Introduction to quantum cohomology

The goal of this seminar is to make sense of the following statement:

The Kontsevich formula: *The number N_d of rational curves of degree d in $\mathbb{P}^r$ passing through $3d-1$ points is given by:*

$$N_d = \sum_{d_A+d_B=d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A-2} - d_A \binom{3d-4}{3d_A-1} \right).$$

Although the counting problem itself is of a classical nature, its resolution requires understanding various topics in modern algebraic geometry. We shall therefore make a tour of the most important ideas necessary to understand the derivation via quantum cohomology of the Kontsevich formula: moduli spaces, stable n-pointed curves, stable maps, and Gromov-Witten invariants.

The ultimate reference on the topic is [3]. We shall refer mostly to the simplified accounts in [1] and [2].

1 Warming up to enumerative geometry

We begin in the first few talks with some background and intuition on enumerative geometry problems and gently introduce the formalism of moduli spaces in algebraic geometry.

Talk 1: What is enumerative geometry?

Date: 23.04.2019

- Briefly discuss the problem of counting the number of solutions to a polynomial equation
- Discuss the question: "How many points in the plane lie on each of two given lines?" and the difficulties in answering it
- Introduce the projective space in order to answer it

References: Chapter 1 of [2].

Talk 2: Enumerative geometry in $\mathbb{P}^2$

Date: 30.04.2019

- Define curves in $\mathbb{P}^2$ of degree d and the correspondence to $\mathbb{P}^{d(d+3)/2}$
- Discuss the examples of the number of lines passing through 2 points and of conics passing through 5 points

References: Chapter 2 of [2] and Section 3.1 of [1].

Talk 3: Cross-ratios in $\mathbb{P}^1$

Date: 07.05.2019

- Describe the notion of quadruples of points in $\mathbb{P}^1$
- Projective equivalence and the cross-ratio
- Introduce the space $\mathcal{M}_{0,4}$
- Discuss projective equivalence in families and the universal property

References: Section 0.1 of [1].

Talk 4: Introduction to moduli spaces

Date: 14.05.2019

- Families and equivalence
- Definition of fine/coarse moduli spaces and examples
- Moduli functor

References: Section 0.2 of [1].

2 Stable n-pointed curves

We now apply our knowledge about moduli spaces to parameter spaces of algebraic curves and their compactifications. Although our main object of study are moduli spaces of stable maps, many of their properties are inherited from the Deligne-Mumford-Knudsen moduli space of stable n-pointed curves, which are the subject of the next few talks.

These talks are of a more technical nature.

Talk 5: Definition of stable n-pointed curves

Date: 21.05.2019

- Define n-pointed smooth rational curves and their families
- Introduce the moduli space $\mathcal{M}_{0,\mathrm{n}}$ and its universal family
- Discuss the cases $n = 3$ and $n = 4$
- Define stable n-pointed curves and introduce their moduli space $\overline{\mathcal{M}}_{0,\mathrm{n}}$

References: Sections 1.1 and 1.2 of [1] up to and including Theorem 1.2.5. Emphasis on Examples 1.1.4, 1.2.1, including its revisiting 1.2.7.

Talk 6: The boundary of $\overline{\mathcal{M}}_{0,\mathrm{n}}$

Date: 28.05.2019

- Introduce the operations of stabilisation and contraction and the forgetful map
- Discuss the stratification of $\overline{\mathcal{M}}_{0,\mathrm{n}}$
- Discuss boundary divisors with emphasis on special boundary divisors
- Recursive structure and pullback under forgetful maps

References: From [1]: Sections 1.3, 1.5.

3 Stable maps and enumerative geometry

We now turn to the main object of study: rational curves in projective space. The crucial property of a rational curve is that it can be parametrised by the projective line $\mathbb{P}^1$. We are therefore led to the study of moduli spaces of such parametrisations $\mathbb{P}^1 \to \mathbb{P}^r$ and their compactifications, the moduli spaces of stable maps.

Talk 7: The moduli space of stable maps $\overline{\mathcal{M}}_{0,\mathrm{n}}(\mathbb{P}^{\mathrm{r}}, \mathrm{d})$

Date: 04.06.2019

- Motivate the definition of stable maps
- Introduce the moduli space of stable maps $\overline{\mathcal{M}}_{0,\mathrm{n}}(\mathbb{P}^{\mathrm{r}}, \mathrm{d})$
- If time permits, sketch the construction of $\overline{\mathcal{M}}_{0,\mathrm{n}}(\mathbb{P}^{\mathrm{r}}, \mathrm{d})$

References: From [1]: Sections 2.2-2.4 with emphasis on Examples 2.2.1 and 2.2.2.

Talk 8: Towards enumerative geometry on $\overline{\mathcal{M}}_{0,\mathrm{n}}(\mathbb{P}^{\mathrm{r}}, \mathrm{d})$

Date: 18.06.2019

- Introduce the evaluation map and the two forgetful maps
- Discuss the recursive structure of the boundary
- Explain why counting rational curves is the same as counting stable maps

References: From [1]: Sections 2.5 (only the main definition), 2.6.1, 2.6.6, 2.7.1, 2.7.3, 2.7.5, 3.4.1, 3.5.1.

4 Counting with quantum cohomology

In these last few talks we construct the Gromov-Witten potential, which is the so-called generating function for the Gromov-Witten invariants, and use it to define the quantum product on $\mathbb{P}^r$. We shall also briefly discuss the "physical" motivation for the definition of the quantum product, namely topological quantum field theories.

The Kontsevich formula is then interpreted as a partial differential equation for the Gromov-Witten potential. The most striking aspect of these constructions is that this PDE amounts precisely to the associativity of the quantum product on $\mathbb{P}^2$.

Talk 9: Introduction to Gromov-Witten invariants

Date: 25.06.2019

- Define Gromov-Witten invariants and exaplin why they are enumerative objects
- Some examples
- Describe their most important properties

References: From [1]: Lemma 4.1.3, Proposition 4.1.5, Corollary 4.1.6, Examples 4.1.7 and 4.1.8, and Section 4.2 (only sketches of the proofs).

Talk 10: Quantum cohomology I

Date: 02.07.2019

- Define generating functions and give examples
- Define topological quantum field theories and use this formalism to introduce the quantum product

References: From [1]: Section 5.1. From [2]: pages 179-183.

Talk 11: Quantum cohomology II

Date: 09.07.2019

- Define the Gromov-Witten potential
- Describe the cohomology of $\mathbb{P}^r$
- Define the quantum product

References: From [1]: Section 5.2.

Talk 12: Quantum cohomology III

Date: 16.07.2019

- Prove the associativity of the quantum product

References: From [1]: Section 5.3.

Talk 13: The Kontsevich formula via quantum cohomology

Date: 23.07.2019

- Define classical and quantum potentials
- Compute the quantum product for $\mathbb{P}^2$
- Prove the Kontsevich formula

References: From [1]: Section 5.4.

References

[1] J. Kock and I. Vainsencher. *An invitation to quantum cohomology*. Birkhäuser, Boston, 2007.

[2] S. Katz. *Enumerative geometry and string theory*. American Mathematical Society, 2006.

[3] W. Fulton and R. Pandharipande. *Notes on stable maps and quantum cohomology*. https://arxiv.org/abs/alg-geom/9608011.