Integrable systems

The goal of this seminar is to develop a mathematical understanding of integrable systems. We will focus on the interplay between algebraic and geometric aspects and on the way discuss different ideas and tools from Lie groups and algebras, symplectic geometry, and complex geometry.

1 Preliminaries

The purpose of the first three talks is to introduce integrable systems through the Liouville theorem in the language of symplectic and Poisson geometry.

Talk 1: Hamiltonian formalism

Date: 22.10.

- Discuss the Hamiltonian formulation of classical mechanics (Hamiltonian equations of motion and Poisson brackets)
- Discuss integrals of motion
- Some examples: particle in a rotationally symmetric potential and the harmonic oscillator

References: Sections 1.1-1.2 of [2] and Section 2.1 of [1].

Talk 2: Introduction to manifolds

Date: 29.10.

- Define the following (use subspaces of the Euclidean space $\mathbb{R}^n$ as toy-models): smooth manifolds, tangent spaces and bundles, differential forms and cotangent bundles, vector fields and their Lie algebras.
- Define Poisson manifolds

References: [5].

Talk 3: Introduction to symplectic and Poisson geometry

Date: 5.11.

- Show how one obtains a Poisson bracket from a symplectic form and define symplectic manifolds
- Explain why Hamiltonian flows are symplectic transformations

- State the Darboux theorem
- Cotangent bundle of a manifold example (if time permits)

References: Section 14.1 of [1]

Talk 4: Liouville integrability

Date: 12.11.

- Define Liouville integrability
- Discuss the phase-space structure, quadratures, and compact level sets
- Example of particle in rotationally symmetric potential.
- If time permits, comparison of chaotic-integrable systems.

References: Sections 1.3-1.4 from [2]. (For a more mathematical treatment of some of the topics above see Sections 2.2-2.3 from [1]).

2 Algebraic side

In what follows we reformulate the notion of integrable systems in terms of algebraic objects. In particular, we present the equations of motion as a matrix equation, where the matrices depend on dynamical variables. Integrability of the system imposes strong constraints on the matrices which we express as a so-called factorization problem.

Talk 5: Lax pairs and r-matrices

Date: 19.11.

- Define Lax pairs and explain how to get conserved quantities
- Involution property equivalent to existence of the function r (sketch of proof if there is time)
- Discuss the special cases for r constant and anti-symmetric
- Discuss harmonic oscillator example

References: Sections 2.4-2.5 of [1].

Talk 6: Examples

Date: 26.11.

- Discuss the example of the Euler top
- Discuss another example. Choose from: Kepler problem, Lagrange top, Kovalevskaya top.

References: Sections 2.7-2.10 of [1].

Talk 7: Lax pairs with spectral parameters

Date: 3.12.

- Discuss the Zakharov-Shabat construction.
- Illustrate the ideas with the Euler top example.

References: From [1]: Sections 3.1 (for the Euler top example) and 3.2 (for the Zakharov-Shabat construction, up to but excluding the remarks and the last Proposition)

Talk 8: (Co-)adjoint actions and loop groups

Date: 10.12.

- Define (co-)adjoint actions of Lie groups and algebras
- Give the interpretation of the Lax formalism in terms of the loop algebra

References: Section 3 of [4] and Section 3.3 of [1] (only up to and including equation (3.21)).

Talk 9: The factorisation problem

Date: 17.12.

- Describe the factorisation problem for integrable systems.

References: Section 4 of [4].

3 The geometric side

We follow Hitchin's lectures "Riemann surfaces and integrable systems" in [3]. We shall see how one can find solutions of certain types of integrable systems by studying line bundles on Riemann surfaces.

Talk 10: Riemann surfaces

Date: 07.01.

- Motivate the need for the theory of line bundles of Riemann surfaces
- Introduce Riemann surfaces, holomorphic line bundles and holomorphic sections
- State Theorem 1.6
- Give examples.

References: Section 1 in Chapter 2 of [3].

Talk 11: Line bundles and sheaves

Date: 14.01.

- Properties of line bundles
- Define: genus of a Riemann surface, sheaves, Cech cohomology
- State Theorems 2.4-2.6
- Many examples

References: Section 2 in Chapter 2 of [3].

Talk 12: Vector bundles

Date: 21.01.

- Define the degree of a line bundle using the exponential sequence and state its properties.
- Discuss the Picard group of a Riemann surface.
- Define vector bundles and their degrees.
- State the Riemann-Roch theorem and, if time permits, sketch the proof.

References: Section 3 in Chapter 2 of [3].

Talk 13: Direct images of line bundles

Date: 28.01.

- State the Birkhoff-Grothendieck theorem (no proof).
- State and prove and corollary immediately following the theorem.
- State Propositions 4.2, 4.3 (sketch proofs if there is time).
- State and prove Proposition 4.4.

References: Section 4 in Chapter 2 of [3].

Talk 14: Lax pair equations and Riemann surfaces

Date: 04.02.

- Show how to obtain a matrix polynomial from a line bundle on a Riemann surface.
- Introduce spectral curves.
- Discuss Proposition 5.1.
- Show how to interpret linear flows on the space of line bundles as solutions to Lax pairs equations.

References: Section 5 in Chapter 2 of [3].

4 Putting everything together

Talk 15: Linearisation and factorisation

Date: 11.02.

- Show that the solution of integrable systems by linearisation can be interpreted as a time evolution of the bundle from the previous talk.

References: Section 5.7 of [1].

References

[1] O.Babelon, D.Bernard, and M.Talon. *Introduction to classical integrable systems* Cambridge University Press, 2003.

[2] N.Beisert. *Introduction to integrability lecture notes.* `https://people.phys.ethz.ch/~nbeisert/lectures/IntHS16-Notes.pdf`

[3] N.J.Hitchin, G.B.Segal, and R.S.Ward. *Integrable systems: Twistors, loop groups, and Riemann surfaces.* Oxford University Press, 1999.

[4] T.Wasserman. *The Riemann-Hilbert factorisation problem in integrable systems.* `https://people.maths.ox.ac.uk/wasserman/pdfs/Riemann-Hilbert.pdf`

[5] L.W.Tu. *An introduction to manifolds.* Springer, 2011.