions appear:

1) In Gromov–Witten theory, computations are *perturbative* in both $\mathrm{e}^{-t/\hbar_{\rm ws}}$, g_s. How do we go beyond perturbation theory? As we will see, in what concerns the first parameter, *mirror symmetry* gives the complete answer. It actually tells us a lot about the analyticity structure of $F_g(t)$.

2) For the g_s parameter, not much is known. But based on our previous discussion of the trans-series structure, we might expect that the true structure of the total free energy is

$$F(t, g_s) = \sum_{k\geq 0} \mathrm{e}^{-kA(t)/g_s} \sum_{n\geq 0} F_n^{(k)}(t) g_s^n, \tag{2.21}$$

Here, the sum is over *spacetime instantons*, labeled by k (we have assumed this to be an integer, but there might be more general labelings), and $\Phi_n^{(k)}(t)$ is the n-th loop correction around this instanton configuration. Standard topological string theory/GW theory captures only the perturbative expansion in spacetime, i.e.

$$F^{(0)}_{n=2g}(t) = F_g(t). \tag{2.22}$$

3 Mirror symmetry

Mirror symmetry asserts that, given a CY threefold X, there is another CY threefold $\widetilde{X}$ where the calculation of $F_0(t)$ (genus zero GW invariants) reduces to the study of *variation of complex structures*. The basic buiding blocks are the *period integrals* of the holomorphic 3-form Ω on $\widetilde{X}$,

$$t_I = \oint_{A_I} \Omega, \qquad \frac{\partial F_0}{\partial t_I} = \oint_{B^I} \Omega, \tag{3.1}$$

where A_I, B^I is a symplectic basis of $H_3(\widetilde{X})$. The Kähler parameters of X are identified with period integrals of $\widetilde{X}$ over a set of cycles. These A-period integrals depend on the complex parameters of the mirror CY $\widetilde{X}$ and it can be seen that they provide a parametrization of the moduli space of complex structures of $\widetilde{X}$. They are sometimes called *flat coordinates*. Notice that (3.1) are analogues in higher dimension of the period integrals of a Riemann surface.

The calculation of $N_{g,Q}$ in generic CYs (for $g > 0$) and of their open counterparts by using mirror symmetry is much more difficult, and there is no complete, algorithmic solution to the problem. The most general method to find $F_g, F_{g,h}$ involves writing differential equations for the generating functionals [7, 38]. But the *initial conditions* for these equations are not given and therefore one has to use extra info (not always available).

3.1 The toric case

A very special case of mirror symmetry occurs when X is *toric*. Toric varieties contain an algebraic torus

$$\mathbb{T} = (\mathbb{C}^*)^r \subset X$$

as an open set, and they admit an action of $\mathbb{T}$ which acts on this set by multiplication. This provides r circle symmetries. Notice that a toric CY is always noncompact.

Example 3.1. The total spaces of the bundles

$$K^{-1} \to S, \tag{3.2}$$

where S is a rational complex surface (for example $\mathbb{P}^2$) and K its canonical line bundle. These CYs are sometimes called *local surfaces*.

The non compact, toric Calabi-Yau threefolds that we will study can be described as symplectic quotients. Let us consider the complex linear space $\mathbf{C}^{N+d}$, described by $N+d$ coordinates $z_1, \cdots, z_{N+d}$, and let us introduce N real equations of the form

$$\mu_A = \sum_{j=1}^{N+3} Q_A^j |z_j|^2 = t_A, \qquad A = 1, \cdots, N. \tag{3.3}$$

In this equation, Q_A^j are integer numbers satisfying

$$\sum_{j=1}^{N+d} Q_A^j = 0. \tag{3.4}$$

Furthermore, we consider the action of the group $G_N = U(1)^N$ on the $z's$ where the A-th $U(1)$ acts on z_j by

$$z_j \rightarrow \exp(\mathrm{i}\, Q_A^j\, \alpha_A) z_j.$$

The space defined by the equations (3.3), quotiented by the group action G_N,

$$X = \bigcap_{A=1}^{N} \mu_A^{-1}(t_A)/G_N \tag{3.5}$$

turns out to be a Calabi-Yau manifold. It can be seen that the condition (3.4) is equivalent to the Calabi-Yau condition. The N parameters t_A are Kähler moduli of the Calabi-Yau. This mathematical description of X appears in the study of two-dimensional linear sigma model with $\mathcal{N} = (2,2)$ supersymmetry [39]. The theory has $N+3$ chiral fields, whose lowest components are the z's and are charged under N vector multiplets with charges Q_A^j. The equations (3.3) are the D-term equations, and after dividing by the $U(1)^N$ gauge group we obtain the Higgs branch of the theory.

Example 3.2. *Resolved conifold.* The simplest Calabi-Yau manifold is probably the so-called resolved conifold, which is the total space of the bundle

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1. \tag{3.6}$$

This manifold has a description of the form (3.5), with $N = 1$, $d = 3$. There is only one constraint given by

$$|z_1|^2 + |z_4|^2 - |z_2|^2 - |z_3|^2 = t \tag{3.7}$$

and the $U(1)$ group acts as

$$z_1, z_2, z_3, z_4 \rightarrow \mathrm{e}^{\mathrm{i}\alpha} z_1, \mathrm{e}^{-\mathrm{i}\alpha} z_2, \mathrm{e}^{-\mathrm{i}\alpha} z_3, \mathrm{e}^{\mathrm{i}\alpha} z_4. \tag{3.8}$$

Notice that, for $z_2 = z_3 = 0$, (3.7) describes a $\mathbb{P}^1$ whose area is proportional to t. Therefore, (z_1, z_4) can be taken as homogeneous coordinates of the $\mathbb{P}^1$ which is the basis of the fibration, while z_2, z_3 can be regarded as coordinates for the fibers.

The GW invariants of the resolved conifold can be computed in closed form at all genera. One finds,

$$\begin{aligned} F_0(t) &= \mathrm{Li}_3(\mathrm{e}^{-t}) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-nt}}{n^3}, \\ F_1(t) &= -\frac{1}{12}\log(1-\mathrm{e}^{-t}), \\ F_g(t) &= \frac{|B_{2g}|}{2g(2g-2)!}\mathrm{Li}_{3-2g}(\mathrm{e}^{-t}). \end{aligned} \tag{3.9}$$

Example 3.3. *Local* $\mathbb{P}^2$. Let us now consider a more complicated example, namely the non-compact Calabi-Yau manifold $\mathcal{O}(-3) \to \mathbb{P}^2$. This is the total space of $\mathbb{P}^2$ together with its anticanonical bundle, and it is often called local $\mathbb{P}^2$. We can describe it again as in (3.5) with $N = 1$. There are four complex variables, $z_0, \cdots, z_3$, and the constraint (3.3) reads now

$$|z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_0|^2 = t. \tag{3.10}$$

The $U(1)$ action on the zs is

$$z_0, z_1, z_2, z_3 \to \mathrm{e}^{-3\mathrm{i}\alpha} z_0, \mathrm{e}^{\mathrm{i}\alpha} z_1, \mathrm{e}^{\mathrm{i}\alpha} z_2, \mathrm{e}^{\mathrm{i}\alpha} z_3. \tag{3.11}$$

Notice that $z_{1,2,3}$ describe the basis $\mathbb{P}^2$, while z_0 parameterizes the complex direction of the fiber.

Example 3.4. *Local curves.* These are the bundles

$$X_p = \mathcal{O}(-p) + \mathcal{O}(p-2) \to \mathbb{P}^1. \tag{3.12}$$

As CY manifolds, they have one single Kähler parameter t corresponding to the size of $\mathbb{P}^1$. The GW invariants of these CY manifolds have to be defined *equivariantly* [11]. The appropriate group action is the two- torus

$$\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^* \tag{3.13}$$

defined by scaling the line bundles over $\mathbb{P}^1$. The resulting invariants are defined in the localized equivariant cohomology ring defined by $\mathbb{T}$, and generated by s_1 and s_2. There is a special case, called *equivariant CY case*, which corresponds to the *antidiagonal* action $s_1 +_2 = 0$. In this case, the GW invariants are independent of the s_i and can be computed in closed form by using for example the topological vertex. One finds,

$$\begin{aligned} F_0^{X_p}(t) &= (-1)^p \mathrm{e}^{-t} + \frac{1}{8}(2\,p^2 - 4\,p + 1)\mathrm{e}^{-2t} + \frac{(-1)^p}{54}(1 - 6\,p + 3\,p^2)(2 - 6\,p + 3\,p^2)\mathrm{e}^{-3t} \\ &\quad + \mathcal{O}(\mathrm{e}^{-4t}), \\ F_1^{X_p}(t) &= -\frac{(-1)^p}{12}\mathrm{e}^{-t} + \frac{1}{48}(p^4 - 4\,p^3 + p^2 + 6\,p - 2)\mathrm{e}^{-2t} \\ &\quad + \frac{(-1)^p}{72}(-2 + 14\,p - 19\,p^2 - 20\,p^3 + 45\,p^4 - 24\,p^5 + 4\,p^6)\mathrm{e}^{-3t} + \mathcal{O}(\mathrm{e}^{-4t}), \end{aligned} \tag{3.14}$$

Example 3.5. *Simple Hurwitz numbers.* One interest of the theory on local curves is that it contains the theory of simple Hurwitz numbers of $\mathbb{P}^1$ as a special case [11] (see also [14]). The Hurwitz number

$$H_{g,d}^{\mathbb{P}^1}(1^d), \tag{3.15}$$

counts (connected) coverings of $\mathbb{P}^1$, with degree d, by a Riemann surface of genus g, and with simple branch points. We recall that a simple branch point is a point whose associated partition is of the form $\mu = (2, 1^{d-2})$, i.e. it has one ramification point in Σ_g, and by the Riemann–Hurwitz formula there must be $2g - 2 + 2d$ such points. This theory of simple Hurwitz numbers can be obtained from GW theory of local curves by taking the limit

$$\lim_{p\to\infty} p^{2-2g} F_g^{X_p}(t) = F_g^H(t_H) \tag{3.16}$$

where

$$\mathrm{e}^{-t_H} = (-1)^p p^2 \mathrm{e}^{-t} \tag{3.17}$$

is kept fixed. The resulting free energy $F_g^H(t_H)$ can be written as

$$F_g^H(t_H) = \sum_{d\geq 0} \frac{H_{g,d}^{\mathbb{P}^1}(1^d)}{(2g-2+2d)!} Q^d, \qquad Q = \mathrm{e}^{-t_H}. \tag{3.18}$$

In that sense, the theory of simple Hurwitz numbers can be regarded as one of the simplest examples of topological string theory/GW theory.

The total free energy is known to satisfy a Toda-like equation [37],

$$\exp\left(F^{\mathrm{H}}(g_s, t_H + g_s) + F^{\mathrm{H}}(g_s, t_H - g_s) - 2F^{\mathrm{H}}(g_s, t_H)\right) = g_s^2 \mathrm{e}^t \partial_{t_H}^2 F^{\mathrm{H}}(g_s, t_H), \tag{3.19}$$

and this will be useful in order to have a hint of the nonperturbative structure beyond perturbation theory.

3.2 Mirror symmetry in the toric case

We will now discuss the mirrors of toric CY manifolds. There are many ways to think about mirror symmetry in the toric case. The subject was initiated in the papers [30, 29], and we will follow the presentation in [27, 3]. As shown by Hori and Vafa [27], the linear sigma model describing a CY manifold has a mirror which is a *Landau–Ginzburg model* for $N + d$ variables

$$Y_i, \quad i = 1, \cdots, N + d \tag{3.20}$$

satisfying the periodicity condition

$$Y_j \equiv Y_j + 2\pi \mathrm{i} \tag{3.21}$$

and the linear constraints

$$\sum_{j=1}^{N+d} Q_A^j Y_j + t_A = 0, \quad A = 1, \cdots, N. \tag{3.22}$$

Equivalently, one can think about this as an algebraic variety defined by the variables

$$\xi_i = \mathrm{e}^{Y_i}, \quad i = 1, \cdots, N+d, \quad \xi \in \mathbb{C}^*. \tag{3.23}$$

Notice that the constraints (3.22) have a d-parameter family of solutions. One of them is trivial in the CY case, and corresponds to a shift

$$Y_j \to Y_j + c, \tag{3.24}$$

which leaves the equation unchanged due to (3.4). Therefore, we can always put one of the Y_j to zero, and parametrize the space of solutions by $d-1$ coordinates (this corresponds to a projectivization of the coordinates ξ_i). Notice that, in the case of $d = 3$, the CY threefold case, this means that the mirror LG model depends on only *two* coordinates. In this case we will write

$$Y_j = Y_j(u, v) = a_j u + b_j v + \tau_j(t_A). \tag{3.25}$$

If we require that the periodicity (3.21) leads to the same periodicity for u, v, we need a_j, b_j to be integers. Finally, there is a symmetry group acting on the solutions as

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \tag{3.26}$$

where

$$G = SL(2, \mathbb{Z}) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.27}$$

is the group of 2×2 integer matrices with determinant ± 1. If we denote

$$X_1 = \mathrm{e}^u, \qquad X_2 = \mathrm{e}^v, \tag{3.28}$$

this is the group that preserves the symplectic form

$$\left| \frac{\mathrm{d}X_1}{X_1} \wedge \frac{\mathrm{d}X_2}{X_2} \right| \tag{3.29}$$

on $\mathbb{C}^* \times \mathbb{C}^*$. The action of G on $X_{1,2}$ is given by

$$(X_1, X_2) \mapsto (X_1^a X_2^b, X_1^c X_2^d). \tag{3.30}$$

Notice that, since $X_{1,2} \in \mathbb{C}^*$, we are allowed to invert the coordinates.

The LG potential for this model is

$$W=\sum_{j=1}^{N+d} \mathrm{e}^{Y_j}. \tag{3.31}$$

Notice that under the symmetry (3.24), the potential changes by an overall function

$$W \to \mathrm{e}^c W. \tag{3.32}$$

Since we have a big symmetry group in our choice of coordinates, we must require that the quantities that we compute are invariant under the symmetries. As we will see, this is the case. We will give in a moment a geometric interpretation for this model, in terms of a mirror CY. For the moment being, we notice that

$$W(X_1,X_2)=0 \tag{3.33}$$

defines an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$, which we will denote by Σ. Let us first discuss some examples.

Example 3.6. *Resolved conifold.* The constraint is here

$$Y_1+Y_2-Y_3-Y_4+t=0. \tag{3.34}$$

We set $Y_4=0$, say, and solve in terms of $Y_1=v$, $Y_3=u$. We obtain then,

$$Y_2=-t+u-v, \tag{3.35}$$

and the superpotential reads

$$W(u,v)=\mathrm{e}^v+\mathrm{e}^u+\mathrm{e}^{-t+u-v}+1 \tag{3.36}$$

After shifting $Y_j \to Y_j+v$, which corresponds to multiplying by X_2, we obtain in terms of X_1, X_2,

$$W(X_1,X_2)=X_2^2+X_2(1+X_1)+z_t X_1. \tag{3.37}$$

Notice that $W=0$ is a curve of genus zero which can be solved as

$$X_2=-\frac{1+X_1 \pm \sqrt{(1+X_1)^2-4z_t X_1}}{2}. \tag{3.38}$$

This curve has two branch points obtained from

$$(1+X_1)^2-4z_t X_1=0, \tag{3.39}$$

which are

$$X_1^{\pm}=2z_t-1\pm 2\sqrt{z_t(z_t-1)} \tag{3.40}$$

Example 3.7. *Local* $\mathbb{P}^2$. The constraint is here

$$Y_1 + Y_2 + Y_3 - 3Y_0 + t = 0. \tag{3.41}$$

We set $Y_1 = 0$,

$$Y_0 = u, \quad Y_2 = v, \quad Y_3 = 3u - v - t, \tag{3.42}$$

so that the superpotential is

$$W = 1 + \mathrm{e}^u + \mathrm{e}^v + z_t \mathrm{e}^{-v+3u}. \tag{3.43}$$

Again, after a global shift we obtain

$$W(X_1, X_2) = X_2^2 + (1 + X_1)X_2 + X_1^3 z_t, \tag{3.44}$$

whose zero locus is a cubic curve with four branch points (one of them at infinity).

As in this example, in most of the cases of interest, the locus $W(X_1, X_2) = 0$ can be described as

$$v(x) = \log\left[\frac{a(x) + \sqrt{\sigma(x)}}{c(x)}\right], \quad \sigma(x) = \prod_{i=1}^{2s}(x - x_i). \tag{3.45}$$

where

$$v = \log X_2, \quad x = X_1. \tag{3.46}$$

This can be further rewritten as,

$$v(x) = \frac{1}{2}\log\left[\frac{a^2(x) - \sigma(x)}{c^2(x)}\right] + \tanh^{-1}\left[\frac{\sqrt{\sigma(x)}}{a(x)}\right] \tag{3.47}$$

As we will see in a moment, only the second term is relevant in the analysis of the theory.

Example 3.8. *Local curves.* The mirror geometry can not be obtained with the standard techniques, but an algebraic curve describing the geometry has been proposed in [32], based on [14]. It can be written as,

$$v(x) = \tanh^{-1}\left[\frac{\sqrt{(x - x_1)(x - x_2)}}{x - \frac{x_1 + x_2}{2}}\right] - p\tanh^{-1}\left[\frac{\sqrt{(x - x_1)(x - x_2)}}{x + \sqrt{x_1 x_2}}\right], \tag{3.48}$$

where x_1, x_2 are known functions of t,

$$x_1 = (1 - \zeta)^{-p}(1 + \zeta^{\frac{1}{2}})^2, \quad x_2 = (1 - \zeta)^{-p}(1 - \zeta^{\frac{1}{2}})^2, \tag{3.49}$$

and

$$(-1)^p \mathrm{e}^{-t} = (1 - \zeta)^{-p(p-2)}\zeta. \tag{3.50}$$

Example 3.9. *Hurwitz theory.* The curve for Hurwitz theory can be obtained as a limit of the previous curve,

$$y(h) = 2\tanh^{-1}\left[2\frac{\sqrt{(a-h)(b-h)}}{2h-(a+b)}\right] - \sqrt{(a-h)(b-h)}, \tag{3.51}$$

where

$$b = \left(1+\chi^{\frac{1}{2}}\right)^2, \quad a = \left(1-\chi^{\frac{1}{2}}\right)^2. \tag{3.52}$$

and

$$\chi \mathrm{e}^{-\chi} = \mathrm{e}^{-t_H}, \tag{3.53}$$

3.3 Periods

So far, from the above construction we obtain a function or LG superpotential $W(X_1, X_2)$, defined up to an overall rescaling, whose zero locus defines an algebraic curve Σ in $\mathbb{C}^* \times \mathbb{C}^*$. The CY mirror corresponding to the LG superpotential, which I will denote by $\widetilde{X}_\Sigma$, might be written as

$$F(U, V, X_1, X_2) = U^2 + V^2 - W(X_1, X_2) = 0, \tag{3.54}$$

This is a threefold. In order to proceed with the standard construction of mirror symmetry, we need a holomorphic 3-form Ω. This is taken to be

$$\Omega = \mathrm{Res}_{F=0}\left(\frac{\mathrm{d}U\mathrm{d}V}{F(U,V,X_1,X_2)}\frac{\mathrm{d}X_1}{X_1}\frac{\mathrm{d}X_2}{X_2}\right). \tag{3.55}$$

Notice that this form respects all the symmetries of the model. A shift (3.24) acts on the coordinates U, V as

$$U,\ V \to \mathrm{e}^{c/2}U,\ \mathrm{e}^{c/2}V, \tag{3.56}$$

and leads to

$$F \to \mathrm{e}^c F, \tag{3.57}$$

but Ω is invariant under it. Similarly, since we are using the symplectic form (3.29), we have invariance under (3.27).

We now want to integrate this 3-form over 3-cycles in the geometry. There is a natural class of 3-cycles which can be obtained from one-cycles in Σ. In order to see this, let us first study a particular example (see [28] for a very useful discussion of this issue, and a detailed analysis of the periods for the mirror of local $\mathbb{P}^2$).

Example 3.10. Let us consider the geometry of the resolved conifold, given by (3.37), which can be written as

$$W(X_1, X_2) = \frac{1}{4}\Big[(2X_2+1+X_1)^2 - (1+X_1)^2 + 4z_t X_1\Big] = \widetilde{X}_2^2 - \sigma(X_1)^2 \tag{3.58}$$

where

$$\widetilde{X}_2 = X_2 + \frac{1+X_1}{2}, \quad \sigma(X_1)^2 = \frac{(1+X_1)^2 - 4z_t X_1}{4} = \frac{1}{4}(X_1 - X_1^+)(X_1 - X_1^-). \quad (3.59)$$

We then have

$$F(U,V,X_1,X_2) = U^2 + V^2 - \tilde{X}_2^2 + \sigma(X_1)^2. \quad (3.60)$$

Let us now take U, V, X_1 to be real, and $\tilde{X}_2 = \mathrm{i}\widehat{X}_2$ to be imaginary and let us consider values of z_t for which $X_1^{\pm}$ are real (negative z_t will do the job). Restricted to this "real slice," the defining equation of our manifold can be written as

$$U^2 + V^2 + \widehat{X}_2^2 + \frac{1}{4}(X_1 - X_1^+)(X_1 - X_1^-) = 0 \quad (3.61)$$

This is, up to a rescaling of the X_1 coordinate, a 3-sphere $\mathbb{S}^3$ of radius

$$R^2 = \frac{1}{4}(X_1^+ - X_1^-)^2 = 4z_t(z_t - 1) > 0 \quad (3.62)$$

in the space

$$\mathbb{R}^4 = (\mathrm{Re}\, U, \mathrm{Re}\, V, \mathrm{Re} X_1, \mathrm{Im}\, \widetilde{X}_2) \quad (3.63)$$

It can be also regarded as a fibration of a two-sphere $\mathbb{S}^2$ over the interval

$$[X_1^-, X_1^+] \quad (3.64)$$

which degenerates at the endpoints, see Fig. 2.

What happens in general is that, for certain values of the parameters defining $W(X_1, X_2)$, and for certain "real slices" of the variables X_1, X_2, the inequality

$$W(X_1, X_2) \geq 0 \quad (3.65)$$

defines a region homeomorphic to a disc. The boundary of this region intersects the X_1 axis at two branch points X_1^+, X_1^- of the curve $W(X_1, X_2) = 0$. In the case considered above, we have

$$W(X_1, X_2) = R^2 - \widehat{X}_2^2 - \frac{1}{4}(X_1 - X_*)^2, \quad X_* = \frac{1}{2}(X_1^- + X_1^+), \quad (3.66)$$

and the region (3.65) defines a disc of radius R. We can then construct a sphere $\mathbb{S}^3$ in the geometry by taking

$$U^2 + V^2 = W(X_1, X_2), \quad U,\, V \in \mathbb{R}, \quad W(X_1, X_2) \geq 0, \quad X_1,\, X_2 \in \mathbb{C}^*. \quad (3.67)$$

This is a fibration of a circle $\mathbb{S}^1 \subset \mathbb{R}^2$, with coordinates (U, V), over the disc defined by the region (3.65). The radius of the circle vanishes at the boundary $W(X_1, X_2) = 0$, and

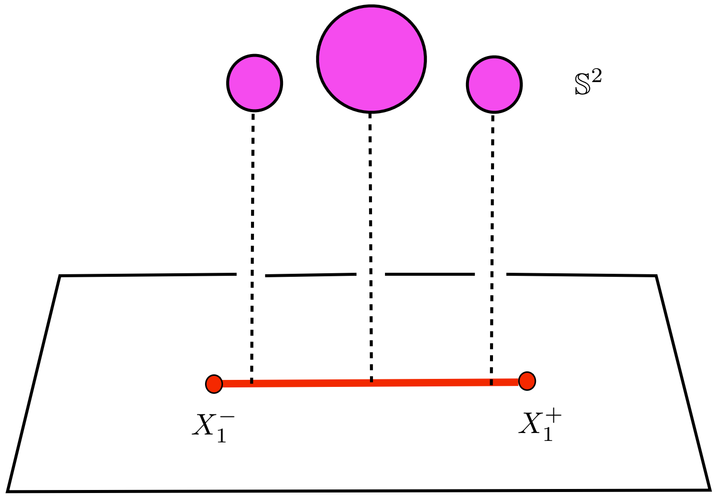

Figure 2: The equation (3.61), which defines an $\mathbb{S}^3$, can be regarded as a fibration of $\mathbb{S}^2$ over the cut $[X_1^-, X_1^+]$.

defines a 3-sphere $\mathbb{S}^3$. The integral of Ω around this 3-sphere can be easily computed. If we parametrize F by V, we have

$$\mathrm{d}V = \frac{1}{2V}\mathrm{d}F, \tag{3.68}$$

therefore

$$\mathrm{Res}_{F=0}\left(\frac{\mathrm{d}U\mathrm{d}V}{F(U,V,X_1,X_2)}\right) = \mathrm{Res}_{F=0}\frac{1}{2}\frac{\mathrm{d}U\mathrm{d}F}{F\sqrt{F+W(X_1,X_2)-U^2}} = \frac{1}{2}\frac{\mathrm{d}U}{\sqrt{W(X_1,X_2)-U^2}}, \tag{3.69}$$

and we integrate

$$\begin{aligned} &\frac{1}{2}\int_{W(X_1,X_2)\geq 0}\frac{\mathrm{d}X_1}{X_1}\frac{\mathrm{d}X_2}{X_2}\int_{-\sqrt{W(X_1,X_2)}}^{\sqrt{W(X_1,X_2)}}\frac{\mathrm{d}U}{\sqrt{W(X_1,X_2)-U^2}} \\ &= \frac{\pi}{2}\int_{W(X_1,X_2)\geq 0}\frac{\mathrm{d}X_1}{X_1}\frac{\mathrm{d}X_2}{X_2} \end{aligned} \tag{3.70}$$

Since the region (3.65) is bounded by the two branches of $\widetilde{X}_2$, as we can see in Fig. 3, and these branches correspond to the two branches of X_2, this integral can be evaluated as

$$\frac{\pi}{2}\int_{A_I}(\log X_2^+ - \log X_2^-)\frac{\mathrm{d}X_1}{X_1} = \frac{\pi}{2}\oint_{A_I}\log X_2\frac{\mathrm{d}X_1}{X_1}, \tag{3.71}$$

where A_I is a cut joining two branch points of the algebraic curve Σ defined by $W(X_1,X_2) = 0$, and $X_2^{\pm}$ are the two branches of X_2.

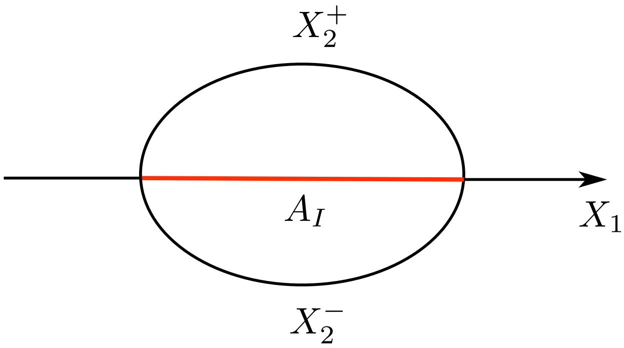

Figure 3: In the compact case, the region $W(X_1, X_2) \geq 0$ encircles a branch cut A_I of the curve $W(X_1, X_2) = 0$ and is bounded by the curves $\widetilde{X}_2^{\pm}$, which correspond to the branches $X_2^{\pm}$.

One can also find in a natural way noncompact cycles which are dual to the cycles above. For the same values of z_t, assume now that U, V are imaginary, while $\tilde{X}_2$ is real. Then the defining equation can be written as

$$U^2 + V^2 + \tilde{X}_2^2 = \sigma(X_1)^2. \tag{3.72}$$

If X_1 belonga to any of the intervals,

$$[X_1^+, \infty), \quad (-\infty, X_1^-], \tag{3.73}$$

the equation above describes a fibration of an $\mathbb{S}^2$ over an infinite interval. The $\mathbb{S}^2$ degenerates at one of the endpoints, and the resulting geometry is a noncompact $\mathbb{S}^3$, see Fig. 4. The two branches of $\widetilde{X}_2$ start now at the endpoints of the cut and go to infinity, Fig. 5.

One can then see that the CY periods become, in the local case, the more elementary periods over Σ of the form

$$\Pi = \oint_{\mathcal{C}} \log X_2 \frac{\mathrm{d}X_1}{X_1}, \tag{3.74}$$

where $\mathcal{C}$ is a one-cycle on Σ. If $v(x)$ has the structure in (3.45), in doing the contour integrals only the second term in (3.45) survives, and we then end up with periods of the form

$$\oint_{\mathcal{C}} y(x) \mathrm{d}x, \tag{3.75}$$

where

$$y(x) = \frac{2}{x} \tanh^{-1} \left[\frac{\sqrt{\prod_{i=1}^{2s} (x - x_i)}}{a(x)} \right]. \tag{3.76}$$

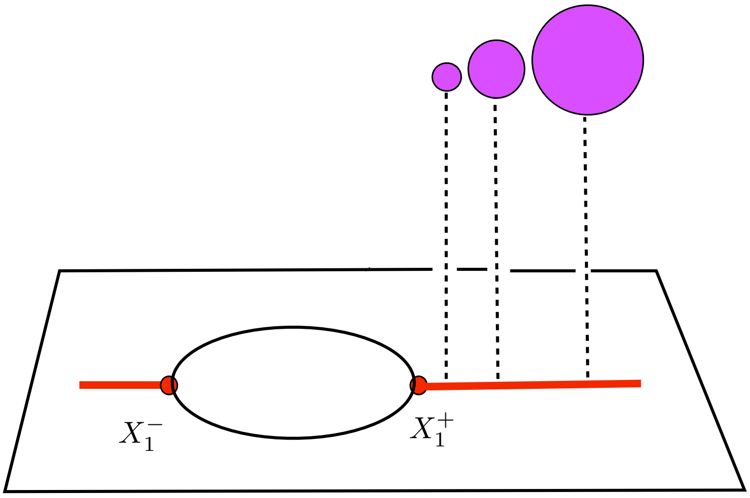

Figure 4: For other slices of the Calabi-Yau threefold we find noncompact $\mathbb{S}^3$s.

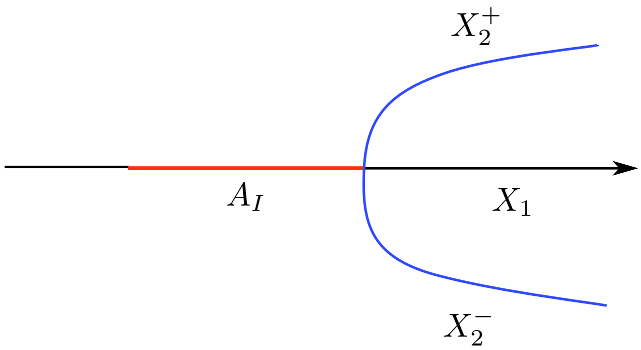

Figure 5: In the noncompact case, the region $W(X_1, X_2) \geq 0$ goes from an endpoint of the branch cut A_I of the curve $W(X_1, X_2) = 0$ to infinity, and it is bounded by the curves $\widetilde{X}_2^{\pm}$, which correspond to the branches $X_2^{\pm}$.

As we have seen, there are two kinds of periods. The first kind corresponds to compact cycles, and they are related to cuts A_I in the spectral curve, while the second kind corresponds to noncompact cycles, and they are related to cycles B^I which go from one endpoint of an A_I cut to infinity. Notice that these two types of cycles are symplectically dual, since

$$A_I \cap B^J = \delta_I^J \tag{3.77}$$

We expect that one set of these periods gives the mirror map relating the Kähler parameters t_I to the complex coordinates of the curve Σ, while the other set gives the derivatives of the prepotential, as in (3.1). In practice, the periods are computed by a system of Picard–Fuchs equations. Let us consider an example.

Example 3.11. *Local* $\mathbb{P}^2$. The PF equation for the mirror of local $\mathbb{P}^2$, which is described by the curve (3.44), can be found to be

$$\theta^3\Pi + 3z(3\theta+2)(3\theta+1)\theta\Pi = 0. \tag{3.78}$$

where

$$z = \mathrm{e}^{-t}, \quad \theta = -\frac{\mathrm{d}}{\mathrm{d}t} = z\frac{\mathrm{d}}{\mathrm{d}z}. \tag{3.79}$$

The solutions to this differential equation can be generated by Frobenius method, which applies to systems of differential equations of the above type with various variables z_i. One first introduces

$$\varpi_0(z,\rho) = \sum_{n\geq 0} a_n(\rho) z^{n+\rho}, \tag{3.80}$$

where we denoted

$$z^{n+\rho} = \prod_i z_i^{n_i+\rho_i} \tag{3.81}$$

as well as the derivatives

$$\varpi_{i_1\cdots i_n}(z) = \frac{\partial^n}{\partial\rho_{i_1}\cdots\partial\rho_{i_n}}\varpi_0(z,\rho)\Big|_{\rho=0}. \tag{3.82}$$

In general one has

$$\varpi_i(z) = \widetilde{\varpi}_i(z) + \varpi_0(z)\log\, z_i, \tag{3.83}$$

where

$$\widetilde{\varpi}_i(z) = \sum_{n\geq 0} d_n z^n, \qquad d_n = \frac{\mathrm{d}a_n(\rho)}{\mathrm{d}\rho_i}\Big|_{\rho=0}. \tag{3.84}$$

It turns out that the periods which are identified to the Kähler parameters of the mirror manifold are just

$$-t^i(z) = \frac{\varpi_i(z)}{\varpi_0(z)} = \log\, z_i + \frac{\widetilde{\varpi}_i(z)}{\varpi_0(z)}. \tag{3.85}$$

This is often called the *mirror map*, since it relates the Kähler parameters of the mirror manifold to the complex coordinate which appears in the algebraic equations describing the family of CY manifolds.

Let us now define

$$\mathcal{D}_i^{(2)} = \frac{1}{2}\kappa_{ijk}\partial_{\rho_j}\partial_{\rho_k}, \tag{3.86}$$

where κ_{ijk} is the *classical* intersection number. Then, we have that

$$\begin{aligned}\partial_{t^i} F &= \frac{1}{\varpi_0(z,\rho)} \mathcal{D}_i^{(2)} \varpi_0(z,\rho)\Big|_{\rho=0} \\ &= \frac{1}{2}\kappa_{ijk} t_j t_k + \frac{1}{2}\kappa_{ijk}\Big\{ \frac{\widetilde{\varpi}_{jk}^{(2)}(z)}{\varpi_0(z)} - \frac{\widetilde{\varpi}_j(z)\widetilde{\varpi}_k(z)}{\varpi_0^2(z)} \Big\}\end{aligned} \tag{3.87}$$

where

$$\widetilde{\varpi}_{jk}^{(2)}(z) = \sum_{n\geq 0} \partial_{\rho_j}\partial_{\rho_k} a_n(\rho)\Big|_{\rho=0} z^n. \tag{3.88}$$

In our case, we have

$$a_n(\rho) = (-1)^n \frac{\Gamma(3\rho+3n)}{\Gamma(3\rho)} \frac{\Gamma^3(\rho+1)}{\Gamma^3(\rho+n+1)}. \tag{3.89}$$

Notice that, since

$$\Gamma(3\rho) = \frac{\Gamma(3\rho+1)}{3\rho}, \tag{3.90}$$

we have that

$$a_n(0) = \delta_{n1} \Rightarrow \omega_0(z) = 1. \tag{3.91}$$

This is typical of the toric case. The mirror map can be obtained from

$$\omega_1(z) = \log\, z + \sum_{n=1}^{\infty} \frac{(3n)!}{n\cdot (n!)^3}(-1)^n z^n. \tag{3.92}$$

We will denote

$$\widetilde{\omega}_1(z) = \sum_{n=1}^{\infty} \frac{(3n)!}{n\cdot (n!)^3}(-1)^n z^n \tag{3.93}$$

Its expansion reads,

$$\widetilde{\omega}_1(z) = -6z + 45z^2 - 560z^3 + \mathcal{O}(z^4). \tag{3.94}$$

We calculate now the remaining period,

$$\omega_2(z) = (\log z)^2 + 2\log z\, \widetilde{\omega}_1(z) + \widetilde{\omega}_2(z). \tag{3.95}$$

where

$$\widetilde{\omega}_2(z) = \sum_{n=1}^{\infty} 18 \frac{(-1)^n}{n!} \frac{\Gamma(3n)}{\Gamma(1+n)^2} \Big\{ \psi(3n) - \psi(n+1) \Big\} z^n, \tag{3.96}$$

and the first terms in the expansion are

$$\widetilde{\omega}_2(z) = -18z + \frac{423}{2} z^2 - 2972 z^3 + \frac{389415}{8} z^4 + \cdots . \tag{3.97}$$

This is the known value at large radius of the derivative of the prepotential. Notice that we can write,

$$\omega_2(z) = t^2 + \widetilde{\omega}_2(z) - \widetilde{\omega}_1(z)^2. \tag{3.98}$$

The general formula of mirror symmetry give us the following structure for the derivative of the prepotential:

$$\partial_t F_0 = \frac{K}{2}\omega_2(z) = \frac{K}{2}\Big(t^2 + \widetilde{\omega}_2(z) - \widetilde{\omega}_1(z)^2\Big). \tag{3.99}$$

It turns out that

$$K = -\frac{1}{3} \tag{3.100}$$

in this model. After integrating w.r.t. t we find the expansion,

$$F_0(t) = -\frac{1}{18}t^3 + 3\mathrm{e}^{-t} - \frac{45}{4}\mathrm{e}^{-2t} + \frac{244}{3}\mathrm{e}^{-3t} + \cdots . \tag{3.101}$$

This gives the correct instanton ezpansion.

3.4 Global structure of $F_g(t)$

For simplicity of the discussion, we will assume $b_2(X) = 1$.

Mirror symmetry gives the genus g amplitudes as holomorphic "functions" on a non-trivial moduli space, parametrized by the periods t_I, and therefore it is a *nonperturbative treatment* in $\mathrm{e}^{-t/\hbar_{\rm ws}}$. The expansion of $F_g(t)$ around $t = \infty$ (large radius point) gives back GW theory. In our calculation above, this corresponds to the $z = 0$ regular singular point of the PF system

However, one finds a very rich *phase structure* as we move in the t-space [39, 6]. The large radius expansion $F_g(t)$ has a *finite radius of convergence* $\mathrm{e}^{-t_*/\ell_s^2}$, and it breaks down at small distances since $t_* \sim \ell_s^2$. The point $t = t_*$ is called the *conifold point*. It can be easily identified with the regular singular point which is closest to $t = \infty$ in the PF system. For example, for local $\mathbb{P}^2$ it is located at

$$z = -\frac{1}{27}. \tag{3.102}$$

One can actually evaluate the value of t corresponding to this value of z [6, 31]. It is given by

$$t_* = \mathrm{Re}\,\frac{G\Big(\frac{1}{3}, \frac{2}{3}, 1; 1\Big)}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \sim 2.9 \tag{3.103}$$

in units of ℓ_s^2, where G is the Meijer G-function. This gives the radius of convergence of the genus zero free energy, and in particular one can check that the GW invariants grow like

$$N_{0,d} \sim \mathrm{e}^{dt_*}, \tag{3.104}$$

at large d [31].

The conifold point can be understood as a *phase transition* characterized by the universal critical behavior of the $c = 1$ string [13, 7, 24]

$$F_g(t) \sim \frac{B_{2g}}{2g(2g-2)}\mu^{2-2g}, \quad \mu \sim t - t_* \tag{3.105}$$

In some toric models the critical behavior is controlled by pure 2d gravity $c = 0$ [14]. Typically, in the small area region $\mathrm{Re}(t) < \mathrm{Re}(t_*)$ there are other "special points" (orbifold/Gepner points) where there is a simple, exact CFT description of the theory.

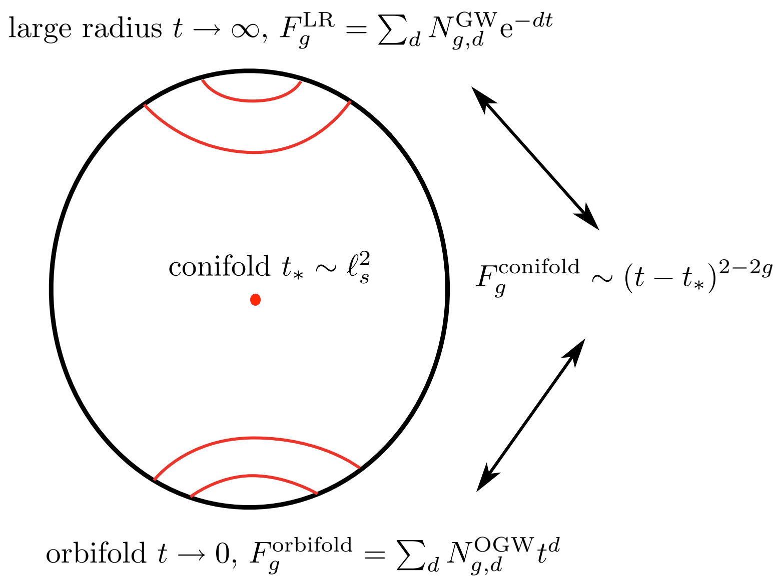

Figure 6: A picture of the global moduli space of a typical CY manifold.

There are many possible parametrizations of the moduli space by the periods, all related by the symplectic group acting on $H_3(\widetilde{X})$. Similarly, there are many F_g, related by suitable actions of the symplectic group. At every special point p (large radius, ...) there is a preferred "frame" and a set of periods t where the amplitude $F_g^p(t)$ has a good expansion.

This picture was developed and exemplified recently in [1]. They showed, building on previous work, that $F_g(t)$ are suitable generalizations of *almost modular forms* for the symplectic group acting on

$$\tau_{IJ} = \frac{\partial^2 F_0}{\partial t^I \partial t^J}$$

Moreover, the expansion of the preferred F_g^p around each special point defines generalizations of GW theory (albeit so far only the orbifold theories have been constructed in detail).

4 The geometry of matrix models

A matrix model for a Hermitian $N \times N$ matrix M and potential $V(M)$ is defined by the partition function

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}V(M)} \tag{4.1}$$

and by the (connected) correlation functions

$$F_h(z_1, \cdots, z_h) = \left\langle \mathrm{Tr}\,\frac{1}{z_1 - M} \cdots \mathrm{Tr}\,\frac{1}{z_h - M} \right\rangle^{(c)} \tag{4.2}$$

When $V(M) = M^2/2$, we have the famous *Gaussian ensemble*, but in most applications one considers general polynomials.

The above quantities turn out to have an *asymptotic expansion* in powers of g_s

$$\log Z = \sum_{g=0}^{\infty} F_g g_s^{2-2g} \tag{4.3}$$

and

$$F_h(z_1, \cdots, z_h) = \sum_{g=0}^{\infty} g_s^{2g+h-2} F_{g,h}(z_1, \cdots, z_h)$$

Notice that the expansion is labeled by a genus g. This is because it is possible to interpret the contributions at order g in terms of diagrams representing two-dimensional surfaces of genus g ['t Hooft]. The "leading" term as $g_s \to 0$ in the g_s asymptotics is called the genus zero or *planar* contribution.

MMs in the $1/N$ expansion have been studied for 30 years now, starting with [BIPZ]. The main result is that all planar quantities in a matrix model can be obtained from a single object, an algebraic curve $y(z)$ called the *classical spectral curve* of the MM.

4.1 Planar limit and spectral curve

Let us consider a general matrix model with action $V(M)$, and let us write the partition function after reduction to eigenvalues as follows:

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi} \mathrm{e}^{N^2 S_{\mathrm{eff}}(\lambda)} \tag{4.4}$$

where the effective action is given by

$$S_{\mathrm{eff}}(\lambda) = -\frac{1}{tN} \sum_{i=1}^{N} V(\lambda_i) + \frac{2}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j|. \tag{4.5}$$

To study the asymptotic expansion in g_s, we will study the limit

$$N \to \infty, \quad g_s \to 0, \quad t = g_s N \text{ fixed}. \tag{4.6}$$

t is called the 't Hooft parameter. Notice that, since a sum over N eigenvalues is roughly of order N, the effective action is of order $\mathcal{O}(1)$. We can now regard N^2 as a sort of $\hbar^{-1}$ in such a way that, as $N \to \infty$ with t fixed, the integral (4.4) will be dominated by a saddle-point configuration that extremizes the effective action. Varying $S_{\text{eff}}(\lambda)$ w.r.t. the eigenvalue λ_i, we obtain the equation

$$\frac{\mathrm{d}S_{\text{eff}}}{\mathrm{d}\lambda_i} = 0 \Rightarrow \frac{1}{2t} V'(\lambda_i) = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \quad i = 1, \cdots, N. \tag{4.7}$$

This equation can be given a simple interpretation: we can regard the eigenvalues as coordinates of a system of N classical particles moving on the real line. The equation (4.7) says that these particles are subject to an effective potential

$$V_{\text{eff}}(\lambda_i) = V(\lambda_i) - \frac{2t}{N} \sum_{j \neq i} \log |\lambda_i - \lambda_j| \tag{4.8}$$

which involves a logarithmic Coulomb repulsion between eigenvalues. For small 't Hooft parameter, the potential term dominates over the Coulomb repulsion, and the particles tend to be in an extremum x_* of the potential $V'(x_*) = 0$. As t grows, the Coulomb repulsion will force the eigenvalues to be apart from each other and to spread out along the real axis.

To encode this information about the equlibrium distribution of the particles, it is convenient to define an *eigenvalue distribution* (for finite N) as

$$\rho(\lambda) = \langle \frac{1}{N} \mathrm{Tr}\, \delta(\lambda - M) \rangle = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i), \tag{4.9}$$

where the λ_i solve (4.7) in the saddle-point approximation. In the large N limit, it is reasonable to expect that this distribution becomes a continuous distribution with compact support. The simplest solution occurs when $\rho(\lambda)$ vanishes outside an interval $\mathcal{C}$. This is the so-called *one-cut solution.* Based on considerations above, we expect $\mathcal{C}$ to be centered around an extremum x_* of the potential. In particular, as $t \to 0$, the interval $\mathcal{C}$ should collapse to the point x_*.

We can now write the saddle-point equation in terms of continuum quantities, by using the standard rule

$$\frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \to \int_{\mathcal{C}} f(\lambda) \rho(\lambda) \mathrm{d}\lambda. \tag{4.10}$$

Notice that the distribution of eigenvalues $\rho(\lambda)$ satisfies the normalization condition

$$\int_{\mathcal{C}} \rho(\lambda) d\lambda = 1. \tag{4.11}$$

The equation (4.7) then becomes

$$\frac{1}{2t} V'(\lambda) = \mathrm{P} \int_{\mathcal{C}} \frac{\rho(\lambda')\mathrm{d}\lambda'}{\lambda - \lambda'} \tag{4.12}$$

where P denotes the principal value of the integral. The above equation is an integral equation that allows one in principle to compute $\rho(\lambda)$, given the potential $V(\lambda)$, as a function of the 't Hooft parameter t and the coupling constants. Once $\rho(\lambda)$ is known, one can easily compute $F_0(t)$: in the saddle-point approximation, the free energy is given by

$$\frac{1}{N^2} F = S_{\text{eff}}(\rho) + \mathcal{O}(N^{-2}), \tag{4.13}$$

where the effective action in the continuum limit is a functional of ρ:

$$S_{\text{eff}}(\rho) = -\frac{1}{t} \int_{\mathcal{C}} \mathrm{d}\lambda \, \rho(\lambda) V(\lambda) + \int_{\mathcal{C}\times\mathcal{C}} \mathrm{d}\lambda \, \mathrm{d}\lambda' \, \rho(\lambda)\rho(\lambda') \log|\lambda - \lambda'|. \tag{4.14}$$

Therefore, the planar free energy is given by

$$F_0(t) = t^2 S_{\text{eff}}(\rho), \tag{4.15}$$

We can obtain (4.7) directly in the continuum formulation by computing the extremum of the functional

$$S(\rho, \xi) = S_{\text{eff}}(\rho) + \xi \Big(\int_{\mathcal{C}} \mathrm{d}\lambda \, \rho(\lambda) - 1 \Big) \tag{4.16}$$

with respect to ρ. Here, ξ is a Lagrange multiplier that imposes the normalization condition of the density. This leads to

$$\frac{1}{t} V(\lambda) = 2 \int \mathrm{d}\lambda' \, \rho(\lambda') \log|\lambda - \lambda'| + \xi, \tag{4.17}$$

which can be also obtained by integrating (4.12) with respect to λ. The Lagrange multiplier ξ appears in this way as an integration constant that only depends on the coupling constants. It can be computed by evaluating (4.17) at a convenient value of λ (say, $\lambda = 0$ if $V(\lambda)$ is a polynomial). Since the effective action is evaluated on the distribution of eigenvalues that solves (4.12), one can simplify the expression to

$$F_0(t) = -\frac{t}{2} \int_{\mathcal{C}} \mathrm{d}\lambda \, \rho(\lambda) V(\lambda) - \frac{1}{2} t^2 \xi(t). \tag{4.18}$$

It is convenient to introduce the *effective potential on an eigenvalue* as

$$V_{\rm eff}(\lambda) = V(\lambda) - 2t\int {\rm d}\,\lambda'\rho(\lambda')\log|\lambda-\lambda'|. \tag{4.19}$$

This is of course the continuum counterpart of (4.8). In terms of this object, the saddle–point equation (4.17) says that the effective potential is *constant* on the interval $\mathcal{C}$:

$$V_{\rm eff}(\lambda) = t\xi(t), \qquad \lambda\in\mathcal{C}. \tag{4.20}$$

The density of eigenvalues is obtained as a solution to the saddle-point equation (4.12). This equation is a singular integral equation which has been studied in detail in other contexts of physics (see, for example, [36]). The way to solve it is to introduce an auxiliary function called the *resolvent*. The resolvent is defined as

$$\omega_0(p) = \int d\lambda\frac{\rho(\lambda)}{p-\lambda} \tag{4.21}$$

and it has three important properties. First of all, due to the normalization property of the eigenvalue distribution (4.11), it has the asymptotic behavior

$$\omega_0(p)\sim\frac{1}{p}, \qquad p\to\infty. \tag{4.22}$$

Second, as a function of p it is an analytic function on the whole complex plane except on the interval $\mathcal{C}$, where it has a discontinuity as one crosses the interval $\mathcal{C}$. This discontinuity can be computed by standard contour deformations. We have

$$\omega_0(p+{\rm i}\epsilon) = \int_{\mathbb{R}}{\rm d}\lambda\frac{\rho(\lambda)}{p+{\rm i}\epsilon-\lambda} = \int_{\mathbb{R}-{\rm i}\epsilon}{\rm d}\lambda\frac{\rho(\lambda)}{p-\lambda} = {\rm P}\int{\rm d}\lambda\frac{\rho(\lambda)}{p-\lambda} + \int_{C_\epsilon}{\rm d}\lambda\frac{\rho(\lambda)}{p-\lambda}, \tag{4.23}$$

where C_ϵ is a contour around $\lambda=p$ in the lower half plane, and counterclockwise. This can be evaluated as a residue, and we finally obtain,

$$\omega_0(p+{\rm i}\epsilon) = {\rm P}\int{\rm d}\lambda\frac{\rho(\lambda)}{p-\lambda} - \pi{\rm i}\rho(p). \tag{4.24}$$

Similarly

$$\omega_0(p-{\rm i}\epsilon) = \int_{\mathbb{R}+{\rm i}\epsilon}{\rm d}\lambda\frac{\rho(\lambda)}{p-\lambda} = {\rm P}\int{\rm d}\lambda\frac{\rho(\lambda)}{p-\lambda} + \pi{\rm i}\rho(p), \tag{4.25}$$

One then finds the key equation

$$\rho(\lambda) = -\frac{1}{2\pi{\rm i}}\big(\omega_0(\lambda+i\epsilon)-\omega_0(\lambda-i\epsilon)\big). \tag{4.26}$$

From these equations we deduce that, if the resolvent at genus zero is known, the planar eigenvalue distribution follows from (4.26), and one can compute the planar free energy. On the other hand, by using again (4.23) and (4.25) we can compute

$$\omega_0(p+i\epsilon)+\omega_0(p-i\epsilon)=2\mathrm{P}\int \mathrm{d}\lambda \frac{\rho(\lambda)}{p-\lambda} \tag{4.27}$$

and we then find the equation

$$\omega_0(p+\mathrm{i}\epsilon)+\omega_0(p-\mathrm{i}\epsilon)=\frac{1}{t}V'(p), \tag{4.28}$$

which determines the resolvent in terms of the potential. In this way we have reduced the original problem of computing $F_0(t)$ to the Riemann-Hilbert problem of computing $\omega_0(\lambda)$. There is in fact a closed expression for the resolvent in terms of a contour integral [35] which is very useful. Let $\mathcal{C}$ be given by the interval $b \le \lambda \le a$. Then, one has

$$\omega_0(p)=\frac{1}{2t}\oint_{\mathcal{C}}\frac{dz}{2\pi i}\frac{V'(z)}{p-z}\left(\frac{(p-a)(p-b)}{(z-a)(z-b)}\right)^{\frac{1}{2}}. \tag{4.29}$$

This equation is easily proved by converting (4.28) into a discontinuity equation:

$$\widehat{\omega}_0(p+i\epsilon)-\widehat{\omega}_0(p-i\epsilon)=\frac{1}{t}\frac{V'(p)}{\sqrt{(p-a)(p-b)}}, \tag{4.30}$$

where $\widehat{\omega}_0(p)=\omega_0(p)/\sqrt{(p-a)(p-b)}$. This equation determines $\omega_0(p)$ to be given by (4.29) up to regular terms, but because of the asymptotics (4.22), these regular terms are absent. The asymptotics of $\omega_0(p)$ also gives two more conditions. By taking $p \to \infty$, one finds that the r.h.s. of (4.29) behaves like $c + d/p + \mathcal{O}(1/p^2)$. Requiring the asymptotic behavior (4.22) imposes $c=0$ and $d=1$, and this leads to

$$\begin{aligned}\oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{V'(z)}{\sqrt{(z-a)(z-b)}}&=0,\\ \oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{zV'(z)}{\sqrt{(z-a)(z-b)}}&=2t.\end{aligned} \tag{4.31}$$

These equations are enough to determine the endpoints of the cuts, a and b, as functions of the 't Hooft coupling t and the coupling constants of the model.

The above expressions are in fact valid for very general potentials (we can for example apply them to logarithmic potentials), but when $V(z)$ is a polynomial, one can find a very convenient expression for the resolvent: if we deform the contour in (4.29) we pick up a pole at $z=p$, and another one at infinity, and we get

$$\omega_0(p)=\frac{1}{2t}V'(p)-\frac{1}{2t}\sqrt{(p-a)(p-b)}M(p), \tag{4.32}$$

where

$$M(p) = \oint_{\infty} \frac{\mathrm{d}z}{2\pi i} \frac{V'(z)}{z-p} \frac{1}{\sqrt{(z-a)(z-b)}} \tag{4.33}$$

which can be written as a contour integral around $z = 0$

$$M(p) = \oint_{0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{V'(1/z)}{1-pz} \frac{1}{\sqrt{(1-az)(1-bz)}}. \tag{4.34}$$

These formulae, together with the expressions (4.31) for the endpoints of the cut, completely solve the one-matrix model with one cut in the planar limit, for polynomial potentials.

A useful way to encode the solution to the matrix model is to define

$$y(p) = V'(p) - 2t\,\omega_0(p) = M(p)\sqrt{(p-a)(p-b)}. \tag{4.35}$$

This is an algebraic curve which is usually called the *spectral curve* of the matrix model. Notice that it is related to the density of eigenvalues as

$$y(p) = 2\pi \mathrm{i} t\rho(p), \quad p \in [a,b], \tag{4.36}$$

and the normalization condition for ρ (4.11) reads, in terms of $y(p)$,

$$t = \frac{1}{4\pi \mathrm{i}} \oint_{\mathcal{C}} y(p)\mathrm{d}p. \tag{4.37}$$

Let us now consider the interval $B^{\Lambda} = [b, \Lambda]$, where b is the endpoint of the cycle $\mathcal{C}$ and $\Lambda \gg b$. One can show that

$$\frac{\partial F_0(t)}{\partial t} = \int_{B^{\Lambda}} y(\lambda)\mathrm{d}\lambda, \tag{4.38}$$

where in the right hand side we take the finite part regularization as $\Lambda \to \infty$ (this depends on the choice of Λ due to logarithmic terms). Notice that (4.37) and (4.38) are very similar to the structure we found in (3.1) (and in particular in local mirror symmetry).

Example 4.1. *The Gaussian matrix model.* Let us now apply this technology to the simplest case, the Gaussian model with $V(M) = M^2/2$. Let us first look for the position of the endpoints from (4.31). Deforming the contour to infinity and changing $z \to 1/z$, we find that the first equation in (4.31) becomes

$$\oint_{0} \frac{dz}{2\pi i} \frac{1}{z^2} \frac{1}{\sqrt{(1-az)(1-bz)}} = 0, \tag{4.39}$$

where the contour is now around $z = 0$. Therefore $a + b = 0$, in accord with the symmetry of the potential. Taking this into account, the second equation becomes:

$$\oint_{0} \frac{dz}{2\pi i} \frac{1}{z^3} \frac{1}{\sqrt{1-a^2z^2}} = 2t, \tag{4.40}$$

and gives

$$a = 2\sqrt{t}. \tag{4.41}$$

We see that the interval $\mathcal{C} = [-a, a] = [-2\sqrt{t}, 2\sqrt{t}]$ opens as the 't Hooft parameter grows up, and as $t \to 0$ it collapses to the minimum of the potential at the origin, as expected. We immediately find from (4.32)

$$\omega_0(p) = \frac{1}{2t}\Big(p - \sqrt{p^2 - 4t}\Big), \tag{4.42}$$

and from the discontinuity equation we derive the density of eigenvalues

$$\rho(\lambda) = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}. \tag{4.43}$$

The graph of this function is a semicircle of radius $2\sqrt{t}$, and the above eigenvalue distribution is the famous Wigner-Dyson semicircle law. Notice also that the equation (4.35) gives in this case

$$y^2 = p^2 - 4t. \tag{4.44}$$

This is the equation for a curve of genus zero, which resolves the singularity $y^2 = p^2$. We then see that the opening of the cut as we turn on the 't Hooft parameter can be interpreted as a deformation of a geometric singularity.

We now compute $\xi(t)$ in the Gaussian case. For $\lambda = 0$, we have

$$\xi(t) = -2\int \mathrm{d}\lambda \rho(\lambda) \log|\lambda| = \frac{1 - \log t}{2} \tag{4.45}$$

and

$$-\frac{t}{4}\int \mathrm{d}\lambda \rho(\lambda)\lambda^2 = -\frac{t^2}{4}. \tag{4.46}$$

Therefore,

$$F_0(t) = \frac{1}{2}t^2 \log t - \frac{3}{4}t^2. \tag{4.47}$$

We can also use the regularized integral,

$$\frac{\partial F_0}{\partial t} = \int_{2\sqrt{t}}^{\Lambda} \mathrm{d}x\, \sqrt{x^2 - 4t} = t(\log t - 1) - 2t\log\Lambda + \frac{1}{2}\Lambda^2 + \mathcal{O}(1/\Lambda^2). \tag{4.48}$$

The finite part gives $t(\log t - 1)$, which is indeed the right result.

4.2 Multicut solutions

We have so far considered the so-called one cut solution to the one-matrix model. This is not, however, the most general solution, and we now will consider the multicut solution in the saddle-point approximation.

Recall from our previous discussion that the cut appearing in the one-matrix model was centered around a minimum of the potential. If the potential has many minima, one can have a solution with various cuts, centered around the different minima. In fact, one can formally consider cuts centered around any critical point of the potential. The most general solution has then s cuts (where s is generically equal to the number of critical points of the potential), and the support of the eigenvalue distribution is a disjoint union of s intervals

$$\mathcal{C} = \cup_{I=1}^{s} A_I, \tag{4.49}$$

where

$$A_I = (x_{2I-1}, x_{2I}) \tag{4.50}$$

and $x_1 < \cdots < x_{2s}$. In order to solve this case, we introduce the *filling fractions*

$$\epsilon_I = \frac{N_I}{N} = \int_{A_I} \mathrm{d}\lambda\, \rho(\lambda), \qquad I = 1, \cdots, s. \tag{4.51}$$

Notice that

$$\sum_{I=1}^{s} \epsilon_I = 1. \tag{4.52}$$

A closely related variable are the partial 't Hooft parameters

$$t_I = t\epsilon_I = g_s N_I, \qquad I = 1, \cdots, s. \tag{4.53}$$

Notice that there are only $g = s - 1$ independent filling fractions, but the partial 't Hooft parameters are all independent.

We now extremize the functional with the condition that the filling fractions are *fixed*,

$$S(\rho, \epsilon^I) = S_{\rm eff}(\rho) + \sum_I \Gamma_I \left(\int_{A_I} \mathrm{d}\lambda\, \rho(\lambda) - \epsilon_I \right), \tag{4.54}$$

where Γ_I are Lagrange multipliers. If we take the variation w.r.t. the density $\rho(\lambda)$ we find the equation

$$\frac{1}{t} V(\lambda) = 2 \int_{\mathcal{C}} \mathrm{d}\lambda' \rho(\lambda') \log|\lambda - \lambda'| + \Gamma_I, \qquad \lambda \in A_I. \tag{4.55}$$

which can be rewritten as

$$V_{\rm eff}(\lambda) = t\Gamma_I, \qquad \lambda \in A_I. \tag{4.56}$$

By taking a derivative w.r.t. λ we get again the equation (4.12). It is easy to see that the way to have multiple cuts is to require $\omega_0(p)$ to have $2s$ branch points corresponding to the roots of the polynomial $V'(z)^2 - R(z)$. Therefore we have

$$\omega_0(p) = \frac{1}{2t}V'(p) - \frac{1}{2t}\sqrt{\prod_{k=1}^{2s}(p - x_k)}M(p), \tag{4.57}$$

which can be solved in a compact way by

$$\omega_0(p) = \frac{1}{2t}\oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{V'(z)}{p-z}\left(\prod_{I=1}^{2s} \frac{p - x_I}{z - x_I}\right)^{\frac{1}{2}}. \tag{4.58}$$

In order to satisfy the asymptotics (4.22) the following conditions must hold:

$$\delta_{\ell s} = \frac{1}{2t}\oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{z^\ell V'(z)}{\prod_{I=1}^{2s}(z - x_I)^{\frac{1}{2}}}, \qquad \ell = 0, 1, \cdots, s. \tag{4.59}$$

In contrast to the one-cut case, these are only $s + 1$ conditions for the $2s$ variables x_k representing the endpoints of the cut. For $s > 1$, there are not enough conditions to determine the solution of the model, and we need extra input to determine the positions of the endpoints x_k. To do that, we can consider the filling fractions as *fixed* data, and solve for the position of the endpoints in terms of them. This gives again $s - 1$ conditions (since there are $s - 1$ independent filling fractions), so at least formally, in the $1/N$ expansion, the model is well–defined. It is however quantum–mechanically unstable.

Notice that the spectral curve is now of the form

$$y(p) = M(p)\sqrt{\prod_{k=1}^{2s}(p - x_k)}, \tag{4.60}$$

and the partial 't Hooft parameters can be written as

$$t_I = \frac{1}{4\pi\mathrm{i}}\oint_{A_I} y(\lambda)\mathrm{d}\lambda. \tag{4.61}$$

We can also define cycles B^I_Λ which go from the endpoint of the A_I cycle to the point Λ, and one can show that

$$\frac{\partial F_0(t)}{\partial t_I} = \int_{B^I_\Lambda} y(\lambda)\mathrm{d}\lambda. \tag{4.62}$$

Notice that this structure is the same we found in our discussion of local mirror symmetry.

4.3 Higher amplitudes

One simple consequence of the analysis above is that, in order to compute the planar free energy, all the information needed is encoded in $y(x)$. It turns out that this is true beyond the planar level as well:

There is a recursive algorithm to compute all the correlation functions and free energies of the theory, at all orders in the $1/N$ *expansion, which only relies on geometric information encoded in the spectral curve.*

This surprising and beautiful result is the culmination of the work started in [5] to solve the loop equations of matrix models, and is due to Eynard and collaborators (see [23] for a synthesis and references). We note for example a formula for F_1 for the one-cut case [5],

$$F_1 = -\frac{1}{24}\log\left[M(a)M(b)(a-b)^4\right]. \tag{4.63}$$

As we can see, this depends only on the moment function $M(p)$ of the curve, evaluated at the branch points, and of the branch points themselves. There is a generalization of this formula to the multicut case [4, 21, 23],

$$F_1 = -\frac{1}{24}\log\left\{\Delta^3(\det A)^{12}\prod_{I=1}^{2s} y'(x_I)\right\}, \tag{4.64}$$

where

$$\Delta = \prod_{I<J}(x_I - x_J) \tag{4.65}$$

is the discriminant of the curve, and A is the matrix of A-periods. The derivative in $y(x)$ means

$$y'(x_I) = \frac{\mathrm{d}y(x_I)}{\mathrm{d}z_I(x_I)}, \qquad z_I(p) = \sqrt{p - x_I}. \tag{4.66}$$

5 Random matrices and mirror symmetry

5.1 A conjecture

The relation between matrix models and topological strings on toric CY manifolds can be motivated by the following observation. On the matrix model side, we noticed that the genus zero free energy $F_0(t)$ as well as the 't Hooft parameters t are determined by period integrals (4.61), (4.62). On the other hand, the genus zero generating functional in topological string theory on toric CY manifolds is determined by period integrals of the form (3.75). We then see that, at this formal level, we can establish the following correspondence between matrix models and topological strings on the mirrors of toric CYs:

matrix models	topological strings
parameters in the potential	complex deformation parameters
filling fractions	flat coordinates
spectral curve	target CY geometry
planar free energy	prepotential
$1/N$ expansion	genus expansion

Notice that the "effective" spectral curve involved in (3.76) can be written in the form (4.60) but with a non–polynomial function $M(p)$:

$$M(p) = \frac{2}{p\sqrt{\sigma(p)}} \tanh^{-1}\left[\frac{\sqrt{\sigma(p)}}{a(p)}\right]. \tag{5.1}$$

We see that, formally, the prepotential of the topological string is described by a formalism similar to that encountered in matrix models, although the spectral curve (3.76) is not of the form found in matrix models with polynomial potentials. It is however not unheard of: the Chern–Simons (CS) matrix model for $\mathbb{S}^3$ [32]

$$Z_{CS} = \frac{\mathrm{e}^{-\frac{g_s}{12}N(N^2-1)}}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\beta_i}{2\pi}\, \mathrm{e}^{-\sum_i \beta_i^2/2g_s} \prod_{i<j} \Big(2\sinh\frac{\beta_i - \beta_j}{2}\Big)^2. \tag{5.2}$$

which computes the partition function of CS theory on $\mathbb{S}^3$ can be solved in the planar limit by various techniques (see [33] for an exposition) and it is characterized by a spectral curve of the form

$$y(p) = \frac{2}{p} \tanh^{-1}\left[\frac{\sqrt{(1+p)^2 - 4z_t p}}{1+p}\right], \qquad z_t = \mathrm{e}^{-t}, \tag{5.3}$$

where $t = g_s N$ is the 't Hooft parameter (in comparing to [33], notice that we have rescaled $p \to \mathrm{e}^t p$, so in the above formula $p = \mathrm{e}^{\beta}$ where β is the original eigenvalue variable in the CS matrix integral). The $1/N$ expansion of the above matrix model reproduces the $1/N$ expansion of CS theory, and in particular can be obtained with the formalism of matrix models as applied to the above curve. On the other hand, notice that the above spectral curve is *exactly* the curve $v(x)$ for the resolved conifold, compare (3.76 and (3.38). It has been shown by Gopakumar and Vafa [25] that the total free energy of CS theory, in the $1/N$ expansion, reproduces the total free energy of the GW theory of the resolved conifold. It follows that the $1/N$ expansion of the CS matrix model, which can be obtained from the curve(5.3) by using the residue calculus of [23], reproduces the g_s expansion of the resolved conifold.

A bold generalization of this observation leads to the following conjecture, which was put forward in this form in [34].

Conjecture. The free energies $F_g(t)$ for topological string on toric CY geometries can be obtained by applying matrix model techniques (in particular the recursion relations of [23]) to the spectral curve (3.76) defined by its mirror.

At this level we don't even have to worry about the matrix integral, since in order to compute the $F_g(t)$ only the spectral curve is needed, and this is provided by mirror symmetry.

This conjecture has been considerably refined in [8], and in particular it also applies to the computation of *open* string amplitudes. It has supporting evidence of different kinds:

- It is closely related to conjecture made by Dijkgraaf and Vafa in [18], which is in turn based on the large N geometric transition of [12].
- It can be seen that the $F_g(t)$ obtained by using the recursion of [23] satisfy the holomorphic anomaly equations of [7], which are also obeyed by the topological string free energies [22].
- It has been argued [19] that the Kodaira–Spencer theory describing topological string theory on local geometries reproduces the recursion of [23].
- Explicit calculations on both sides agree in all examples that have been considered so far [32, 8, 9].

5.2 An example

In order to illustrate this conjecture, let us consider the example of local $\mathbb{P}^2$. In the "large radius limit $t \to 0$ the mirror map was computed in (3.92), and we will write it as

$$Q = z_t \exp\left[3 \sum_{k\geq 0} \frac{(3k-1)!}{(k!)^3} (-1)^k z_t^k\right], \qquad Q = \mathrm{e}^{-t}. \tag{5.4}$$

Its inversion gives

$$z_t = Q + 6\,Q^2 + 9\,Q^3 + 56\,Q^4 - 300\,Q^5 + 3942\,Q^6 + \cdots . \tag{5.5}$$

The spectral curve of local $\mathbb{P}^2$ was written in (3.44). In the form (3.47) it reads,

$$y = \frac{2}{p} \tanh^{-1}\left[\frac{\sqrt{p\big(p(p+1)^2 - 4z_t\big)}}{p(p+1)}\right], \tag{5.6}$$

where $p = X_1^{-1}$. Of course, doing a computation in terms of p does not change anything, since $X_1 \to X_1^{-1}$ is a symmetry of our problem. It has the advantage however that the branch point at infinity is now at $x_1 = 0$. The other branch points are the roots of the cubic equation

$$x(x+1)^2 - 4z_t = 0. \tag{5.7}$$

In terms of

$$\xi = \Big(1 + 54\, z_t + 6\sqrt{3}\sqrt{z_t\ (1+27\, z_t)}\Big)^{\frac{1}{3}} \tag{5.8}$$

they are given by

$$x_2 = \frac{(\xi-1)^2}{3\xi}, \quad x_3 = -\frac{2}{3} + \frac{1}{3}\Big(\omega\xi + \frac{1}{\omega\xi}\Big), \quad x_4 = -\frac{2}{3} + \frac{1}{3}\Big(\omega^*\xi + \frac{1}{\omega^*\xi}\Big), \tag{5.9}$$

where $\omega = \exp(2\mathrm{i}\pi/3)$. The cuts are (x_1, x_2), (x_3, x_4).

We can now compute a simple amplitude, namely the genus one topological string amplitude $F_1(t)$. According to the conjecture above, this is given by the genus one free energy of a matrix model with spectral curve (5.6), and we will use the formula (4.64). In the case of local $\mathbb{P}^2$ the computation is subtle since the curve (5.6) is singular at the branchpoint $x_1 = 0$, and the corresponding singularity in $y'(x_1)$ leads to a logarithmic divergence for F_1. To make sense of this we should compute instead the derivative of F_1 w.r.t. t, since at genus one one needs to insert a puncture and the object which is well defined is in fact $\mathrm{d}F_1/\mathrm{d}t$. We have to regularize it by setting $x_1 = \epsilon$ and then take $\epsilon \to 0$. One finds,

$$\lim_{\epsilon\to 0} \frac{1}{y'(\epsilon)}\frac{\mathrm{d}}{\mathrm{d}t} y'(\epsilon) = 0, \tag{5.10}$$

therefore in the appropriately regularized version of (4.64) $y'(x_1)$ contributes 1. For the remaining branchpoints, one finds

$$y'(x_i) = M(x_i)\prod_{j\neq i}(x_i - x_j)^{1/2}, \qquad M(x_i) = \frac{2}{x_i^2(x_i+1)}, \quad i = 2, \cdots, 4. \tag{5.11}$$

Finally,the A period matrix can be computed as an integral of the unique Abelian differential on the spectral curve, and this integral can be written in terms of elliptic functions as

$$A = \frac{2}{\sqrt{(x_1-x_3)(x_2-x_4)}} K(k), \quad k^2 = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_2-x_4)}. \tag{5.12}$$

Putting everything together one obtains,

$$\begin{aligned} F_1(t) = &-\frac{1}{24}\sum_{i=2}^{4}\ln M(x_i) - \frac{1}{2}\ln K(k) + \frac{1}{8}\ln(x_1-x_3)^2 + \frac{1}{8}\ln(x_2-x_4)^2 \\ &- \frac{1}{12}\sum_{1<i<j}\ln(x_i-x_j)^2 - \frac{1}{16}\sum_{i=2}^{4}\log(x_i^2) \end{aligned} \tag{5.13}$$

Plugging now all the data, and reexpressing the result in terms of the flat coordinate t, we obtain the instanton expansion

$$F_1(t) = -\frac{1}{12}\log t + \frac{Q}{4} - \frac{3\,Q^2}{8} - \frac{23\,Q^3}{3} + \frac{3437\,Q^4}{16} - \frac{43107\,Q^5}{10} + 79522\,Q^6 + \mathcal{O}(Q^7) \quad (5.14)$$

The coefficients of the Q expansion are Gromov–Witten invariants and count worldsheet instantons of genus one in the local $\mathbb{P}^2$ geometry. This expansion is in agreement with the results obtained with other methods, like for example the topological vertex [2] and the holomorphic anomaly equations (applied to this case in [31]).

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