GEOMETRIC QUANTIZATION SEMINAR

1. Introduction

Classical physics does not predict the behavior of atoms and molecules correctly. Indeed, classically, Coulomb's law implies that the electron of the hydrogen atom should orbit around the proton, and thus the electron continuously radiates energy and causes the hydrogen atom to collapse. This contradicts the observed stability of the hydrogen atom. One of the major triumphs of quantum mechanics is its explanation for the stability of atoms.

Mathematically, a classical mechanical system can be described by a so–called symplectic manifold M called the state space, and the observables are functions on M. A quantum mechanical system, on the other hand, is described by a Hilbert space, and the observables are "operators" on this Hilbert space. A process which roughly associates to a classical theory a quantum theory is called "quantization". Ideally, one would like to associate to each classical observable a quantum observable, but it is impossible to achieve this: there are no go theorems. In practice, one has to lower one's expectation so that a reasonable quantization process can be constructed.

The goal of this seminar is to study one particular method of quantization called geometric quantization. Position space quantization, momentum space quantization and holomorphic quantization are particular instances of geometric quantization. Geometric quantization is performed in three steps. The first step is called "prequantization", and it consists in the construction of a Hilbert space. This Hilbert space is roughly given by the square integrable sections of a complex line bundle over M. The smooth functions on M are mapped to operators on the Hilbert space. It turns out that this Hilbert space is "too big". Therefore, in the next step one considers only polarized square integrable sections of the complex line bundle. It may happen that there exist no polarized square integrable sections at all, but the metaplectic correction solves this problem by replacing the square integrable sections by half densities.

2. Seminar plan

Geometric Quantization of Symplectic vector spaces

2.1. **Recap of Classical Mechanics.** The main goal of this discussion is to briefly review the Newtonian and Hamiltonian formulation of classical mechanics with more emphasis on the Hamiltonian formulation. The discussion of Newtonian mechanics will include Newton's law of motion and conservation laws. The discussion of the Hamiltonian formulation of classical mechanics will include reformulation of Newtonian mechanics, it will also

give a hint that how it naturally leads to a beautiful mathematical subject called Symplectic geometry.
Seminar Date: 17.10.2017.
Note: Background in physics will be very helpful for this talk.
References: Hall [3], chapter 2, section 2.1 and section 2.5.

2.2. **Linear Symplectic Geometry.** Symplectic geometry naturally appears in the study of classical mechanics. In this talk, we will discuss linear symplectic geometry. In particular, we will introduce real symplectic vector spaces. A symplectic vector space is a real vector space equipped with a nondegenerate skew-symmetric bilinear form. Due to the presence of the bilinear form, we can define the notion of orthogonality and this allows to talk about isotropic, coisotropic and Lagrangian subspaces. Largrangian subspaces will play an important role later in the construction of quantum Hilbert spaces. We will also discuss automorphisms of symplectic vector spaces which are known as symplectomorphisms. Finally, we will discuss a Darboux theorem for symplectic vector spaces which says that any symplectic vector space is symplectomorphic to $\mathbb{R}^{2n}$ for some n.
Seminar Date: 24.10.2017.
References: Libermann and Marle [7] chapter I, section 1.1-1.3. Woodhouse [10] chapter 1.

2.3. **Obervables in classical mechanics and Poisson bracket.** We will introduce the notion of observables in the classical mechanics. Observables are simply functions on the phase space. As the phase space is a "symplectic manifold", the functions have a new multiplication structure called the Poisson bracket. We will discuss the Poisson bracket for the functions on a symplectic vector space. Interestingly, conservation laws in the classical mechanics can phrased using the Poisson bracket. Time permitting (which is optional), we will discuss Liouville's theorem which roughly states that the solutions of Hamilton's equation preserve the "symplectic volume".
Seminar Date: 07.11.2017.
References: Hall [3] chapter 2, section 2.5.

2.4. **Prequantization I.** Here, we will discuss the notion of prequantization on symplectic vector spaces. Prequantization is a first step towards geometric quantization. More precisely, we will discuss background materials such as Hilbert spaces and operators on Hilbert spaces in a nontechnical way and then introduce the notion of prequantization. We will also briefly discuss shortcomings of prequantization.
Seminar Date: 14.11.2017.
Reference: Hall [3] chapter 22, section $22.1-22.3$, Woodhouse [10] chapter 5, section 5.2, Kostant [5].

2.5. **(Geometric) Quantization I.** We will discuss geometric quantization of $\mathbb{R}^{2n}$. The idea is to start with prequantization of a symplectic vector space and then modify the construction to construct reasonable Hilbert

spaces for the quantum theory. The Hilbert space coming from a quantization is too big and loosely speaking, it consists of "functions" on the sylplectic vector space. A polarization is a geometric object that allows to choose functions depending only on "half" of the variables and this results a construction of a reasonable quantum Hilbert space of the theory. This talk will introduce all necessary ingredients for geometric quantization of a symplectic vector space such as polarization and discuss geometric quantization of $\mathbb{R}^{2n}$.
Seminar Date: 21.11.2017.
Note: Background in quantum mechanics is very helpful for this talk.
References: Hall [3] chapter 22, section 22.3, 22.4, Woodhouse [10], chapter 5, Kirwin and Wu [4].

Geometric quantization of sympletic manifolds

This part of the seminar will concentrate on geometric quantization of general symplectic manifolds.

2.6. **Introduction to Manifolds.** This talk will be a brief introduction to manifolds. We will give definition of smooth manifolds and discuss some examples such as sphere, real projective spaces and torus. We will also discuss the notion of vector fields and differential forms on a manifold.
Seminar Date: 28.11.2017.
Note: Familiarity with manifolds will have great advantage for this talk.
References: O'Neill [8] chapter 1.

2.7. **Vector Bundles on Manifolds.** Here, we will introduce the notion of vector bundles on a manifold. Tangent bundle, Cotangent bundle and line bundles will be the main examples. We will also discuss metric and connection on a vector bundle with more emphasis on line bundles.
Seminar Date: 05.12.2018.
References: Lee [6] selected materials from chapter 4, chapter 5, chapter 6.

2.8. **Symplectic manifolds and Prequantization II.** We will introduce symplectic manifolds and give few basic examples such as symplectic vector spaces and the cotangent bundle of a manifold. We will also discuss prequantization of a symplectic manifold.
Seminar Date: 12.12.2017.
References: Da Silva [2], chapter1-3, Hall [3] chapter 23, section 23.3, Kostant [5], Libermann and Marle [7], chapter 3, section 3.1-3.5.

2.9. **Polarizations of a symplectic manifolds with examples.** Recall from the linear case that we need the notion of polarization to construct a quantum theory from prequantization. Here, we will discuss the notion of polarization in a more general context. We will also discuss different types of polarizations.

Seminar Date: 19.12.2017.
Reference: Hall [3] chapter 23,section 23.4, Woodhouse [10] chapter 4, section 4.4 and 4.10, Kirwin and Wu [4].

2.10. **Geometric quantization II (without half form).** Here, we will sketch a general framework for geometric quantization on a symplectic manifold. As in the linear case, a Hilbert space of polarized sections can be constructed from prequantization and a choice of a polarization. We will mainly concentrate on geometric quantization of the cotangent bundle of a manifold. We will also see that geometric quantization of contangent bundle of a manifold with vertical polarization gives rise to the "position" Hilbert space which is also known as Hilbert space for Schrödinger representation.
Seminar Date: 09.01.2018.
Reference: Hall [3] chapter 23, section 23.5, Woodhouse [10] chapter 5, section 5.5.

2.11. **Geometric quantization III*.** Generally a polarized section for a real polarization tends to have infinite norm even for the cotangent bundle of a manifold. Hence, there may not exist any nonzero square integrable polarized sections for a real polarization. One can overcome this problem by using the notion of "half-forms". In this talk, we will introduce half-forms and discuss quantization with half forms for real and complex polarizations.
Seminar Date: 16.01.2018.
Note: This talk will be slightly challenging in the sense that it requires some new concepts such as half-forms and pairing of half-forms. In particular, the pairing of half forms is tricky and very sensitive to the choice of polarization.
Reference: Hall [3] chapter 23, section 23.6, Woodhouse [10], chapter 5, section 5.10.

2.12. **Geometric Quantization IV*.** Construction of Hilbert spaces via geometric quantization depends on a choice of a polarization. Using the notion of the so called pairing map, it is possible to compare half-form Hilbert spaces corresponding to two different polarizations when the polarizations are transverse to each other. In this talk, we will introduce the pairing map and discuss in detail the pairing map for the vertical and the horizental polarization of $\mathbb{R}^{2n}$. Interestingly, it turns out this pairing map between the Hilbert spaces corresponding to the vertical and the horizontal polarization of $\mathbb{R}^{2n}$ is the Fourier transform.
Seminar Date: 23.01.2018.
Note: This talk can be thought as an extension of the previous talk. It will be as challenging as the last talk because it essentially uses similar ideas.
Reference: Bates and Weinstein [1], example 7.20, Hall [3] chapter 23, section 23.8, Woodhouse [10] chapter 5, section 5.8 and chapter 6.

2.13. **Pairing maps.** In this talk, we will continue the discussion of the pairing map. In particular, we compute the pairing map associated to the vertical polarization and the canonical complex polarization of $T^*\mathbb{R}^n$. We

will see that this pairing map is the Segal-Bargmann transform.
Seminar Date: 30.01.2018.
Note: This talk depends heavily on the previous talk.
Reference: Rawnsley [9].

2.14. **Fermionic geometric quantization.** There is also a version of geometric quantization of a vector space with an inner product which is known as fermionic geometric quantization. As in the symplectic case, we can introduce the notion of polarization and construct Hilbert space of "polarized " sections. In this talk, we will explore this odd version of geometric quantization.
Seminar Date: 06.02.2018.
Note: Background on linear algebra and inner product spaces is sufficient for this talk.
Reference: Wu [11].

3. Prerequisites

A background on linear algebra and a background at the level of introduction to real analysis is required. A knowledge of introduction to manifolds is a plus but these concepts will be reviewed as needed.

References

[1] Sean Bates and Alan Weinstein. *Lectures on the geometry of quantization*, volume 8 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1997.
[2] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
[3] Brian C. Hall. *Quantum theory for mathematicians*, volume 267 of *Graduate Texts in Mathematics*. Springer, New York, 2013.
[4] William D. Kirwin and Siye Wu. Geometric quantization, parallel transport and the Fourier transform. *Comm. Math. Phys.*, 266(3):577–594, 2006.
[5] Bertram Kostant. Quantization and unitary representations. I. Prequantization. pages 87–208. Lecture Notes in Math., Vol. 170, 1970.
[6] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.
[7] Paulette Libermann and Charles-Michel Marle. *Symplectic geometry and analytical mechanics*, volume 35 of *Mathematics and its Applications*. 1987.
[8] Barrett O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity.
[9] J. H. Rawnsley. A nonunitary pairing of polarizations for the Kepler problem. *Trans. Amer. Math. Soc.*, 250:167–180, 1979.
[10] Nicholas Woodhouse. *Geometric quantization*. The Clarendon Press, Oxford University Press, New York, 1980. Oxford Mathematical Monographs.
[11] Siye Wu. Projective flatness in the quantisation of bosons and fermions. *J. Math. Phys.*, 56(7), 2015.