SEMINAR ON LIE GROUPS, LIE ALGEBRAS AND THEIR REPRESENTATIONS

1. Introduction

The concept of a Lie group arises naturally by putting together the algebraic notion of a group with the geometric notion of a smooth manifold. A Lie group is a smooth manifold with a group structure such that the group operations are smooth. Lie groups arise in a natural way as symmetries of a geometric object. The general linear group $\mathrm{GL}_n(\mathbb{R})$ is our guiding example of a Lie group. A representation of a Lie group G on a vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. Similarly, the notion of Lie algebra appears naturally. For example, the tangent space $\mathfrak{g}$ at the identity element of a Lie group G has this structure. For $G = \mathrm{GL}_n(\mathbb{R})$, this yields $\mathfrak{g} = \mathrm{Mat}_{\mathbb{R}}(n \times n)$ with the composition rule given by the commutator $[X, Y] = XY - YX$. A vector space with such a composition rule is called a Lie algebra.

In this seminar, we will study (matrix) Lie groups, Lie algebras and their representations. We will introduce the notion of Lie groups and Lie algebras and discuss the correspondence between them. Since finite dimensional "semisimple" Lie algebras can be viewed as elementary building blocks of more complicated Lie algebras, we will study them with an emphasis on the structure theory and their representations. Finally, we will discuss representations of Lie groups and prove a version of the Peter-Weyl Theorem, which is a statement about using irreducible representations of a compact Lie group G to study the Hilbert space of square integrable functions on G with respect to the so-called Haar measure. As a corollary of the Peter-Weyl theorem, it follows that every compact Lie group can be realized as a matrix Lie group.

2. Introduction to matrix Lie groups and Lie algebras

2.1. **Introduction to matrix Lie groups and Lie algebras.** We will introduce matrix Lie groups and Lie algebras and give examples. We will briefly discuss the exponential of a matrix which we will use to define the Lie algebra $\mathfrak{g}$ associated to a matrix Lie group G. This assignment $G \mapsto \mathfrak{g}$ allows us to construct examples of (matrix) Lie algebras.

Seminar date: 17.04.18

References: [2], Chapter 1, sections 1.1, 1.2.1-1.2.3, Chapter 2, section 2.1, section 2.4, Theorem 2.11 (without proof), Chapter 3, section 3.1, pages 49-51, section 3.3 and section 3.4, pages 57-58.

Additional References: [5], Chapter 2, section 2.1, pages 30-38 and section 2.2, pages 44-48.

2.2. **Lie group homomorphisms to homomorphisms of Lie algebras.** We will introduce the notion of homomorphism of matrix Lie groups and homomorphism of Lie algebras. Moreover, we will show that a homomorphism $\Phi : G \to H$ of matrix Lie groups gives rise to a homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ of the associated Lie algebras. As an example, we will discuss the induced Lie algebra homomorphism associated to the Adjoint map, which will give rise to the so-called adjoint representation. We will also examine how the construction of Lie algebra homomorphism from a Lie group homomorphism interacts with composition of Lie group homomorphisms. Finally, we will introduce the notion of the exponential map of the Lie algebra of a matrix Lie group and we will sketch how the exponential map can be used to show a Lie group homomorphism $\Phi : G \to H$ is uniquely determined by $\phi : \mathfrak{g} \to \mathfrak{h}$ when G is connected.

Seminar date: 24.04.18

References: [2], Chapter 3, sections 3.5, 3.7 and 3.8.

Additional references: [5], Chapter 2, section 2.6, pages 78-81.

Note: This talk assumes familiarity with the notion of connectedness of a topological space. However, we only need a working definition of connectedness which can be discussed during the preparation of the talk.

2.3. **Homomorphism of Lie algebras to Lie group homomorphisms.** We saw that a homomorphism of matrix Lie groups $\Phi : G \to H$ induces a homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$. In this talk, we will investigate the converse of this statement. Our main technical tool will be the so-called Baker-Campbell-Hausdorff (BCH) formula which is roughly a statement about expressing the group structure of G in a small neighborhood of the identity in terms of Lie algebra structure of $\mathfrak{g}$. Finally, we will sketch a proof of the following statement: Given a homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$, there is a Lie group homomorphism $\Phi : G \to H$ whose induced Lie algebra homomorphism is ϕ when G is "simply connected".

Seminar date: 08.05.18

References: [2], Chapter 5, sections 5.3-5.7.

Additional references: [5], Chapter 2, section 2.6, pages 81-83.

Note: This talk is slightly challenging. It also requires familiarity with simply connectedness of a topological space.

3. Finite dimensional Lie algebras and their representations

3.1. **Basics of Lie algebras and Nilpotent Lie algebras.** We will introduce some basic objects such as ideals, center, centralizer and normalizer which are useful to study a Lie algebra. We will discuss abelian and nilpotent Lie algebras. Finally, we will prove Engel's theorem which states that if $\mathfrak{g}$ is a Lie algebra consisting of nilpotent endomorphisms on a finite dimensional vector space $V \neq \{0\}$, then there is a nonzero $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$. From this it follows that, in a suitable basis of V, all $X \in \mathfrak{g}$ can be represented as strictly upper triangular matrices. In particular, $\mathfrak{g}$ is a nilpotent Lie algebra.

Seminar date: 15.05.18

References: [3], Chapter I, sections 1.1, 1.2, 2.1, 2.3, 3.2, 3.3; [4], Chapter I, section 2, pages 7-9 and section 6.

3.2. **Solvable Lie algebras.** In this talk, we will discuss solvable Lie algebras. We will prove Lie's theorem for a solvable Lie algebra $\mathfrak{g}$ which is a Lie subalgebra of $\mathfrak{gl}(V)$ of a finite dimensional complex vector space V. Lie's theorem says that if $\mathfrak{g}$ is a solvable Lie algebra as in the previous line, then V contains a common eigenvector for all $X \in \mathfrak{g}$. Consequently, all $X \in \mathfrak{g}$ can be represented as upper triangular matrices in a suitable basis of V.

Seminar date: 29.05.18

Reference: [3], Chapter I, section 3.1 and Chapter II, section 4.1; [4], Chapter I, section 5.

3.3. **Simple and Semisimple Lie algebras.** We will introduce simple and semisimple Lie algebras. Our main goal will be to prove Cartan's semisimplicity criterion which states that semisimplicity of a finite dimensional complex Lie algebra $\mathfrak{g}$ is equivalent to nondegeneracy of the so-called Killing form which is a symmetric bilinear form on $\mathfrak{g}$. We will discuss Cartan's criterion for solvability, which we will need in the proof of Cartan's semisimplicity criterion, for a Lie subalgebra of $\mathfrak{gl}(V)$ where V is a finite dimensional complex vector space. We will sketch a proof of the following statement: A finite dimensional complex Lie algebra $\mathfrak{g}$ is semisimple if and only if it can be written as a direct sum (of Lie algebras) of simple (as Lie algebra) ideals. We will also give examples of simple and semisimple Lie algebras.

Seminar date: 05.06.18

Reference: [3], Chapter II, sections 4.3, 5.1 and 5.2; [4], Chapter I, section 3, pages 13-14 and section 7, pages 24-30.

Note: This talk is slightly challenging because the proof of Cartan's solvability criterion is slightly technical. We will also need the Jordan decomposition theorem for this talk which we will use without proof. However, we will need the notion of semisimple elements of $\mathfrak{gl}(V)$ which we will introduce during the talk.

3.4. **Structure theory of semisimple Lie algebras (root space decomposition).** For this talk, we assume that $\mathfrak{g}$ is a finite dimensional semisimple complex Lie algebra. We will discuss structure theory of $\mathfrak{g}$. The main goal will be to show that $\mathfrak{g}$ has a root space decomposition. The root space decomposition plays a central role in the classification of semisimple Lie algebras. We will introduce the notion of toral subalgebras of $\mathfrak{g}$ and we will see that a toral algebra of $\mathfrak{g}$ is always abelian. We will show that if $\mathfrak{g} \neq \{0\}$, then it has a nontrivial toral subalgebra $\mathfrak{h}$. We will use a maximal toral subalgebra $\mathfrak{h}$ to show that $\mathfrak{g}$ has a root space decomposition: $\mathfrak{g} = \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where Δ is a finite of subset of $\mathfrak{h}^*$ and $\mathfrak{g}_\alpha$ is the "eigenspace" associated to the "root" α. Finally, we will show that $\mathfrak{g}_0 = \mathfrak{h}$.

Seminar date: 12.06.18

Reference: [3], Chapter II, sections 8.1-8.2.

3.5. **Structure theory of semisimple Lie algebras (orthogonality of root spaces).** In this talk, we will study the root space decomposition $\mathfrak{g} = \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ associated to a maximal toral Lie subalgebra $\mathfrak{h}$ in detail. We will examine interactions between a root space decomposition and the Killing form of $\mathfrak{g}$. We will show that $\mathfrak{g}_\alpha$ is one dimensional for nonzero α. As an example, we will also discuss the root space decomposition of $\mathfrak{sl}(2, \mathbb{C})$. Finally, we will explain in what sense $\mathfrak{g}$ contains many copies of $\mathfrak{sl}(2, \mathbb{C})$ and briefly outline how $\mathfrak{sl}(2, \mathbb{C})$ can be used to study roots and root spaces of $\mathfrak{g}$.

Seminar date: 19.06.18

Reference:: [3], Chapter II, sections 8.3-8.4; [4], Chapter II, section 4, pages 94-100.

3.6. **Representations of a Lie algebra.** We will introduce representations (modules) of a Lie algebra $\mathfrak{g}$ and the notions of irreducible representations and completely reducible representations. We will discuss Schur's lemma. We will also discuss the Casimir element of a finite dimensional representation. Both Schur's lemma and the Casimir of a representation will be used in proving Weyl's theorem which states that every finite dimensional complex representation of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ is completely reducible. This means that irreducible finite dimensional complex representations of $\mathfrak{g}$ are the building blocks for finite dimensional complex representations of $\mathfrak{g}$.

Seminar date: 26.06.18

Reference:: [3], Chapter II, section 6.1-6.3; [4], Chapter V, section 1, proposition 5.1 and corollary 5.2, section 4, pages 241-243.

3.7. **Weight space decomposition and representations of $\mathfrak{sl}(2, \mathbb{C})$.** We will introduce the notion of weights and weight spaces of a representation of a Lie algebra. We will show that a complex representation of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ always has weight space decomposition. We will then discuss representations of $\mathfrak{sl}(2, \mathbb{C})$, in particular, we will classify it's finite dimensional irreducible representations. We will briefly indicate how the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ plays a crucial role for the classification of irreducible representations of all semisimple Lie algebras.

Seminar date: 03.07.18

Reference: [3], Chapter II, section 7; [4], Chapter I, section 9, Chapter II, section 2, page 86 and Chapter V, section 2, pages 225-226.

4. Representation of Compact Lie groups

4.1. **Representations of a compact Lie group.** In this talk, we will discuss representations of a Lie group G. We will mainly concentrate on representations of a compact Lie group. We will introduce the notion of character of a representation of G and discuss their orthogonality relations with respect to the so-called Haar measure on G. We will show that a representation of G is determined up to isomorphism by its character.

Seminar date: 10.07.18

Reference: [1], Chapter II, section 1, section 3 and section 4; [4], Chapter IV, section 2.

Note: This talk requires some knowledge of measure theory, in particular the L^2 space of a measure on a measurable space. We will use the existence of the Haar measure on a compact Lie group as a fact (without proof). We also will use the following fact without proof: If V is a representation of a compact Lie group G, then V possesses a G-invariant inner product.

4.2. **Peter-Weyl Theorem.** The main goal of this talk is to prove the Peter-Weyl theorem for a compact Lie group. The theorem says that the characters of irreducible representations generate a dense subspace of $L^2(G)$ the Hilbert space of square integrable functions with respect to the Haar measure on G. We will recap necessary ingredients from Analysis and give a detailed sketch of a proof. As an application of this theorem, we will show that a compact Lie group can be realized as a matrix Lie group.

Seminar date: 17.07.18

Reference: [1], Chapter III, section 2 and section 3; [4], Chapter IV, section 3.

Note: This talk is challenging, it requires good knowledge of Analysis e.g. compact operators on a Hilbert space.

References

[1] T. Bröcker & T. Dieck. *Representations of Compact Lie Groups*. Graduate Texts in Mathematics. Springer, 2003.
[2] Brian Hall. *Lie groups, Lie algebras, and representations*. Graduate Texts in Mathematics. Springer, second edition, 2015.
[3] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer, 1974.
[4] A. Knapp. *Lie groups Beyond an Introduction*. Birkhäuser, second edition, 2002.
[5] Wulf Rossmann. *Lie groups*. Oxford Graduate Texts in Mathematics. Oxford University Press, second edition, 2002.