# Algebraic Curves — Exercises

July 20, 2020

## 1 Sheet 1 — 12 May 2020

**Exercise 1.1** (1.8 of [1])**.** Let $k$ be a field. Show that the algebraic subsets of $\mathbb{A}^1(k)$ are just the finite subsets, together with $\mathbb{A}^1(k)$ itself.

**Exercise 1.2** (1.9 of [1])**.** If $k$ is a finite field, show that every subset of $\mathbb{A}^1(k)$ is algebraic.

**Exercise 1.3** (1.14 of [1])**.** Let $F$ be a nonconstant polynomial in $k[X_1, \dots, X_n]$, $k$ algebraically closed. Show that $\mathbb{A}^n(k) \setminus V(F)$ is infinite if $n \geq 1$, and $V(F)$ is infinite if $n \geq 2$. Conclude that the complement of any proper algebraic set is infinite. (*Hint*: see problem 1.4 of [1].)

**Exercise 1.4** (1.15 of [1])**.** Let $V \subset \mathbb{A}^n(k)$, $W \subset \mathbb{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) \mid (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbb{A}^{n+m}(k)$. It is called the *product* of $V$ and $W$.

**Exercise 1.5** (1.16 of [1])**.** Let $V, W$ be algebraic sets in $\mathbb{A}^n(k)$. Show that $V = W$ if and only if $I(V) = I(W)$.

**Exercise 1.6** (1.17 of [1])**.**

1. Let $V$ be an algebraic set in $\mathbb{A}^n(k)$, $P \in \mathbb{A}^n(k)$ a point not in $V$. Show that there is a polynomial $F \in k[X_1, \dots, X_n]$ such that $F(Q) = 0$ for all $Q \in V$, but $F(P) = 1$. (*Hint*: $I(V) \neq I(V \cup \{P\})$.)
2. Let $P_1, \dots, P_r$ be distinct points in $\mathbb{A}^n(k)$, not in an algebraic set $V$. Show that there are polynomials $F_1, \dots, F_r \in I(V)$ such that $F_i(P_j) = 0$ if $i \neq j$, and $F_i(P_i) = 1$. (*Hint*: Apply the preceding part to the union of $V$ and all but one point.)
3. With $P_1, \dots, P_r$ and $V$ as in the preceding point, and $a_{ij} \in k$ for $1 \leq i, j \leq r$, show that there are $G_i \in I(V)$ with $G_i(P_j) = a_{ij}$ for all $i$ and $j$. (*Hint*: Consider $\sum_j a_{ij} F_j$.)

**Exercise 1.7** (1.22 of [1])**.** Let $I$ be an ideal in a ring $R$, $\pi\colon R \to R/I$ the natural homomorphism.

1. Show that for every ideal $J'$ of $R/I$, $\pi^{-1}(J') = J$ is an ideal of $R$ containing $I$, and for every ideal $J$ of $R$ containing $I$, $\pi(J) = J'$ is an ideal of $R/I$. This sets up a natural one-to-one correspondence between $\{$ideals of $R/I\}$ and $\{$ideals of $R$ that contain $I\}$.
2. Show that $J'$ is a radical ideal if and only if $J$ is radical. Similarly for prime and maximal ideals.
3. Show that $J'$ is a finitely generated if $J$ is. Conclude that $R/I$ is Noetherian if $R$ is Noetherian. Any ring of the form $k[X_1, \dots, X_n]/I$ is Noetherian.

## 2 Sheet 2 — 19 May 2020

Let $k$ be an algebraically closed field.

**Exercise 2.1** (2.4 of [1])**.** Let $V \subset \mathbb{A}^n$ be a nonempty variety. Show that the following are equivalent: (i) $V$ is a point; (ii) $\Gamma(V) = k$; (iii) $\dim_k \Gamma(V) < \infty$.

**Exercise 2.2** (2.14 of [1], important)**.** A set $V \subset \mathbb{A}^n(k)$ is called a *linear subvariety* of $\mathbb{A}^n(k)$ if $V = V(F_1, \dots, F_r)$ for some polynomials $F_i$ of degree 1. (a) Show that if $T$ is an affine change of coordinates on $\mathbb{A}^n$, then $V^T$ is also a linear subvariety of $\mathbb{A}^n(k)$. (b) If $V \neq \varnothing$, show that there is an affine change of coordinates $T$ of $\mathbb{A}^n$ such that $V^T = V(X_{m+1}, \dots, X_n)$. (*Hint*: use induction on $r$.) So $V$ is a variety. (c) Show that the $m$ that appears in part (b) is independent of the choice of $T$. It is called the *dimension* of $V$. Then $V$ is isomorphic (as a variety) to $\mathbb{A}^m(k)$. (*Hint*: Suppose there were an affine change of coordinates $T$ such that $V(X_{m+1}, \dots, X_n)^T = V(X_{s+1}, \dots, X_n)$, $m < s$; show that $T_{m+1}, \dots, T_n$ would be dependent.)

**Exercise 2.3** (2.15 of [1], important)**.** Let $P = (a_1, \dots, a_n)$, $Q = (b_1, \dots, b_n)$ be distinct points of $\mathbb{A}^n$. The *line* through $P$ and $Q$ is defined to be $\{a_1 + t(b_1 - a_1), \dots a_n + t(b_n - a_n) \mid t \in k\}$. (a) Show that if $L$ is the line through $P$ and $Q$, and $T$ is an affine change of coordinates, then $T(L)$ is the line through $T(P)$ and $T(Q)$. (b) Show that a line is a linear subvariety of dimension 1, and that a linear subvariety of dimension 1 is the line through any two of its points. (c) Show that, in $\mathbb{A}^2$, a line is the same thing as a hyperplane. (d) Let $P, P' \in \mathbb{A}^2$, $L_1, L_2$ two distinct lines through $P$, $L_1', L_2'$ distinct lines through $P'$. Show that there is an affine change of coordinates $T$ of $\mathbb{A}^2$ such that $T(P) = P'$ and $T(L_i) = L_i'$, $i = 1, 2$.

**Exercise 2.4** (2.44 of [1], important)**.** Let $V$ be a variety in $\mathbb{A}^n$, $I = I(V) \subset k[X_1, \dots, X_n]$. $P \in V$, and let $J$ be an ideal of $k[X_1, \dots, X_n]$ that contains $I$. Let $J'$ be the image of $J$ in $\Gamma(V)$. Show that there is a natural homomorphism $\phi$ from $\mathcal{O}_P(\mathbb{A}^n)/J\mathcal{O}_P(\mathbb{A}^n)$ to $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$, and that $\phi$ is an isomorphism. In particular, $\mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n)$ is isomorphic to $\mathcal{O}_P(V)$.

## 3 Sheet 3 — 26 May 2020

**Exercise 3.1** (2.29 of [1])**.** Let $R$ be a DVR with quotient field $K$, ord the order function on $K$. (a) If $\mathrm{ord}(a) < \mathrm{ord}(b)$, show that $\mathrm{ord}(a + b) = \mathrm{ord}(a)$. (b) If $a_1, \dots, a_n \in K$, and for some $i$, $\mathrm{ord}(a_i) < \mathrm{ord}(a_j)$ (all $j \neq i$), then $a_1 + \cdots + a_n \neq 0$.

**Exercise 3.2** (3.6 of [1])**.** Irreducible curves with given tangent lines $L_i$ of multiplicity $r_i$ may be constructed as follows: if $\sum r_i = m$, let $F = \prod L_i^{r_i} + F_{m+1}$, where $F_{m+1}$ is chosen to make $F$ irreducible (see Problem 2.34 of [1]).

**Exercise 3.3** (3.11 of [1])**.** Let $V \subset \mathbb{A}^n$ be an affine variety, $P \in V$. The *tangent space* $T_P(V)$ is defined to be $\{(v_1, \dots, v_n) \in \mathbb{A}^n \mid \text{for all } G \in I(V), \sum G_{X_i}(P)v_i = 0\}$. If $V = V(F)$ is a hypersurface, $F$ irreducible, show that $T_P(V) = \{(v_1, \dots, v_n) \mid \sum F_{X_i}(P)v_i = 0\}$. How does the dimension of $T_P(V)$ relate to the multiplicity of $F$ at $P$?

**Exercise 3.4** (3.12 of [1])**.** A simple point $P$ on a curve $F$ is called a *flex* if $\mathrm{ord}_P^F(L) \geq 3$, where $L$ is the tangent to $F$ at $P$. The flex is called *ordinary* if $\mathrm{ord}_P^F(L) = 3$, a *higher* flex otherwise. (a) Let $F = Y - X^n$. For which $n$ does $F$ have a flex at $P = (0,0)$, and what kind of flex? (b) Suppose $P = (0,0)$, $L = Y$ is the tangent line, $F = Y + aX^2 + \dots$. Show that $P$ is a flex on $F$ if and only if

$a = 0$. Give a simple criterion for calculating $\mathrm{ord}_P^F(Y)$, and therefore determining if $P$ is a higher flex.

**Exercise 3.5** (3.16 of [1])**.** Let $F \in k[X_1, \ldots, X_r]$ define a hypersurface in $\mathbb{A}^r$. Write $F = F_m + F_{m-1} + \ldots$, and let $m = \nu_P(F)$ where $P = (0,0)$. Suppose that $F$ is irreducible, and let $\mathcal{O} = \mathcal{O}_P(V(F))$, $\mathfrak{m}$ its maximal ideal. Show that $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$ is a polynomial of degree $r-1$ for sufficiently large $n$, and that the leading coefficient of $\chi$ is $\nu_P(F)/(r-1)!$.

Can you find a definition for the multiplicity of a local ring that makes sense in all the cases you know?

**Exercise 3.6.** Let $K$ be a field with a valuation as in Definition 5.3. Prove that

1. $R = \{z \in K \mid v(z) >= 0\}$ is a ring.
2. For all $z \in K^*$ we have $z \in R$ or $z^{-1} \in R$.
3. For all $z, w \in R$ we have $v(z) \leq v(w)$ iff $w \in (z)$.
4. The group of units of $R$ is $R^* = \{z \in R \mid v(z) = 0\}$
5. $R$ is a local ring with maximal ideal $\mathfrak{m} = \{z \in R \mid v(z) > 0\}$ and quotient field $K$.
6. $R$ is a PID.

# 4 Sheet 4 — 02 June 2020

**Exercise 4.1** (3.14 of [1])**.** Let $V = V(X^2 - Y^3, Y^2 - Z^3) \subset \mathbb{A}^3$, $P = (0,0,0)$, $\mathfrak{m} = \mathfrak{m}_P(V)$. Find $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$. (See problem 1.40 of [1].)

**Exercise 4.2** (3.17 of [1])**.** Find the intersection numbers of various pairs of curves from the examples of Section 1 of [1], at the point $P = (0,0)$.

**Exercise 4.3** (3.23 of [1])**.** A point $P$ on a curve $F$ is called a *hypercusp* if $\nu_P(F) > 1$, $F$ has only one tangent line $L$ at $P$, and $I(P, L \cap F) = \nu_P(f) + 1$. Generalize the results of problem 3.22 of [1] to this case.

**Exercise 4.4** (3.24 of [1])**.** The object of this problem is to find a property of the local ring $\mathcal{O}_P(F)$ that determines whether or not $P$ is an ordinary multiple point on $F$.

Let $F$ be an irreducible plane curve, $P = (0,0)$, $\nu = \nu_P(F) > 1$. Let $\mathfrak{m} = \mathfrak{m}_P(F)$. For $G \in k[X,Y]$, denote its residue in $\Gamma(F)$ by $g$; and for $g \in \mathfrak{m}$, denote its residue in $\mathfrak{m}/\mathfrak{m}^2$ by $\bar{g}$. (a) Show that the map from {forms of degree 1 in $k[X,Y]$} to $\mathfrak{m}/\mathfrak{m}^2$ taking $aX + bY$ to $\overline{ax+by}$ is an isomorphism of vector spaces (see problem 3.13 of [1]). (b) Suppose $P$ is an ordinary multiple point, with tangents $L_1, \ldots, L_\nu$. Show that $I(P, F \cap L_i) > \nu$ and $\bar{l}_i \neq \lambda \bar{l}_j$ for all $i \neq j$, all $\lambda \in k$. (c) Suppose there are $G_1, \ldots, G_\nu \in k[X,Y]$ such that $I(P, F \cap G_i) > \nu$ and $\bar{g}_i \neq \lambda \bar{g}_j$ for all $i \neq j$, and all $\lambda \in k$. Show that $P$ is an ordinary multiple point on $F$. (*Hint*: Write $G_i = L_i +$higher terms, $\bar{l}_i = \bar{g}_i \neq 0$, and $L_i$ is the tangent to $G_i$, so $L_i$ is tangent to $F$ by property (5) of intersection numbers. Thus $F$ has $\nu$ tangents at $P$.) (d) Show that $P$ is an ordinary multiple point on $F$ if and only if there are $g_1, \ldots g_\nu \in \mathfrak{m}$ such that $\bar{g}_i \neq \lambda \bar{g}_j$ for all $i \neq j$, $\lambda \in k$, and $\dim \mathcal{O}_P(F)/(g_i) > \nu$.

**Exercise 4.5** (important)**.** Let $F$ be an irreducible cubic curve. Show that $F$ has at most one multiple point. Show that such a multiple point must be either a node or a cusp.

**Exercise 4.6.** Let $F$ and $G$ be irreducible plane curves, and let $P$ be a point. Check that the intersection product $I(P, F \cap G)$ as given in Definition 6.2 of Lecture 7 satisfies properties (3), (4), and (7) given at the end of Lecture 7.

# 5 Sheet 5 — 09 June 2020

**Exercise 5.1** (important, 4.28 of [1])**.** For simplicity of notation, in this problem we let $X_0, \ldots, X_n$ be coordinates for $\mathbb{P}^n$, $Y_0, \ldots, Y_m$ be coordinates for $\mathbb{P}^m$, $T_{00}, T_{01}, \ldots, T_{0m}, T_{10}, \ldots, T_{nm}$ coordinates for $\mathbb{P}^N$, where $N = (n+1)(m+1) - 1 = n + m + nm$.

Define $S \colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by the formula:

$$S([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) = [x_0 y_0 : x_0 y_1 : \cdots : x_n y_m.$$

$S$ is called the *Segre embedding* of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{n+m+nm}$. (a) Show that $S$ is a well-defined, one-to-one mapping. (b) Show that if $W$ is an algebraic subset of $\mathbb{P}^N$, then $S^{-1}(W)$ is an algebraic subset of $\mathbb{P}^n \times \mathbb{P}^m$. (c) Let $V = V(\{T_{ij}T_{kl} - T_{il}T_{kj} \mid i, k = 0, \ldots, n; j, l = 0, \ldots, m\}) \subset \mathbb{P}^N$. Show that $S(\mathbb{P}^n \times \mathbb{P}^m) = V$. In fact, $S(U_i \times U_j) = V \cap U_{ij}$, where $U_{ij} = \{[t] \mid t_{ij} \neq 0\}$. (d) Show that $V$ is a variety.

**Exercise 5.2** (important)**.** Show that the Segre embedding $S(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ is the quadric surface. On the quadric surface, there are two families of lines. Show that two lines intersect if they come from different families, and are parallel if they are from the same family.

**Exercise 5.3** (important, after 4.26 of [1])**.** (a) Define maps $\phi_{i,j} \colon \mathbb{A}^{n+m} \to U_i \times U_j \subset \mathbb{P}^n \times \mathbb{P}^m$. Using $\phi_{n+1,m+1}$, define the "biprojective closure" of an algebraic set in $\mathbb{A}^{n+m}$. Choose two items of Proposition 3 of §4.3 in [1], and prove their analogues in the current setting. (b) Generalize part (a) to maps $\phi \colon \mathbb{A}^{n_1} \times \mathbb{A}^{n_r} \times \mathbb{A}^m \to \mathbb{P}^{n_1} \times \mathbb{P}^{n_r} \times \mathbb{A}^m$. Show that this sets up a correspondence between {nonempty affine varieties in $\mathbb{A}^{n_1 + \cdots + m}$} and {varieties in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{A}^m$ that intersect $U_{n_1+1} \times \cdots \times \mathbb{A}^m$}. Show that this correspondence preserves function fields and local rings.

**Exercise 5.4** (4.27 of [1])**.** Show that the pole set of a rational function on a variety in any multispace is an algebraic subset.

**Exercise 5.5** (4.19 of [1])**.** If $I = (F)$ is the ideal of an affine hypersurface, show that $I^* = (F^*)$.

**Exercise 5.6** (4.20 of [1])**.** Let $V = V(Y - X^2, Z - X^3) \subset \mathbb{A}^3$. Prove:

(a) $I(V) = (Y - X^2, Z - X^3)$.

(b) $ZW - XY \in I(V)^* \subset k[X, Y, Z, W]$, but $ZW - XY \notin ((Y - X^2)^*, (Z - X^3)^*)$.

So if $I(V) = (F_1, \ldots, F_r)$, it does not follow that $I(V)^* = (F_1^*, \ldots, F_r^*)$.

**Exercise 5.7** (4.11 of [1])**.** A set $V \subset \mathbb{P}^n(k)$ is called a *linear subvariety* of $\mathbb{P}^n(k)$ if $V = V(H_1, \ldots, H_r)$, where each $H_i$ is a form of degree 1. (a) Show that if $T$ is a projective change of coordinates, then $V^T = T^{-1}(V)$ is also a linear subvariety. (b) Show that there is a projective change of coordinates $T$ of $\mathbb{P}^n$ such that $V^T = V(X_{m+2}, \ldots, X_{n+1})$, so $V$ is a variety. (c) Show that the $m$ that appears in part (b) is independent of the choice of $T$. It is called the *dimension* of $V$ ($m = -1$ if $V = \varnothing$).

**Exercise 5.8** (5.2 of [1], see also Lecture 9, part of Example 7.14)**.** Show that the following curves are irreducible; find their multiple points, and the multiplicities and tangents at the multiple points.

(a) $XY^4 + YZ^4 + XZ^4$.

(b) $X^2Y^3 + X^2Z^3 + Y^2Z^3$.

(c) $Y^2Z - X(X-Z)(X-\lambda Z)$, $\lambda \in k$.

(d) $X^n + Y^n + Z^n$, $n > 0$.

# 6 Sheet 6 — 16 June 2020

**Exercise 6.1** (5.5 of [1])**.** Let $P = [0:1:0]$, $F$ a curve of degree $n$, $F = \sum F_i(X,Z)Y^{n-i}$, $F_i$ a form of degree $i$. Show that $\nu_P(F)$ is the smallest $\nu$ such that $F_\nu \neq 0$, and the factors of $F_\nu$ determine the tangents to $F$ at $P$.

**Exercise 6.2** (5.18 of [1])**.** Show that there is only one conic passing through the five points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$, $[1:1:1]$, and $[1:2:3]$; show that it is nonsingular.

**Exercise 6.3** (5.23 of [1], important, slightly changed)**.** A problem about flexes (see Problem 3.12 of [1]): Let $F$ be a projective plane curve of degree $n$, and assume $F$ contains no lines.

Let $F_i = F_{X_i}$ and $F_{ij} = F_{X_iX_j}$, forms of degree $n-1$ and $n-2$ respectively. Form a $3 \times 3$ matrix with the entry in the $(i,j)$th place being $F_{ij}$. Let $h$ be the determinant of this matrix, a form of degree $3(n-2)$. This $H$ is called the *Hessian* of $F$.

1. Show that $H$ vanishes identically on an irreducible plane curve iff the curve is a line.

The following theorem shows the relationship between flexes and the Hessian.

**Theorem.** ($\mathrm{char}(k) = 0$)

(i) $P \in H \cap F$ if and only if $P$ is either a flex or a multiple point of $F$.

(ii) $I(P, H \cap F) = 1$ if and only if $P$ is an ordinary flex.

Outline of the proof.

2. Let $T$ be a projective change of coordinates. Then the Hessian of $F^T = (\det(T))^2(H^T)$. So we can assume $P = [0:0:1]$; write $f(X,Y) = F(X,Y,1)$ and $h(X,Y) = H(X,Y,1)$.
3. $(n-1)F_j = \sum_i X_iF_{ij}$. (Use Euler's Theorem.)
4. $I(P, f \cap h) = I(P, f \cap g)$ where $g = f_y^2 f_{xx} + f_x^2 f_{yy} - 2f_xf_yf_{xy}$. (*Hint*: Perform row and column operations on the matrix for $h$. Add $x$ times the first row plus $y$ times the second row to the third row, then apply the preceding part. Do the same with the columns. Then calculate the determinant.)
5. If $P$ is a multiple point on $F$, then $I(P, f \cap g) > 1$.

6. Suppose $P$ is a simple point, $Y = 0$ is the tangent line to $F$ at $P$, so $f = y + ax^2 + bxy + cy^2 + ex^2y + \dots$. Then $P$ is a flex if and only if $a = 0$, and $P$ is an ordinary flex if and only if $a = 0$ and $d \neq 0$. A short calculation shows that $g = 2a + 6dx + (8ac - 2b^2 + 2e)y +$ higher terms, which concludes the proof.

**Corollary.**

($i$) A nonsingular curve of degree $> 2$ always has a flex.

($ii$) A nonsingular cubic has nine flexes, all ordinary.

**Exercise 6.4** (5.26 of [1])**.** (char$(k) = 0$) Let $F$ be an irreducible curve of degree $n$ in $\mathbb{P}^2$. Suppose $P \in \mathbb{P}^2$, with $\nu_P(F) = r \geq 0$. Then for all but a finite number of lines $L$ through $P$, $L$ intersects $F$ in $n - r$ distinct points other than $P$. We outline a proof:

1. We may assume $P = [0 : 1 : 0]$. If $L_\lambda = \{[\lambda : t : 1] \mid t \in k\} \cup \{P\}$, we need only consider the $L_\lambda$. Then $F = A_r(X, Z)Y^{n-r} + \dots + A_n(X, Z)$, $A_r \neq 0$. (See Problems 4.24, 5.5 of [1].)
2. Let $G_\lambda(t) = F(\lambda, t, 1)$. It is enough to show that for all but a finite number of $\lambda$, $G_\lambda$ has $n - r$ distinct points.
3. Show that $G_\lambda$ has $n-r$ distinct roots if $A_r(\lambda, 1) \neq 0$, and $F \cap F_Y \cap L_\lambda = \{P\}$ (see Problem 1.53 of [1]).

# 7 Sheet 7 — 23 June 2020

**Exercise 7.1** (5.33 of [1])**.** Let $C$ be an irreducible cubic, $L$ a line such that $L \bullet C = P_1 + P_2 + P_3$, $P_i$ distinct. Let $L_i$ be the tangent line to $C$ at $P_i$: $L_i \bullet C = 2P_i + Q_i$ for some $Q_i$. Show that $Q_1$, $Q_2$, $Q_3$ lie on a line. ($L^2$ is a conic!)

**Exercise 7.2** (5.37 of [1])**.** Suppose $\mathcal{O}$ is a flex on $C$. (a) Show that the flexes form a subgroup of $C$; as an abelian group, this subgroup is isomorphic to $\mathbb{Z}/(3) \times \mathbb{Z}/(3)$. (b) Show that the flexes are exactly the elements of order three in the group. (I.e., exactly those elements $P$ such that $P \oplus P \oplus P = \mathcal{O}$.) (c) Show that a point $P$ is of order two in the group if and only if the tangent to $C$ at $P$ passes through $\mathcal{O}$. (d) Let $C = Y^2Z - X(X - Z)(X - \lambda Z)$, $\lambda \neq 0, 1$, $\mathcal{O} = [0 : 1 : 0]$. Find the points of order two. (e) Show that the points of order two on a nonsingular cubic form a group isomorphic to $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$. (f) Let $C$ be a nonsingular cubic, $P \in C$. How many lines through $P$ are tangent to $C$ at some point $Q \neq P$? (The answer depends on whether $P$ is a flex.)

**Exercise 7.3** (5.41 of [1])**.** Let $C$ be a nonsingular cubic, $\mathcal{O}$ a flex on $C$. Let $P_1, \dots, P_{3m} \in C$. Show that $P_1 \oplus \dots \oplus P_{3m} = \mathcal{O}$ if and only if there is a curve $F$ of degree $m$ such that $F \bullet C = \sum_{i=1}^{3m} P_i$. (*Hint*: Use induction on $m$. Let $L \bullet C = P_1 + P_2 + Q$, $L' \bullet C = P_3 + P_4 + R$, $L'' \bullet C = Q + R + S$, and apply induction to $S, P_5, \dots, P_{3m}$; use Noether's Theorem.)

**Exercise 7.4** (5.43 of [1])**.** For which points $P$ on a nonsingular cubic $C$ does there exist a nonsingular conic that intersects $C$ only at $P$.

**Exercise 7.5** (6.14 of [1])**.** Let $X$, $Y$ be varieties, $F\colon X \to Y$ a mapping. Let $X = \bigcup_\alpha U_\alpha$, $Y = \bigcup_\alpha V_\alpha$, with $U_\alpha$, $V_\alpha$ open subvarieties, and suppose $f(U_\alpha) \subset V_\alpha$ for all $\alpha$. (a) Show that $f$ is a morphism if and only if each restriction $f_\alpha\colon U_\alpha \to V_\alpha$ of $f$ is a morphism. (b) If each $U_\alpha$, $V_\alpha$ is affine, $f$ is a morphism if and only if each $\tilde{f}(\Gamma(V_\alpha)) \subset \Gamma(U_\alpha)$.

**Exercise 7.6** (6.16 of [1])**.** Let $f\colon X \to Y$ be a morphism of varieties, $X' \subset X$, $Y' \subset Y$ subvarieties (open or closed). Assume $f(X') \subset Y'$. Then the restriction of $f$ to $X'$ is a morphism from $X'$ to $Y'$. (Use Problems 6.14 and 2.9 of [1].)

# 8 Sheet 8 — 30 June 2020

**Exercise 8.1.** Show that the twisted cubic (projective or affine, whichever you prefer) has dimension 1 and is therefore a curve (i.e. variety of dimension 1).

**Exercise 8.2** (6.40 of [1])**.** If there is a dominating rational map from $X$ to $Y$, then $\dim(Y) \leq \dim(X)$.

**Exercise 8.3** (6.41 of [1])**.** Every $n$-dimensional variety is birationally equivalent to a hypersurface in $\mathbb{A}^{n+1}$ (or $\mathbb{P}^{n+1}$).

**Exercise 8.4** (6.43 of [1])**.** Let $C$ be a projective curve, $P \in C$. Then there is a birational morphism $f\colon C \to C'$, $C'$ a projective plane curve, such that $f^{-1}(f(P)) = \{P\}$. We outline a proof:

(a) We can assume: $C \subset \mathbb{P}^{n+1}$. Let $T, X_1, \ldots, X_n, Z$ be coordinates for $\mathbb{P}^{n+1}$; Then $C \cap V(T)$ is finite; $C \cap V(T, Z) = \varnothing$; $P = [0 : \cdots : 0 : 1]$; and $k(C)$ is algebraic over $k(u)$, where $u = \bar{T}/\bar{Z} \in k(C)$.

(b) For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n$, let $\phi_\lambda\colon C \to \mathbb{P}^2$ be defined by the formula $\phi([t : x_1 : \cdots : x_n : z]) = [t : \sum \lambda_i x_i : z]$. Then $\phi_\lambda$ is a well-defined morphism, and $\phi_\lambda(P) = [0 : 0 : 1]$. Let $C'$ be the closure of $\phi_\lambda(C)$.

(c) The variable $\lambda$ can be chosen so $\phi_\lambda$ is a birational morphism from $C$ to $C'$, and $\phi_\lambda^{-1}([0 : 0 : 1]) = \{P\}$. (Use Problem 6.32 of [1] and the fact that $C \cap V(T)$ is finite.)

# 9 Sheet 9 — 07 July 2020

Assume $k$ is algebraically closed of characteristic 0.

**Exercise 9.1** (6.43 of [1], important)**.** Let $C$ be a projective curve, $P \in C$. Then there is a birational morphism $f\colon C \to C'$, $C'$ a projective plane curve, such that $f^{-1}(f(P)) = \{P\}$. We outline a proof:

(a) We can assume: $C \subset \mathbb{P}^{n+1}$. Let $T, X_1, \ldots, X_n, Z$ be coordinates for $\mathbb{P}^{n+1}$; Then $C \cap V(T)$ is finite; $C \cap V(T, Z) = \varnothing$; $P = [0 : \cdots : 0 : 1]$; and $k(C)$ is algebraic over $k(u)$, where $u = \bar{T}/\bar{Z} \in k(C)$.

(b) For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n$, let $\phi_\lambda\colon C \to \mathbb{P}^2$ be defined by the formula $\phi([t : x_1 : \cdots : x_n : z]) = [t : \sum \lambda_i x_i : z]$. Then $\phi_\lambda$ is a well-defined morphism, and $\phi_\lambda(P) = [0 : 0 : 1]$. Let $C'$ be the closure of $\phi_\lambda(C)$.

(c) The variable $\lambda$ can be chosen so $\phi_\lambda$ is a birational morphism from $C$ to $C'$, and $\phi_\lambda^{-1}([0 : 0 : 1]) = \{P\}$. (Use Problem 6.32 of [1] and the fact that $C \cap V(T)$ is finite.)

**Exercise 9.2** (6.46 of [1], important)**.** Let $k(\mathbb{P}^1) = k(T)$, $T = X/Y$ (see Problem 4.8 of [1]). For any variety $V$, and $f \in k(V)$, $f \notin k$, the subfield $k(f)$ generated by $f$ is naturally isomorphic to $K(T)$. Thus a nonconstant $f \in k(V)$ corresponds to a homomorphism from $k(T)$ to $k(V)$, and hence to a dominating rational map from $V$ to $\mathbb{P}^1$. The corresponding map is usualy denoted also by $f$. If this rational map is a morphism, show that the pole set of $f$ is $f^{-1}([1:0])$.

**Exercise 9.3** (7.2 of [1])**.** (a) For each of the curves $F$ in §3.1 of [1], find $F'$; show that $F'$ is nonsingular in the first five examples, but not in the sixth. (b) Let $F = Y^2 - X^5$. What is $F'$? What is $(F')'$? What must be done to resolve the singularity of the curve $Y^2 = X^{2n+1}$?

**Exercise 9.4** (7.6 of [1])**.** If $P$ is an ordinary cusp on $C$, show that $f^{-1}(P) = \{P_1\}$, where $P_1$ is a simple point on $C'$.

## 10 Sheet 10 — 14 July 2020

**Exercise 10.1.** Show that a nonconstant morphism between two curves has finite fibre.

**Exercise 10.2.** Show that principal divisors form a subgroup of $\operatorname{Div} C$.

**Exercise 10.3.** Make sure you understand the sequence of equalities in the last step of the proof of Lemma 12.9, ii), Lecture 20 (specifically the third "=").

**Exercise 10.4.** Show that if $C$ is in good position, then so is $C'$ (see notation from Lecture 19 and hints therein).

**Exercise 10.5** (7.12 of [1])**.** Find a quadratic transformation of $Y^2Z^2 - X^4 - Y^4$ with only ordinary multiple points. Do the same with $Y^4 + Z^4 - 2X^2(Y - Z)^2$.

## 11 Sheet 11 — 21 July 2020

**Exercise 11.1.**

1. Show that if a rational map of curves has degree one, then it is a birational map.
2. If $f_0, \dots, f_r$ are functions in $k(C)$ for some smooth curve $C$ that are regular and do not vanish simultaneously on points of $C$, then $[f_0 : \cdots : f_r]$ is a morphism from $C$ to $\mathbb{P}^r$.

**Exercise 11.2.** Prove Lemma 12.16 from Lecture 22.

**Exercise 11.3.** Prove parts i) and iii) of Proposition 12.18 from Lecture 22.

**Exercise 11.4.** If $P$ is a base point of a divisor $D$ on a curve, then $L(D) = L(D - P)$.

## References

[1] Fulton, William. Algebraic curves. An introduction to algebraic geometry. Notes written with the collaboration of Richard Weiss. Reprint of 1969 original. Advanced Book Classics. *Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA*, 1989. xxii+226 pp. ISBN: 0-201-51010-3